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Exact Controllability and Asymptotic Analysis for Shallow Shells

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Abstract The authors consider the exact controllability of the vibrations of a thin shallow shell, of thickness 2ϵ with controls imposed on the lateral surface and at the top and bottom of the shell. Apart from proving the existence of exact controls, it is shown that the solutions of the three dimensional exact controllability problems converge, as the thickness of the shell goes to zero, to the solution of an exact controllability problem in two dimensions.

Keywords Exact controllability, Asymptotic analysis, Shallow shells
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1 Introduction

The problem of exact controllability has been studied extensively by J. L. Lions [11, 12] and the asymptitic behaviour of thin plates and shells has been studied by P. G. Ciarlet, V. Lods, B. Miara and others (cf. [1–6]). I. Figueiredo and E. Zuazua [7] have studied the exact controllability and asymptotic behavior for thin plates and in this paper, we study the exact controllability problem for thin shallow shells and the limiting behaviour of the solutions.

We begin with a brief description of the problem and describe the results obtained.

Let $\widehat{\Omega}^\epsilon = \Phi^\epsilon(\Omega^\epsilon),\ \Omega^\epsilon = \omega \times (-\epsilon, \epsilon)$ with $\omega \subset \mathbb{R}^2$, and the mapping $\Phi^\epsilon : \overline{\Omega}^\epsilon \to \mathbb{R}^3$ be given by

$$\Phi^\epsilon(x^\epsilon) = (x_1, x_2, \epsilon\varphi(x_1, x_2)) + x_3^\epsilon a_3^\epsilon(x_1, x_2)$$

for all $x^\epsilon = (x_1, x_2, x_3^\epsilon) \in \overline{\Omega}^\epsilon$, where φ is an injective mapping of class C^3 and a_3^ϵ is a unit normal vector to the middle surface $\Phi^\epsilon(\overline{\omega})$ of the shell. Let γ_0 be the boundary of ω and let $\widehat{\Gamma}_0^\epsilon = \Phi^\epsilon(\gamma_0 \times (-\epsilon, \epsilon))$ and $\widehat{\Gamma}_\pm^\epsilon = \Phi^\epsilon(\omega \times \{\pm\epsilon\})$.

The exact controllability problem may be formulated as follows: given initial data $\{\hat{\psi}_0^\epsilon, \hat{\psi}_1^\epsilon\}$ in a suitable energy space, does there exists a time $T > 0$ and controls $\hat{u}^\epsilon = (\hat{u}_i^\epsilon)$ on $\widehat{\Gamma}_0^\epsilon$ and $\hat{v}^\epsilon = (\hat{v}_i^\epsilon)$ on $\widehat{\Gamma}_\pm^\epsilon$ such that the unique solution $\hat{\psi}^\epsilon$ of the problem (2.7) reaches equilibrium at time T, that is $\hat{\psi}^\epsilon(T) = \dot{\hat{\psi}}^\epsilon(T) = 0$.

In this article we first show, using Hilbert Uniqueness Method (HUM), that this problem is exactly controllable by assuming the validity of the regularity result described in Lemma 3.1. We then make appropriate scalings on the data and the unknowns and transfer the problem to a domain $\Omega = \omega \times (-1, 1)$ which is independent of ϵ and study the asymptotic behaviour

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of the scaled controlled solutions. The key to the asymptotic analysis lies in establishing the weak convergence (4.32) of the scaled solutions $\phi(\epsilon)$ of the homogeneous problem (3.9) and the strong convergence (4.55) of the scaled solutions $\theta(\epsilon)$ of the forward Cauchy problem (3.1). We then show that the limit (ψ_i) of the scaled controlled solutions $(\psi(\epsilon))$ of (3.18) is of the Kirchhoff-Love form; that is, ψ_3 is independent of x_3,

$$\psi_\alpha = \hat{\psi}_\alpha - x_3\partial_\alpha\psi_3, \quad \hat{\psi}_\alpha \text{ is independent of } x_3.$$

Moreover ψ_3 is the solution (in the transposition sense) of a two dimensional problem with controls on the boundary and interior of the shell and the functions $\hat{\psi}_\alpha$ can be uniquely determined in terms of a known function.

2 The Three-Dimensional Problem

Throughout this paper, Latin indices vary over the set $\{1, 2, 3\}$ and Greek indices over the set $\{1, 2\}$ for the components of vectors and tensors. The summation over repeated indices will be used.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary γ_0 and let ω lie locally on one side of γ_0. For each $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \quad \Gamma_\pm^\epsilon = \omega \times \{\pm\epsilon\}, \quad \Gamma_0^\epsilon = \gamma_0 \times (-\epsilon, \epsilon).$$

Let $x^\epsilon = (x_1, x_2, x_3^\epsilon)$ be a generic point on Ω^ϵ and let $\partial_\alpha = \partial_\alpha^\epsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\epsilon = \frac{\partial}{\partial x_3^\epsilon}$.

We assume that for each ϵ, we are given a function $\varphi^\epsilon : \omega \to \mathbb{R}$ of class C^3. We then define the map $h^\epsilon : \omega \to \mathbb{R}^3$ by

$$h^\epsilon(x_1, x_2) = (x_1, x_2, \varphi^\epsilon(x_1, x_2)) \quad \text{for all } (x_1, x_2) \in \omega. \tag{2.1}$$

At each point of the middle surface $S^\epsilon = h^\epsilon(\omega)$, we define the normal vector

$$a^\epsilon = (|\partial_1\varphi^\epsilon|^2 + |\partial_2\varphi^\epsilon|^2 + 1)^{-\frac{1}{2}}(-\partial_1\varphi^\epsilon, -\partial_2\varphi^\epsilon, 1).$$

For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$ by

$$\Phi^\epsilon(x^\epsilon) = (x_1, x_2, \varphi^\epsilon(x_1, x_2)) + x_3^\epsilon a^\epsilon(x_1, x_2) \quad \text{for all } x^\epsilon \in \Omega^\epsilon. \tag{2.2}$$

It can be shown that there exists an $\epsilon_0 > 0$ such that the mapping $\Phi^\epsilon : \Omega^\epsilon \to \Phi^\epsilon(\Omega^\epsilon)$ is a C^1 diffeomorphism for all $0 < \epsilon \leq \epsilon_0$. The set $\widehat{\Omega}^\epsilon = \Phi^\epsilon(\Omega^\epsilon)$ is the reference configuration of the shell. We denote by $\hat{e}^i$ the standard basis in $\mathbb{R}^3$.

For $0 < \epsilon \leq \epsilon_0$, we define the sets

$$\widehat{\Gamma}_\pm^\epsilon = \Phi^\epsilon(\Gamma_\pm^\epsilon), \quad \widehat{\Gamma}_0^\epsilon = \Phi^\epsilon(\Gamma_0^\epsilon)$$

and we define vectors g_i^ϵ and $g^{i,\epsilon}$ by the relations

$$g_i^\epsilon = \partial_i^\epsilon\Phi^\epsilon \quad \text{and} \quad g^{j,\epsilon}.g_i^\epsilon = \delta_i^j$$

which form the covariant and contravariant basis respectively at $\Phi^\epsilon(x^\epsilon)$. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\epsilon = g_i^\epsilon.g_j^\epsilon \quad \text{and} \quad g^{ij,\epsilon} = g^{i,\epsilon}.g^{j,\epsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon}.\partial_j^\epsilon g_i^\epsilon.$$

Note however that when the set Ω^ϵ is of the special form $\Omega^\epsilon = \omega \times (-\epsilon, \epsilon)$ and the mapping Φ^ϵ is of the form (2.2), the following relations hold

$$\Gamma_{\alpha 3}^{3,\epsilon} = \Gamma_{33}^{p,\epsilon} = 0.$$

The volume element is given by $\sqrt{g^\epsilon}dx^\epsilon$ where

$$g^\epsilon = \det(g_{ij}^\epsilon).$$

It can be shown that for ϵ sufficiently small, there exist constants g_1 and g_2 such that

$$0 < g_1 \le g^\epsilon \le g_2. \tag{2.3}$$

Let $\widehat{A}^{ijkl,\epsilon}$ denote the elastic tensors. We assume that the material of the shell is homogeneous and isotropic. Then the elasticity tensor is given by

$$\widehat{A}^{ijkl,\epsilon} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \tag{2.4}$$

where λ and μ are the Lamè constants of the material.

It satisfies the following coercive and symmetry relations. There exists a constant $c > 0$ such that for all symmetric tensors (t_{ij})

$$\widehat{A}^{ijkl,\epsilon} t_{kl} t_{ij} \ge c \sum_{i,j=1}^{3} (t_{ij})^2, \tag{2.5}$$

$$\widehat{A}^{ijkl,\epsilon} = \widehat{A}^{klij,\epsilon} = \widehat{A}^{jikl,\epsilon}. \tag{2.6}$$

Then the system of equations which govern the vibrations of the medium $\widehat{\Omega}^\epsilon$ is:

$$\begin{aligned}
&\hat{\rho}^\epsilon \ddot{\hat{\psi}}_i^\epsilon - \hat{\partial}_j^\epsilon \hat{\sigma}_{ij}^\epsilon(\hat{\psi}^\epsilon) = 0 && \text{in } \widehat{Q}^\epsilon = \widehat{\Omega}^\epsilon \times (0,T),\\
&\hat{\psi}_i^\epsilon = \hat{u}_i^\epsilon && \text{on } \widehat{\Sigma}_0^\epsilon = \widehat{\Gamma}_0^\epsilon \times (0,T),\\
&\hat{\sigma}_{ij}^\epsilon(\hat{\psi}^\epsilon)\hat{\nu}_j^\epsilon = \hat{v}_i^\epsilon && \text{on } \widehat{\Sigma}_\pm^\epsilon = \widehat{\Gamma}_\pm^\epsilon \times (0,T),\\
&\hat{\psi}^\epsilon(0) = \hat{\psi}_0^\epsilon, \quad \dot{\hat{\psi}}^\epsilon(0) = \hat{\psi}_1^\epsilon && \text{in } \widehat{\Omega}^\epsilon,
\end{aligned} \tag{2.7}$$

where $\hat{\nu}^\epsilon$ is the unit normal vector along the boundary of $\widehat{\Omega}^\epsilon$, $\hat{\rho}^\epsilon$ is the density of mass and

$$\hat{\sigma}_{ij}^\epsilon(\hat{\psi}^\epsilon) = \widehat{A}^{ijkl,\epsilon}\hat{e}_{kl}^\epsilon(\hat{\psi}^\epsilon), \quad \hat{e}_{ij}^\epsilon(\hat{\psi}^\epsilon) = \frac{1}{2}(\hat{\partial}_i^\epsilon \hat{\psi}_j^\epsilon + \hat{\partial}_j^\epsilon \hat{\psi}_i^\epsilon). \tag{2.8}$$

The controls are $\hat{u}^\epsilon$ on the lateral surface $\widehat{\Gamma}_0^\epsilon$ through Dirichlet action and $\hat{v}^\epsilon$ on the upper and lower faces $\widehat{\Gamma}_\pm^\epsilon$ through Neumann action.

We define the spaces

$$H^1_{\widehat{\Gamma}_0^\epsilon}(\widehat{\Omega}^\epsilon) = \{\hat{v}^\epsilon \in H^1(\widehat{\Omega}^\epsilon) : \hat{v}^\epsilon_{\widehat{\Gamma}_0^\epsilon} = 0\}, \tag{2.9}$$

$$V(\widehat{\Omega}^\epsilon) = [H^1_{\widehat{\Gamma}_0^\epsilon}(\widehat{\Omega}^\epsilon)]^3, \tag{2.10}$$

$$X(\widehat{\Omega}^\epsilon) = \{\hat{g}^\epsilon \in L^1(0,T; L^2(\widehat{\Omega}^\epsilon)) : \dot{\hat{g}}^\epsilon \in L^1(0,T; [H^1_{\widehat{\Gamma}_0^\epsilon}(\widehat{\Omega}^\epsilon)]')\}. \tag{2.11}$$

We introduce the function $\hat{q}^\epsilon = (\hat{q}_i^\epsilon)$ by

$$\hat{q}^\epsilon(\hat{x}^\epsilon) = \hat{x}^\epsilon - \hat{x}_0^\epsilon = \Phi^\epsilon(x^\epsilon) - \Phi^\epsilon(x_0^\epsilon) \quad \forall\, \hat{x}^\epsilon \in \widehat{\Omega}^\epsilon,$$

where $\hat{x}_0^\epsilon$ is a fixed point in the middle surface of the shell, and the constants $R(\hat{x}_0^\epsilon)$ and T^ϵ are

$$R(\hat{x}_0^\epsilon) = \|\hat{x}^\epsilon - \hat{x}_0^\epsilon\|_{L^\infty(\widehat{\Omega}^\epsilon)}, \quad T^\epsilon = \frac{2\sqrt{\hat{\rho}^\epsilon}}{\sqrt{\mu}} \max\Big\{R(\hat{x}_0^\epsilon), \frac{C(\widehat{\Omega}^\epsilon)}{R(\hat{x}_0^\epsilon)}\Big\},$$

where $C(\widehat{\Omega}^\epsilon)$ is the constant of continuity of the trace map tr : $V(\widehat{\Omega}^\epsilon) \to [L^2(\hat{\partial}^\epsilon\widehat{\Omega}^\epsilon)]^3$. We denote by $\hat{\tau}_j^\epsilon$ the j-th component of the tangential gradient on $\hat{\partial}^\epsilon\widehat{\Omega}^\epsilon$.

Throughout this paper, we denote by $C_i, i = 1, 2, 3, \cdots$ various constants which are independent of ϵ.

3 Preliminary Results

In this section, we will first recall some existence, regularity and energy estimate results for the forward Cauchy problem associated with (2.7) and then we will deduce some identities for the 3D problem.

Let $\hat{\theta}^\epsilon$ be the solution of the following forward Cauchy problem; that is,

$$\begin{aligned}
&\hat{\rho}^\epsilon \ddot{\hat{\theta}}_i^\epsilon - \hat{\partial}_j^\epsilon \hat{\sigma}_{ij}^\epsilon(\hat{\theta}^\epsilon) = \hat{f}^\epsilon && \text{in } \widehat{Q}^\epsilon,\\
&\hat{\theta}_i^\epsilon = 0 && \text{on } \widehat{\Sigma}_0^\epsilon,\\
&\hat{\sigma}_{ij}^\epsilon(\hat{\theta}^\epsilon)\hat{\nu}_j^\epsilon = 0 && \text{on } \widehat{\Sigma}_\pm^\epsilon,\\
&\hat{\theta}^\epsilon(0) = \hat{\theta}_0^\epsilon, \quad \dot{\hat{\theta}}^\epsilon(0) = \hat{\theta}_1^\epsilon && \text{in } \widehat{\Omega}^\epsilon.
\end{aligned} \tag{3.1}$$

If $\hat{f}^\epsilon = 0$ then we use $\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon$ and $\hat{\phi}^\epsilon$ in place of $\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon$ and $\hat{\theta}^\epsilon$ respectively.

Let $E^{\hat{\theta}^\epsilon}(t)$ denote the energy of the solution $\hat{\theta}^\epsilon$ at time $t \in [0, T]$; that is,

$$E^{\hat{\theta}^\epsilon}(t) = \frac{1}{2}\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon \sum_i |\dot{\hat{\theta}}_i^\epsilon(t)|^2 d\widehat{\Omega}^\epsilon + \frac{1}{2}\hat{a}^\epsilon(\hat{\theta}^\epsilon(t), \hat{\theta}^\epsilon(t)), \tag{3.2}$$

where

$$\hat{a}^\epsilon(\hat{\theta}^\epsilon, \hat{\phi}^\epsilon) = \int_{\widehat{\Omega}^\epsilon} \hat{\sigma}_{ij}^\epsilon(\hat{\theta}^\epsilon)\hat{\partial}_j^\epsilon(\hat{\phi}_i^\epsilon) d\widehat{\Omega}^\epsilon. \tag{3.3}$$

When $t = 0$, we have

$$E^{\hat{\theta}^\epsilon}(0) = \frac{1}{2}\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon \sum_i |\hat{\theta}_{1i}^\epsilon|^2 d\widehat{\Omega}^\epsilon + \frac{1}{2}\hat{a}^\epsilon(\hat{\theta}_0^\epsilon, \hat{\theta}_0^\epsilon). \tag{3.4}$$

Remark 3.1 For $u \in C^2([0, T], [V(\Omega)]')$ and $v \in V(\Omega)$ we denote by $\displaystyle\int_\Omega \ddot{u}v dx$ the duality product between $\ddot{u} \in [V(\Omega)]'$ and $V(\Omega)$.

Lemma 3.1 (a) *Assume that $\hat{\theta}_0^\epsilon \in V(\widehat{\Omega}^\epsilon)$, $\hat{\theta}_1^\epsilon \in [L^2(\widehat{\Omega}^\epsilon)]^3$ and $\hat{f}^\epsilon \in L^1(0, T; (L^2(\widehat{\Omega}^\epsilon))^3)$. Then there exists a unique solution $\hat{\theta}^\epsilon$ of* (3.1) *with*

$$\hat{\theta}^\epsilon \in C^0([0, T], V(\widehat{\Omega}^\epsilon)) \cap C^1([0, T], (L^2(\widehat{\Omega}^\epsilon))^3) \cap W^{2,1}([0, T], V(\widehat{\Omega}^\epsilon)'). \tag{3.5}$$

(b) *If* $\hat{\theta}_0^\epsilon \in H^2(\widehat{\Omega}^\epsilon) \cap V(\widehat{\Omega}^\epsilon), \hat{\theta}_1^\epsilon \in V(\widehat{\Omega}^\epsilon)$ *and* $\hat{f}^\epsilon \in L^1(0,T;V(\widehat{\Omega}^\epsilon))$ *then*

$$\hat{\theta}^\epsilon \in C^0([0,T], H^{\frac{3}{2}+\delta}(\widehat{\Omega}^\epsilon) \cap V(\widehat{\Omega}^\epsilon)) \cap C^1([0,T], V(\widehat{\Omega}^\epsilon)) \cap W^{2,1}([0,T], (L^2(\widehat{\Omega}^\epsilon)^3)) \tag{3.6}$$

for some $\delta > 0$.

(c) *The following energy estimate holds:*

$$E^{\hat{\theta}^\epsilon}(t) \leq C_1 \Big\{ E^{\hat{\theta}^\epsilon}(0) + \frac{1}{2\hat{\rho}^\epsilon} \sum_{i=1}^{3} \Big[\int_0^T ||\hat{f}^{i,\epsilon}||_{L^2(\widehat{\Omega}^\epsilon)} dt \Big]^2 \Big\}. \tag{3.7}$$

Moreover, if $\hat{f}^{\alpha,\epsilon} \in X(\widehat{\Omega}^\epsilon)$ *and* $\hat{f}^{3,\epsilon} \in L^1(0,T;L^2(\widehat{\Omega}^\epsilon))$, *then*

$$E^{\hat{\theta}^\epsilon}(0) \leq C_2 \Big\{ E^{\hat{\theta}^\epsilon}(0) + \sum_{\alpha=1}^{2} \|\hat{f}^{\alpha,\epsilon}\|^2_{X(\widehat{\Omega}^\epsilon)} + \frac{1}{2\hat{\rho}^\epsilon} \Big[\int_0^T \|\hat{f}^{3,\epsilon}\|_{L^2(\widehat{\Omega}^\epsilon)} dt \Big]^2 \Big\}. \tag{3.8}$$

Proof The proof of (a) is classical and for (b) we refer the reader to the work of Grisward [8] and Nicaise [14]. The proof of (c) is similar to the proof of [7, Lemma 2.1].

3.1 Identities related to the 3D shell problem

Letting $\hat{f}^\epsilon = 0$ in (3.1), we see that $\hat{\phi}^\epsilon$ satisfies

$$\begin{aligned}
&\hat{\rho}^\epsilon \ddot{\hat{\phi}}_i^\epsilon - \hat{\partial}_j^\epsilon \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon) = 0 && \text{in } \widehat{Q}^\epsilon, \\
&\hat{\phi}_i^\epsilon = 0 && \text{on } \widehat{\Sigma}_0^\epsilon, \\
&\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\nu}_j^\epsilon = 0 && \text{on } \widehat{\Sigma}_\pm^\epsilon, \\
&\hat{\phi}^\epsilon(0) = \hat{\phi}_0^\epsilon, \quad \dot{\hat{\phi}}^\epsilon(0) = \hat{\phi}_1^\epsilon && \text{in } \widehat{\Omega}^\epsilon.
\end{aligned} \tag{3.9}$$

Then we have the following identity.

Lemma 3.2 *Let* $\hat{\theta}^\epsilon$ *be the solution of* (3.1) *with* $\hat{\theta}_0^\epsilon \in H^2(\widehat{\Omega}^\epsilon) \cap V(\widehat{\Omega}^\epsilon)$, $\hat{\theta}_1^\epsilon \in V(\widehat{\Omega}^\epsilon)$ *and* $\hat{f}^\epsilon \in L^1(0,T;V(\widehat{\Omega}^\epsilon))$ *and* $\hat{\phi}^\epsilon$ *be the solution of* (3.9) *with* $\hat{\phi}_0^\epsilon \in H^2(\widehat{\Omega}^\epsilon) \cap V(\widehat{\Omega}^\epsilon)$, $\hat{\phi}_1^\epsilon \in V(\widehat{\Omega}^\epsilon)$. *Then*

$$\begin{aligned}
&\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt + \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)[\hat{\rho}^\epsilon \dot{\hat{\theta}}_i^\epsilon \dot{\hat{\phi}}_i^\epsilon - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon] d\widehat{\Gamma}_\pm^\epsilon dt \\
&= \Big[\int_{\widehat{\Omega}}^{\epsilon} \hat{\rho}^\epsilon \dot{\hat{\theta}}_i^\epsilon (\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon) d\widehat{\Omega}^\epsilon \Big]_0^T + \Big[\int_{\widehat{\Omega}}^{\epsilon} \hat{\rho}^\epsilon \dot{\hat{\phi}}_i^\epsilon (\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon \hat{\theta}_i^\epsilon) d\widehat{\Omega}^\epsilon \Big]_0^T + \int_0^T \int_{\widehat{\Omega}}^{\epsilon} 3\hat{\rho}^\epsilon \dot{\hat{\theta}}_i^\epsilon \dot{\hat{\phi}}_i^\epsilon d\widehat{\Omega}^\epsilon dt \\
&\quad - \int_0^T \int_{\widehat{\Omega}^\epsilon} \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon d\widehat{\Omega}^\epsilon dt - \int_0^T \int_{\widehat{\Omega}^\epsilon} \hat{f}^{i,\epsilon}(\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon) d\widehat{\Omega}^\epsilon dt.
\end{aligned} \tag{3.10}$$

Proof The proof follows by multiplying the first equation of (3.1) by $\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon$ and the first equation of (3.9) by $\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon \hat{\theta}_i^\epsilon$ and integrating by parts.

Notice that

$$\Big[\int_{\widehat{\Omega}}^{\epsilon} \hat{\rho}^\epsilon \hat{\theta}_i^\epsilon \dot{\hat{\phi}}_i^\epsilon d\widehat{\Omega}^\epsilon \Big]_0^T = \int_0^T \int_{\widehat{\Omega}}^{\epsilon} \hat{\rho}^\epsilon \dot{\hat{\theta}}_i^\epsilon \dot{\hat{\phi}}_i^\epsilon d\widehat{\Omega}^\epsilon dt - \int_0^T \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon d\widehat{\Omega}^\epsilon dt. \tag{3.11}$$

Hence the equation (3.10) can be written as

$$\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt + \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)[\hat{\rho}^\epsilon\dot{\hat{\theta}}_i^\epsilon\dot{\hat{\phi}}_i^\epsilon - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon]d\widehat{\Gamma}_\pm^\epsilon dt$$
$$= \Big[\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon\dot{\hat{\theta}}_i^\epsilon(\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon)d\widehat{\Omega}^\epsilon\Big]_0^T + \Big[\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon\dot{\hat{\phi}}_i^\epsilon(\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon)d\widehat{\Omega}^\epsilon\Big]_0^T + 2\Big[\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon\hat{\theta}_i^\epsilon\dot{\hat{\phi}}_i^\epsilon d\widehat{\Omega}^\epsilon\Big]_0^T$$
$$- \int_0^T\int_{\widehat{\Omega}^\epsilon} \hat{f}^{i,\epsilon}(\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon)d\widehat{\Omega}^\epsilon dt + \int_0^T\int_{\widehat{\Omega}^\epsilon} \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\theta}_i^\epsilon d\widehat{\Omega}^\epsilon dt + \int_0^T\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon\dot{\hat{\theta}}_i^\epsilon\dot{\hat{\phi}}_i^\epsilon d\widehat{\Omega}^\epsilon dt. \tag{3.12}$$

Corollary 3.1 *Let $\hat{\phi}^\epsilon$ be the solution of* (3.9) *with initial data in $H^2(\widehat{\Omega}^\epsilon)\cap V(\widehat{\Omega}^\epsilon)\times V(\widehat{\Omega}^\epsilon)$. Then the following identity holds.*

$$\frac{1}{2}\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt + \frac{1}{2}\int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\Big[\hat{\rho}^\epsilon\sum_i(\dot{\hat{\phi}}_i^\epsilon)^2 - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon\Big]d\widehat{\Gamma}_\pm^\epsilon dt$$
$$= \Big[\int_{\widehat{\Omega}}^\epsilon \hat{\rho}^\epsilon\dot{\hat{\phi}}_i^\epsilon(\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon + \hat{\phi}_i^\epsilon)d\widehat{\Omega}^\epsilon\Big]_0^T + \int_0^T E^{\hat{\phi}^\epsilon}(0)dt. \tag{3.13}$$

Proof The proof follows by taking $\hat{f}^\epsilon = 0$ and $\hat{\theta}^\epsilon = \hat{\phi}^\epsilon$ in (3.12) and noting that $E^{\hat{\phi}^\epsilon}(t) = E^{\hat{\phi}^\epsilon}(0)$.

3.2 The exact controllability problem for 3D shell

We will now prove the exact controllability result for 3D shell using the Hilbert Uniqueness Method. In order to do that we will first establish some a priori estimates for the energy $E^{\hat{\phi}^\epsilon}(t)$ of problem (3.9). We will also introduce the transposition formulation and define the HUM operator and show that it is an isomorphism between $V(\widehat{\Omega}^\epsilon)\times L^2(\widehat{\Omega}^\epsilon)$ and its dual.

Theorem 3.1 (Direct Inequality) *Let $0<\epsilon\le 1$ and $T>0$ be fixed. Assume that $\hat{\phi}_0^\epsilon \in H^2(\widehat{\Omega}^\epsilon)\cap V(\widehat{\Omega}^\epsilon)$ and $\hat{\phi}_1^\epsilon \in V(\widehat{\Omega}^\epsilon)$. Then the solution $\hat{\phi}^\epsilon$ of* (3.9) *with initial data $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}$ satisfies*

$$\Big|\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt + \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\Big[\hat{\rho}^\epsilon\sum_i(\dot{\hat{\phi}}_i^\epsilon)^2 - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon\Big]d\widehat{\Gamma}_\pm^\epsilon dt\Big| \le C_3 E^{\hat{\phi}^\epsilon}(0). \tag{3.14}$$

Proof The proof follows from the above corollary.

Theorem 3.2 (Inverse Inequality) *Let $0<\epsilon\le 1$ and $T>T^\epsilon$. Then for every solution $\hat{\phi}^\epsilon$ of* (3.9) *with initial data $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}\in H^2(\widehat{\Omega}^\epsilon)\cap V(\widehat{\Omega}^\epsilon)\times V(\widehat{\Omega}^\epsilon)$, we have*

$$[T-T^\epsilon]E^{\hat{\phi}^\epsilon}(0) \le C_4\Big\{\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt$$
$$+ \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\Big[\hat{\rho}^\epsilon\sum_i(\dot{\hat{\phi}}_i^\epsilon)^2 - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon\Big]d\widehat{\Gamma}_\pm^\epsilon dt\Big\}. \tag{3.15}$$

Proof Proceeding as in [7, Theorem 3.2], it can be shown that

$$\Big|\int_{\Omega^\epsilon} \hat{\rho}^\epsilon\dot{\hat{\phi}}_i^\epsilon(\hat{q}_j^\epsilon.\hat{\partial}_j^\epsilon\hat{\phi}_i^\epsilon + \hat{\phi}_i^\epsilon)d\widehat{\Omega}^\epsilon\Big| \le \sqrt{\frac{\hat{\rho}^\epsilon}{\mu}}\max\Big\{R(\hat{x}_0^\epsilon), \frac{C(\widehat{\Omega}^\epsilon)}{R(\hat{x}_0^\epsilon)}\Big\}. \tag{3.16}$$

The result then follows from (3.13) and the above estimate.

From the above two theorems it follows that for a fixed ϵ and T with $0 < \epsilon \le 1$ and $T > T^\epsilon$, the mapping

$$\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\} \in H^2(\widehat{\Omega}^\epsilon) \cap V(\widehat{\Omega}^\epsilon) \times V(\widehat{\Omega}^\epsilon) \to \|\!|\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}|\!\|,$$

where

$$\begin{aligned}\|\!|\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}|\!\| = \Big\{ &\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt \\ &+ \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\Big[\hat{\rho}^\epsilon \sum_i (\dot{\hat{\phi}}_i^\epsilon)^2 - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon\Big] d\widehat{\Gamma}_\pm^\epsilon dt\Big\}^{\frac{1}{2}}\end{aligned} \tag{3.17}$$

is a norm in $H^2(\widehat{\Omega}^\epsilon) \cap V(\widehat{\Omega}^\epsilon) \times [V(\widehat{\Omega}^\epsilon)]^3$, is equivalent to the usual norm in $V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3$.

3.3 Transposition formulation

For a given $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\} \in V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3$, we first solve the homogeneous problem (3.9) for $\hat{\phi}^\epsilon$ with initial data $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}$. Then we introduce the backward Cauchy problem: find $\hat{\psi}^\epsilon$ such that

$$\begin{aligned}
&\hat{\rho}^\epsilon \ddot{\hat{\psi}}_i^\epsilon - \hat{\partial}_j^\epsilon \hat{\sigma}_{ij}^\epsilon(\hat{\psi}^\epsilon) = 0 && \text{in } \widehat{Q}^\epsilon,\\
&\hat{\psi}_i^\epsilon = \hat{q}_j^\epsilon \hat{\nu}_j^\epsilon \frac{\partial^\epsilon \hat{\phi}_i^\epsilon}{\partial^\epsilon \hat{\nu}^\epsilon} && \text{on } \widehat{\Sigma}_0^\epsilon,\\
&\hat{\sigma}_{ij}^\epsilon(\hat{\psi}^\epsilon)\hat{\nu}_j^\epsilon = \hat{q}_j^\epsilon \hat{\nu}_j^\epsilon [\hat{\rho}^\epsilon \ddot{\hat{\phi}}_i^\epsilon - \hat{\tau}_j^\epsilon \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)] && \text{on } \widehat{\Sigma}_\pm^\epsilon,\\
&\hat{\psi}^\epsilon(T) = 0, \quad \dot{\hat{\psi}}^\epsilon(T) = 0 && \text{in } \widehat{\Omega}^\epsilon.
\end{aligned} \tag{3.18}$$

The transposition formulation of (3.18) can be obtained as follows.

We multiply the first equation of (3.18) by $\hat{\theta}_i^\epsilon$, the solution of (3.1), and integrate by parts on $\widehat{Q}^\epsilon$ and we obtain the following identity

$$\begin{aligned}\int_{\widehat{\Omega}^\epsilon} [\hat{\rho}^\epsilon \dot{\hat{\psi}}_i^\epsilon(0)\hat{\theta}_{0i}^\epsilon - \hat{\rho}^\epsilon \hat{\psi}_i^\epsilon(0)\hat{\theta}_{1i}^\epsilon] d\widehat{\Omega}^\epsilon = &\int_{\widehat{Q}^\epsilon} \hat{\psi}_i^\epsilon \hat{f}^{i,\epsilon} d\hat{x}^\epsilon + \int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\theta}^\epsilon)\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt \\ &+ \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)[\hat{\rho}^\epsilon \dot{\hat{\phi}}_i^\epsilon \dot{\hat{\theta}}_i^\epsilon - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon \hat{\theta}_i^\epsilon] d\widehat{\Gamma}_\pm^\epsilon dt.\end{aligned} \tag{3.19}$$

Definition 3.1 *The function $\hat{\psi}^\epsilon$ is a solution of the problem* (3.18) *in the sense of transposition if $\hat{\psi}^\epsilon \in L^\infty(0,T;(L^2(\widehat{\Omega}^\epsilon))^3)$, the traces $\{\hat{\psi}^\epsilon(0), \dot{\hat{\psi}}^\epsilon(0)\}$ makes sense in $[L^2(\widehat{\Omega}^\epsilon)]^3 \times V(\widehat{\Omega}^\epsilon)'$ and $\hat{\psi}^\epsilon$ satisfies*

$$\begin{aligned}\langle\{\dot{\hat{\psi}}^\epsilon(0), -\hat{\psi}^\epsilon(0)\}, \{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\}\rangle_\epsilon - \int_{\widehat{Q}^\epsilon} \hat{\psi}_i^\epsilon \hat{f}^{i,\epsilon} d\hat{x}^\epsilon = &\int_{\widehat{\Sigma}_0^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)\hat{\sigma}_{ij}^\epsilon(\hat{\theta}^\epsilon)\hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon d\widehat{\Gamma}_0^\epsilon dt \\ &+ \int_{\widehat{\Sigma}_\pm^\epsilon} (\hat{q}^\epsilon.\hat{\nu}^\epsilon)[\hat{\rho}^\epsilon \dot{\hat{\phi}}_i^\epsilon \dot{\hat{\theta}}_i^\epsilon - \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)\hat{\partial}_j^\epsilon \hat{\theta}_i^\epsilon] d\widehat{\Gamma}_\pm^\epsilon dt\end{aligned} \tag{3.20}$$

for any $\hat{f}^\epsilon \in L^1(0,T;L^2(\widehat{\Omega}^\epsilon)^3)$ and for any $\{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\} \in V(\widehat{\Omega}^\epsilon) \times (L^2(\widehat{\Omega}^\epsilon))^3$ with

$$\langle\{\dot{\hat{\psi}}^\epsilon(0), -\hat{\psi}^\epsilon(0)\}, \{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\}\rangle_\epsilon = \langle\{\hat{\rho}^\epsilon \dot{\hat{\psi}}_i^\epsilon(0), -\hat{\rho}^\epsilon \hat{\psi}_i^\epsilon(0)\}, \{\hat{\theta}_{0i}^\epsilon, \hat{\theta}_{1i}^\epsilon\}\rangle, \tag{3.21}$$

where $\langle\cdot\,,\,\cdot\rangle$ denote the dual product between $V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3$ and its dual.

Theorem 3.3 *Let $0 < \epsilon \le 1$ and $T > T^\epsilon$ be fixed. Then there exists a unique solution $\hat{\psi}^\epsilon \in L^\infty(0, T; (L^2(\Omega^\epsilon))^3)$ of (3.18) in the sense of transposition.*

Proof The proof follows by duality arguments.

3.4 The HUM operator

Let $0 < \epsilon \le 1$ be fixed and $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\} \in V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3$. First we solve the problem (3.9) with initial data $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}$ and then we solve the problem (3.18) in the transposition sense, i.e., we solve (3.19). Let $\hat{\psi}^\epsilon$ be the solution of (3.19) and let $\hat{\psi}_0^\epsilon = \hat{\psi}^\epsilon(0)$ and $\hat{\psi}_1^\epsilon = \dot{\hat{\psi}}^\epsilon(0)$. Then we define

$$\widehat{\Lambda}^\epsilon(\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}) = \{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\},$$

that is,

$$\langle \widehat{\Lambda}^\epsilon(\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\}, \{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\})\rangle_\epsilon = \langle \{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\}, \{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\}\rangle_\epsilon$$

for any $\{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\} \in V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3$.

Theorem 3.4 *Let $0 < \epsilon \le 1$ and $T > T^\epsilon$ be fixed. Then the operator $\widehat{\Lambda}^\epsilon$ is a continuous isomorphism between $V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3$ and its dual. Moreover if $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\} = (\widehat{\Lambda}^\epsilon)^{-1}(\{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\})$ where $\{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\} \in V(\widehat{\Omega}^\epsilon)' \times [L^2(\widehat{\Omega}^\epsilon)]^3$, we have*

$$\Big\{\hat{\rho}^\epsilon \sum_i \|\hat{\phi}_{1i}^\epsilon\|^2_{L^2(\widehat{\Omega})} + \hat{a}^\epsilon(\hat{\phi}_0^\epsilon, \hat{\phi}_0^\epsilon)\Big\}^{\frac{1}{2}} \le \frac{C_6}{T - T^\epsilon} \|\{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\}\|_{V(\widehat{\Omega}^\epsilon) \times [L^2(\widehat{\Omega}^\epsilon)]^3}. \tag{3.22}$$

Proof The proof is similar to the proof of [7, Theorem 3.4].

Theorem 3.5 (Controllability Result) *Let $0 < \epsilon \le 1$ be fixed. If $T > T^\epsilon$, then the elasticity system (2.7) is exactly controllable. More precisely, if $\{\hat{\psi}_0^\epsilon, \hat{\psi}_1^\epsilon\} \in [L^2(\widehat{\Omega}^\epsilon)]^3 \times [V(\widehat{\Omega}^\epsilon)]'$ then there exist controls of the form*

$$\begin{aligned} \hat{u}_i^\epsilon &= \hat{q}_j^\epsilon \hat{\nu}_j^\epsilon \frac{\hat{\partial}^\epsilon \hat{\phi}_i^\epsilon}{\hat{\partial}^\epsilon \hat{\nu}^\epsilon} && \text{on } \widehat{\Sigma}_0^\epsilon, \\ \hat{v}_i^\epsilon &= \hat{q}_j^\epsilon \hat{\nu}_j^\epsilon [\hat{\rho}^\epsilon \ddot{\hat{\phi}}_i^\epsilon - \hat{\tau}_j^\epsilon \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon)] && \text{on } \widehat{\Sigma}_\pm^\epsilon, \end{aligned} \tag{3.23}$$

where $\hat{\phi}^\epsilon$ is the solution of (3.9) with initial data

$$\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\} = (\widehat{\Lambda}^\epsilon)^{-1}(\{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\}) \tag{3.24}$$

such that the solution $\hat{\psi}^\epsilon$ of (2.7) satisfies $\hat{\psi}^\epsilon(T) = \dot{\hat{\psi}}^\epsilon(T) = 0$.

Proof We first solve the problem (3.9) with initial data $\{\hat{\phi}_0^\epsilon, \hat{\phi}_1^\epsilon\} = (\widehat{\Lambda}^\epsilon)^{-1}(\{\hat{\psi}_1^\epsilon, -\hat{\psi}_0^\epsilon\})$ and we obtain the function $\hat{\phi}^\epsilon$. Then we define the controls $\hat{u}^\epsilon$ and $\hat{v}^\epsilon$ as in (3.23). In view of (3.24), the solution $\hat{\psi}^\epsilon$ of (3.18) satisfies

$$\hat{\psi}^\epsilon(0) = \hat{\psi}_0^\epsilon, \quad \dot{\hat{\psi}}^\epsilon(0) = \hat{\psi}_1^\epsilon,$$

or, equivalently, the solution of (2.7) satisfies $\hat{\psi}^\epsilon(T) = \dot{\hat{\psi}}^\epsilon(T) = 0$.

Note that because of the relation (3.10), the equation (3.20) is equivalent to

$$
\begin{aligned}
&\langle\{\dot{\hat{\psi}}^\epsilon(0), -\hat{\psi}^\epsilon(0)\}, \{\hat{\theta}_0^\epsilon, \hat{\theta}_1^\epsilon\}\rangle_\epsilon - \int_{\hat{Q}}^{\epsilon} \hat{\psi}_i^\epsilon \hat{f}^{i,\epsilon} d\hat{x}^\epsilon \\
&= \Big[\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon \dot{\hat{\theta}}_i^\epsilon (\hat{q}_j^\epsilon . \hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon) d\widehat{\Omega}^\epsilon\Big]_0^T + \Big[\int_{\widehat{\Omega}^\epsilon} \hat{\rho}^\epsilon \dot{\hat{\phi}}_i^\epsilon (\hat{q}_j^\epsilon . \hat{\partial}_j^\epsilon \hat{\theta}_i^\epsilon) d\widehat{\Omega}^\epsilon\Big]_0^T + \int_0^T \int_{\widehat{\Omega}^\epsilon} 3\hat{\rho}^\epsilon \dot{\hat{\theta}}_i^\epsilon \dot{\hat{\phi}}_i^\epsilon d\widehat{\Omega}^\epsilon dt \\
&\quad - \int_0^T \int_{\widehat{\Omega}^\epsilon} \hat{\sigma}_{ij}^\epsilon(\hat{\phi}^\epsilon) \hat{\partial}_j^\epsilon \hat{\theta}_i^\epsilon d\widehat{\Omega}^\epsilon dt - \int_0^T \int_{\widehat{\Omega}^\epsilon} \hat{f}^{i,\epsilon} (\hat{q}_j^\epsilon . \hat{\partial}_j^\epsilon \hat{\phi}_i^\epsilon) d\widehat{\Omega}^\epsilon dt. \qquad (3.25)
\end{aligned}
$$

4 The Scaled Problem

To study the asymptotic behaviour of the solutions $\hat{\psi}^\epsilon$ as $\epsilon \to 0$, we first transform the problem (2.7) to $\Omega^\epsilon = \omega \times (-\epsilon, \epsilon)$ and then to the domain $\Omega = \omega \times (-1, 1)$ which is independent of ϵ.

Since the mappings $\Phi^\epsilon : \Omega^\epsilon \to \widehat{\Omega}^\epsilon$ are assumed to be C^1-diffeomorphisms, the correspondence

$$v_i^\epsilon(x^\epsilon, t) g^{i,\epsilon} = \hat{v}_i^\epsilon(\hat{x}^\epsilon, t) \hat{e}^i$$

induces a bijection between $V(\widehat{\Omega}^\epsilon)$ and $V(\Omega^\epsilon)$ where

$$V(\Omega^\epsilon) = \{v^\epsilon \in (H^1(\Omega^\epsilon))^3 : v^\epsilon = 0 \text{ on } \Gamma_0^\epsilon\}.$$

Then we have (cf. [2])

$$
\begin{aligned}
\hat{\partial}_j^\epsilon \hat{v}_i^\epsilon &= v_{k||l}^\epsilon (g^{k,\epsilon})_i (g^{l,\epsilon})_j, & v_{k||l}^\epsilon &= \partial_l^\epsilon v_k^\epsilon - \Gamma_{lk}^{q,\epsilon}(x^\epsilon) v_q^\epsilon, \\
\hat{e}_{ij}^\epsilon(\hat{v}^\epsilon) &= e_{k||l}^\epsilon(v^\epsilon) (g^{k,\epsilon})_i (g^{l,\epsilon})_j, & e_{i||j}^\epsilon(v^\epsilon) &= e_{ij}^\epsilon(v^\epsilon) - \Gamma_{ij}^{p,\epsilon} v_p^\epsilon.
\end{aligned}
$$

With the function $\hat{q}^\epsilon$, we associate the function q^ϵ by

$$q^\epsilon(x^\epsilon) = \Phi^\epsilon(x^\epsilon) - \Phi^\epsilon(x_0^\epsilon) = \hat{x}^\epsilon - \hat{x_0}^\epsilon = \hat{q}^\epsilon(\hat{x}^\epsilon).$$

We define

$$A^{ijkl,\epsilon} = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu(g^{ik,\epsilon} g^{kl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}).$$

Then the equation (3.1) posed in variational form over Ω^ϵ becomes: find $\theta^\epsilon(t) \in V(\Omega^\epsilon)$ a.e. $\forall\, t \in [0, T]$ such that $\theta^\epsilon(0) = \theta_0^\epsilon$, $\dot{\theta}^\epsilon(0) = \theta_1^\epsilon$ and

$$
\begin{aligned}
&\int_{\Omega^\epsilon} \rho^\epsilon \ddot{\theta}_i^\epsilon v_j^\epsilon g^{ij}(\epsilon) \sqrt{g^\epsilon}\, dx^\epsilon + \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k||l}^\epsilon(\theta^\epsilon) e_{k||l}^\epsilon(v^\epsilon) \sqrt{g^\epsilon}\, dx^\epsilon \\
&= \int_{\Omega^\epsilon} f^{i,\epsilon} v_i^\epsilon \sqrt{g^\epsilon}\, dx^\epsilon \quad \forall\, v^\epsilon \in V(\Omega^\epsilon). \qquad (4.1)
\end{aligned}
$$

The equations (3.10), (3.19) and (3.25) posed over Ω^ϵ become

$$
\begin{aligned}
&\int_{\Sigma_0^\epsilon} (q^\epsilon . \nu^\epsilon) A^{ijkl,\epsilon} e_{k||l}^\epsilon(\theta^\epsilon) e_{i||j}^\epsilon(\phi^\epsilon) \sqrt{g^\epsilon}\, d\Gamma_0^\epsilon dt \\
&+ \int_{\Sigma_\pm^\epsilon} (q^\epsilon . \nu^\epsilon) [\rho^\epsilon \dot{\phi}_i^\epsilon \dot{\theta}_j^\epsilon g^{ij,\epsilon} - A^{ijkl,\epsilon} e_{k||l}^\epsilon(\theta^\epsilon) e_{k||l}^\epsilon(\phi^\epsilon)] \sqrt{g^\epsilon}\, d\Gamma_\pm^\epsilon dt \\
&= \Big\{\int_{\Omega^\epsilon} \rho^\epsilon \dot{\theta}_m^\epsilon (g^{m,\epsilon})_i [q_j^\epsilon . \phi_{k||l}^\epsilon(x^\epsilon) (g^{k,\epsilon})_i (g^{l,\epsilon})_j] \sqrt{g^\epsilon}\, dx^\epsilon\Big\}_0^T
\end{aligned}
$$

$$+\Big\{\int_{\Omega^\epsilon}\rho^\epsilon\dot{\phi}^\epsilon_m(g^{m,\epsilon})_i[q^\epsilon_j.\theta^\epsilon_{k||l}(x^\epsilon)(g^{k,\epsilon})_i(g^{l,\epsilon})_j]\sqrt{g^\epsilon}\,dx^\epsilon\Big\}_0^T$$
$$+3\int_0^T\int_{\Omega^\epsilon}\rho^\epsilon\dot{\theta}^\epsilon_i\dot{\phi}^\epsilon_j g^{ij,\epsilon}\sqrt{g^\epsilon}\,dx^\epsilon dt-\int_0^T\int_{\Omega^\epsilon}A^{ijkl,\epsilon}e^\epsilon_{k||l}(\theta^\epsilon)e^\epsilon_{k||l}(\phi^\epsilon)\sqrt{g^\epsilon}\,dx^\epsilon dt$$
$$-\int_0^T\int_{\Omega^\epsilon}f^{m,\epsilon}(g^\epsilon_m)^i[q^\epsilon_j.\phi^\epsilon_{k||l}(x^\epsilon)(g^{k,\epsilon})_i(g^{l,\epsilon})_j]\sqrt{g^\epsilon}\,dx^\epsilon dt, \tag{4.2}$$

$$\int_{\Omega^\epsilon}[\rho^\epsilon\dot{\psi}^\epsilon_i(0)\theta^\epsilon_{0j}-\rho^\epsilon\psi^\epsilon_i(0)\theta^\epsilon_{1j}]g^{ij,\epsilon}\sqrt{g^\epsilon}\,dx^\epsilon-\int_{Q^\epsilon}\psi^\epsilon_i f^{i,\epsilon}\sqrt{g^\epsilon}\,dx^\epsilon dt$$
$$=\int_{\Sigma^\epsilon_0}(q^\epsilon.\nu^\epsilon)A^{ijkl,\epsilon}e^\epsilon_{k||l}(\theta^\epsilon)e^\epsilon_{k||l}(\phi^\epsilon)\sqrt{g^\epsilon}\,d\Gamma^\epsilon_0 dt$$
$$+\int_{\Sigma^\epsilon_\pm}(q^\epsilon.\nu^\epsilon)[\rho^\epsilon\dot{\phi}^\epsilon_i\dot{\theta}^\epsilon_j g^{ij,\epsilon}-A^{ijkl,\epsilon}e^\epsilon_{k||l}(\theta^\epsilon)e^\epsilon_{k||l}(\phi^\epsilon)]\sqrt{g^\epsilon}\,d\Gamma^\epsilon_\pm dt, \tag{4.3}$$

$$\int_{\Omega^\epsilon}[\rho^\epsilon\dot{\psi}^\epsilon_i(0)\theta^\epsilon_{0j}-\rho^\epsilon\psi^\epsilon_i(0)\theta^\epsilon_{1j}]g^{ij,\epsilon}\sqrt{g^\epsilon}\,dx^\epsilon-\int_{Q^\epsilon}\psi^\epsilon_i f^{i,\epsilon}\sqrt{g^\epsilon}\,dx^\epsilon dt$$
$$=\Big\{\int_{\Omega^\epsilon}\rho^\epsilon\dot{\theta}^\epsilon_m(g^{m,\epsilon})_i[q^\epsilon_j.\phi^\epsilon_{k||l}(x^\epsilon)(g^{k,\epsilon})_i(g^{l,\epsilon})_j]\sqrt{g^\epsilon}\,dx^\epsilon\Big\}_0^T$$
$$+\Big\{\int_{\Omega^\epsilon}\rho^\epsilon\dot{\phi}^\epsilon_m(g^{m,\epsilon})_i[q^\epsilon_j.\theta^\epsilon_{k||l}(x^\epsilon)(g^{k,\epsilon})_i(g^{l,\epsilon})_j]\sqrt{g^\epsilon}\,dx^\epsilon\Big\}_0^T$$
$$+3\int_0^T\int_{\Omega^\epsilon}\rho^\epsilon\dot{\theta}^\epsilon_i\dot{\phi}^\epsilon_j g^{ij,\epsilon}\sqrt{g^\epsilon}\,dx^\epsilon dt-\int_0^T\int_{\Omega^\epsilon}A^{ijkl,\epsilon}e^\epsilon_{k||l}(\theta^\epsilon)e^\epsilon_{k||l}(\phi^\epsilon)\sqrt{g^\epsilon}\,dx^\epsilon dt$$
$$-\int_0^T\int_{\Omega^\epsilon}f^{m,\epsilon}(g_{m,\epsilon})^i[q^\epsilon_j.\phi^\epsilon_{k||l}(x^\epsilon)(g^{k,\epsilon})_i(g^{l,\epsilon})_j]\sqrt{g^\epsilon}\,dx^\epsilon dt. \tag{4.4}$$

Let $\Omega=\omega\times(-1,1),\ \Gamma_0=\gamma_0\times[-1,1],\ \Gamma_\pm=\omega\times\{\pm1\}$.

Let $x=(x_i)$ denote a generic point in the set Ω and let $\partial_i=\frac{\partial}{\partial x_i}$. With each point $x^\epsilon=(x^\epsilon_i)\in\Omega^\epsilon$, we associate the point $x=(x_i)\in\Omega$ by $x_\alpha=x^\epsilon_\alpha$ and $x_3=\frac{1}{\epsilon}x^\epsilon_3$.

We assume that the shell is a shallow shell, i.e., there exists a function $\varphi\in C^3(\omega)$ such that

$$\varphi^\epsilon(x_1,x_2)=\epsilon\varphi(x_1,x_2). \tag{4.5}$$

We make the following scalings on the unknowns and the intial data.

$$\begin{aligned}
&\psi^\epsilon_\alpha(x^\epsilon)=\epsilon^2\psi_\alpha(\epsilon)(x),\quad \psi^\epsilon_3(x^\epsilon)=\epsilon\psi_3(\epsilon)(x),\\
&\phi^\epsilon_\alpha(x^\epsilon)=\epsilon^2\phi_\alpha(\epsilon)(x),\quad \phi^\epsilon_3(x^\epsilon)=\epsilon\phi_3(\epsilon)(x),\\
&\theta^\epsilon_\alpha(x^\epsilon)=\epsilon^2\theta_\alpha(\epsilon)(x),\quad \theta^\epsilon_3(x^\epsilon)=\epsilon\theta_3(\epsilon)(x),\\
&f^{\alpha,\epsilon}(x^\epsilon)=\epsilon^2 f^\alpha(\epsilon)(x),\quad f^{3,\epsilon}(x^\epsilon)=\epsilon^3 f^3(\epsilon)(x),\\
&\rho^\epsilon=\epsilon^2\rho,\quad \lambda^\epsilon=\lambda,\quad \mu^\epsilon=\mu,\\
&g^\epsilon_i(x^\epsilon)=g_i(\epsilon)(x),\quad g^\epsilon_{ij}(x^\epsilon)=g_{ij}(\epsilon)(x),\quad g^\epsilon(x^\epsilon)=g(\epsilon)(x),\\
&A^{ijkl,\epsilon}(x^\epsilon)=A^{ijkl}(\epsilon)(x),\quad \Gamma^{p,\epsilon}_{ij}(x^\epsilon)=\Gamma^p_{ij}(\epsilon)(x),\\
&q^\epsilon(x^\epsilon)=q(\epsilon)(x).
\end{aligned} \tag{4.6}$$

With the functions $\hat{e}^\epsilon_{i||j}(v^\epsilon)(x^\epsilon)$ we associate the functions $e_{i||j}(\epsilon;v)(x)$ through the following relation

$$e^\epsilon_{i||j}(v^\epsilon)(x^\epsilon)=\epsilon^2 e_{i||j}(\epsilon;v)(x). \tag{4.7}$$

We define the spaces

$$V(\Omega) = \{v = (v_i) \in (H^1(\Omega))^3 \mid v = 0 \text{ on } \Gamma_0\}, \tag{4.8}$$

$$X(\Omega) = \{g \in L^1(0,T;L^2(\Omega)) : \dot{g} \in L^1(0,T;[H^1_{\Gamma_0}(\Omega)]')\}, \tag{4.9}$$

$$V_{KL}(\Omega) = \{v = (v_i) : v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3,\ v_3 = \eta_3,\ \eta_\alpha \in H^1_0(\omega),\ \eta_3 \in H^2_0(\omega)\}. \tag{4.10}$$

We need the following Lemma which is proved in [1].

Lemma 4.1 *The functions $e_{i||j}(\epsilon, v)$ defined in (4.7) are of the form*

$$e_{\alpha||\beta}(\epsilon; v) = \tilde{e}_{\alpha\beta}(v) + \epsilon^2 e^\sharp_{\alpha||\beta}(\epsilon; v), \tag{4.11}$$

$$e_{\alpha||3}(\epsilon; v) = \frac{1}{\epsilon}\{\tilde{e}_{\alpha 3}(v) + \epsilon^2 e^\sharp_{\alpha||3}(\epsilon; v)\}, \tag{4.12}$$

$$e_{3||3}(\epsilon; v) = \frac{1}{\epsilon^2}\tilde{e}_{33}(v), \tag{4.13}$$

where

$$\tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - v_3\partial_{\alpha\beta}\varphi, \tag{4.14}$$

$$\tilde{e}_{\alpha 3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \tag{4.15}$$

$$\tilde{e}_{33}(v) = \partial_3 v_3 \tag{4.16}$$

and there exists constant C_7 such that

$$\sup_{0<\epsilon\le\epsilon_0} \max_{\alpha,j} \|e^\sharp_{\alpha,j}(\epsilon; v)\|_{0,\Omega} \le C_7\|v\|_{1,\Omega} \quad \textit{for all } v \in V. \tag{4.17}$$

Also there exist constants C_8, C_9 and C_{10} such that

$$\sup_{0<\epsilon\le\epsilon_0} \max_{x\in\Omega} |g(x) - 1| \le C_7\epsilon^2, \tag{4.18}$$

$$\sup_{0<\epsilon\le\epsilon_0} \max_{x\in\Omega} |A^{ijkl}(\epsilon) - A^{ijkl}(0)| \le C_9\epsilon^2, \tag{4.19}$$

where

$$A^{ijkl}(0) = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{4.20}$$

and

$$A^{ijkl}(\epsilon)t_{kl}t_{ij} \ge C_{10}t_{ij}t_{ij} \tag{4.21}$$

for $0 < \epsilon \le \epsilon_0$ and for all symmetric tensors (t_{ij}).

Proof A simple computation using the assumption (4.5) shows that

$$g_\alpha(\epsilon) = \begin{pmatrix} \delta_{\alpha 1} - \epsilon^2 x_3\partial_{\alpha 1}\varphi + O(\epsilon^2) \\ \delta_{\alpha 2} - \epsilon^2 x_3\partial_{\alpha 2}\varphi + O(\epsilon^2) \\ \epsilon\partial_\alpha\varphi + O(\epsilon^4) \end{pmatrix}, \quad g_3(\epsilon) = \begin{pmatrix} -\epsilon\partial_1\varphi + O(\epsilon^3) \\ -\epsilon\partial_2\varphi + O(\epsilon^3) \\ 1 + O(\epsilon^2) \end{pmatrix}, \tag{4.22}$$

$$g^\alpha(\epsilon) = \begin{pmatrix} \delta_{\alpha 1} + O(\epsilon^2) \\ \delta_{\alpha 2} + O(\epsilon^2) \\ \epsilon\partial_\alpha\varphi + O(\epsilon^2) \end{pmatrix}, \quad g^3(\epsilon) = \begin{pmatrix} -\epsilon\partial_1\varphi + O(\epsilon^3) \\ -\epsilon\partial_2\varphi + O(\epsilon^3) \\ 1 + O(\epsilon^2) \end{pmatrix}, \tag{4.23}$$

$$g_{\alpha\beta}(\epsilon) = \delta_{\alpha\beta} + \epsilon^2[\partial_\alpha\varphi\partial_\beta\varphi - 2x_3\partial_{\alpha\beta}\varphi] + O(\epsilon^4), \quad g_{\alpha 3}(\epsilon) = O(\epsilon), \quad g_{33}(\epsilon) = 1 + O(\epsilon^2), \tag{4.24}$$

$$\Gamma^\sigma_{\alpha\beta}(\epsilon) = O(\epsilon^2), \quad \Gamma^3_{\alpha\beta}(\epsilon) = \epsilon\partial_{\alpha\beta}(\varphi) + O(\epsilon^3), \quad \Gamma^\sigma_{\alpha 3} = O(\epsilon). \tag{4.25}$$

The announced results follow from the above relations.

The variational formulation (4.1) posed over the domain Ω becomes: find $\theta(\epsilon)(t) \in V(\Omega)$ a.e. $\forall\, t \in [0,T]$ such that $\theta(\epsilon)(0) = \theta_0(\epsilon),\ \dot\theta(\epsilon)(0) = \theta_1(\epsilon)$ and

$$\begin{aligned}
&\rho\Big[\int_\Omega \epsilon^2 \ddot\theta_\alpha(\epsilon) v_\beta g^{\alpha\beta}(\epsilon) + \int_\Omega \epsilon\ddot\theta_\alpha(\epsilon) v_3 g^{\alpha 3}(\epsilon) + \int_\Omega \epsilon\ddot\theta_3(\epsilon) v_\beta g^{3\beta}(\epsilon)\\
&+ \int_\Omega \ddot\theta_3(\epsilon) v_3 g^{33}(\epsilon)\Big]\sqrt{g(\epsilon)}\,dx + \int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(\theta(\epsilon)) e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}\,dx\\
&= \int_\Omega f^i(\epsilon) v_i \sqrt{g(\epsilon)}\,dx, \quad \forall\, v \in V(\Omega).
\end{aligned} \tag{4.26}$$

The equations (4.2)–(4.4) posed over the domain Ω become

$$\begin{aligned}
&\int_{\Sigma_0} (q(\epsilon).\nu) A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(\theta(\epsilon)) e_{i||j}(\epsilon)(\phi(\epsilon))\sqrt{g(\epsilon)}\,dxdt\\
&+ \int_{\Sigma_\pm} (q(\epsilon).\nu)[\epsilon^2\dot\phi_\alpha(\epsilon)\dot\theta_\beta(\epsilon) g^{\alpha\beta}(\epsilon) + \epsilon\dot\phi_\alpha(\epsilon)\dot\theta_3(\epsilon) g^{\alpha 3}(\epsilon)]\sqrt{g(\epsilon)}\,dxdt\\
&+ \int_{\Sigma_\pm} (q(\epsilon).\nu)[\epsilon\dot\phi_3(\epsilon)\dot\theta_\alpha(\epsilon) g^{\alpha 3}(\epsilon) + \dot\phi_3(\epsilon)\dot\theta_3(\epsilon) g^{33}(\epsilon)\\
&- A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(\phi(\epsilon)) e_{i||j}(\epsilon)(\theta(\epsilon))]\sqrt{g(\epsilon)}\,dxdt\\
&= \rho\Big\{\int_\Omega (\epsilon^2\dot\theta_\alpha(\epsilon)[q_j(\epsilon).\phi_{k||l}(\epsilon) g^{\alpha k}(\epsilon)(g^l(\epsilon))_j] + \epsilon\dot\theta_3(\epsilon)[q_j(\epsilon).\phi_{k||l}(\epsilon) g^{3k}(\epsilon)(g^l(\epsilon))_j])\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&+ \rho\Big\{\int_\Omega (\epsilon^2\dot\phi_\alpha(\epsilon)[q_j(\epsilon).\theta_{k||l}(\epsilon) g^{\alpha k}(\epsilon)(g^l(\epsilon))_j] + \epsilon\dot\phi_3(\epsilon)[q_j(\epsilon).\theta_{k||l}(\epsilon) g^{3k}(\epsilon)(g^l(\epsilon))_j])\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&+ 3\rho\int_0^T\int_\Omega [\epsilon^2\dot\phi_\alpha(\epsilon)\dot\theta_\beta(\epsilon) g^{\alpha\beta}(\epsilon) + \epsilon\dot\phi_\alpha(\epsilon)\dot\theta_3(\epsilon) g^{\alpha 3}(\epsilon)\\
&+ \epsilon\dot\phi_3(\epsilon)\dot\theta_\alpha(\epsilon) g^{\alpha 3}(\epsilon) + \dot\phi_3(\epsilon)\dot\theta_3(\epsilon) g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx\\
&+ \Big\{\int_\Omega f^\alpha(\epsilon)[q_j(\epsilon).\phi_{\alpha||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big\}_0^T + \Big\{\rho\int_\Omega \epsilon f^3(\epsilon)[q_j(\epsilon).\phi_{3||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&- \int_0^T\int_\Omega A^{ijkl}(\epsilon) e_{k||l}(\theta(\epsilon)) e_{i||j}(\phi(\epsilon))\sqrt{g(\epsilon)}\,dxdt,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
&\rho\int_\Omega [\epsilon^2\dot\psi_\alpha(\epsilon)(0)\theta_{0\beta}(\epsilon) g^{\alpha\beta}(\epsilon) + \epsilon\dot\psi_\alpha(\epsilon)(0)\theta_{03}(\epsilon) g^{\alpha 3}(\epsilon) + \epsilon\dot\psi_3(\epsilon)(0)\theta_{0\alpha}(\epsilon) g^{3\alpha}(\epsilon)\\
&+ \dot\psi_3(\epsilon)(0)\theta_{03}(\epsilon) g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx - \rho\int_\Omega [\epsilon^2\psi_\alpha(\epsilon)(0)\theta_{1\alpha}(\epsilon) g^{\alpha\beta}(\epsilon) + \epsilon\psi_\alpha(\epsilon)(0)\theta_{13}(\epsilon) g^{\alpha 3}(\epsilon)\\
&- \epsilon\psi_3(\epsilon)(0)\theta_{1\alpha}(\epsilon) g^{3\alpha}(\epsilon) + \psi_3(\epsilon)(0)\theta_{13}(\epsilon) g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx - \int_Q f^i(\epsilon)\psi_i(\epsilon)\sqrt{g(\epsilon)}\,dxdt\\
&= \int_{\Sigma_0} (q(\epsilon).\nu)[A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(\theta(\epsilon)) e_{i||j}(\epsilon)(\phi(\epsilon)) + \epsilon^2\dot\phi_\alpha(\epsilon)\dot\theta_\beta(\epsilon) g^{\alpha\beta}(\epsilon)\\
&+ \epsilon\dot\phi_\alpha(\epsilon)\dot\theta_3(\epsilon) g^{\alpha 3}(\epsilon)]\sqrt{g(\epsilon)}\,dxdt + \int_{\Sigma_\pm} (q(\epsilon).\nu)[\epsilon\dot\phi_3(\epsilon)\dot\theta_\alpha(\epsilon) g^{\alpha 3}(\epsilon) + \dot\phi_3(\epsilon)\dot\theta_3(\epsilon) g^{33}(\epsilon)\\
&- A^{ijkl}(\epsilon) e_{k||l}(\epsilon)(\phi(\epsilon)) e_{i||j}(\epsilon)(\theta(\epsilon))]\sqrt{g(\epsilon)}\,dxdt,
\end{aligned} \tag{4.28}$$

$$\rho\int_\Omega [\epsilon^2\dot\psi_\alpha(\epsilon)(0)\theta_{0\beta}(\epsilon) g^{\alpha\beta}(\epsilon) + \epsilon\dot\psi_\alpha(\epsilon)(0)\theta_{03}(\epsilon) g^{\alpha 3}(\epsilon) + \epsilon\dot\psi_3(\epsilon)(0)\theta_{0\alpha}(\epsilon) g^{3\alpha}(\epsilon)$$

$$
\begin{aligned}
&+ \dot\psi_3(\epsilon)(0)\theta_{03}(\epsilon)g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx - \rho\int_\Omega [\epsilon^2\psi_\alpha(\epsilon)(0)\theta_{1\alpha}(\epsilon)g^{\alpha\beta}(\epsilon) + \epsilon\psi_\alpha(\epsilon)(0)\theta_{13}(\epsilon)g^{\alpha 3}(\epsilon)\\
&- \epsilon\psi_3(\epsilon)(0)\theta_{1\alpha}(\epsilon)g^{3\alpha}(\epsilon) + \psi_3(\epsilon)(0)\theta_{13}(\epsilon)g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx - \int_Q f^i(\epsilon)\psi_i(\epsilon)\sqrt{g(\epsilon)}\,dxdt\\
=\;& \rho\Big\{\int_\Omega (\epsilon^2\dot\theta_\alpha(\epsilon)[q_j(\epsilon).\phi_{k||l}(\epsilon)g^{\alpha k}(\epsilon)(g^l(\epsilon))_j] + \epsilon\dot\theta_3(\epsilon)[q_j(\epsilon).\phi_{k||l}(\epsilon)g^{3k}(\epsilon)(g^l(\epsilon))_j])\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&+ \rho\Big\{\int_\Omega (\epsilon^2\dot\phi_\alpha(\epsilon)[q_j(\epsilon).\theta_{k||l}(\epsilon)g^{\alpha k}(\epsilon)(g^l(\epsilon))_j] + \epsilon\dot\phi_3(\epsilon)[q_j(\epsilon).\theta_{k||l}(\epsilon)g^{3k}(\epsilon)(g^l(\epsilon))_j])\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&+ 3\rho\int_0^T\int_\Omega [\epsilon^2\dot\phi_\alpha(\epsilon)\dot\theta_\beta(\epsilon)g^{\alpha\beta}(\epsilon) + \epsilon\dot\phi_\alpha(\epsilon)\dot\theta_3(\epsilon)g^{\alpha 3}(\epsilon) + \epsilon\dot\phi_3(\epsilon)\dot\theta_\alpha(\epsilon)g^{\alpha 3}(\epsilon)\\
&+ \dot\phi_3(\epsilon)\dot\theta_3(\epsilon)g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx + \Big\{\int_\Omega f^\alpha(\epsilon)[q_j(\epsilon).\phi_{\alpha||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&+ \Big\{\int_\Omega \epsilon f^3(\epsilon)[q_j(\epsilon).\phi_{3||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big\}_0^T\\
&- \int_0^T\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\theta(\epsilon))e_{i||j}(\phi(\epsilon))\sqrt{g(\epsilon)}\,dxdt,
\end{aligned}
\tag{4.29}
$$

where

$$
\begin{aligned}
\phi_{\alpha||\beta}(\epsilon)(x) &= \partial_\beta\phi_\alpha(\epsilon) - \phi_3(\epsilon)\partial_{\alpha\beta}\varphi + O(\epsilon^2),\\
\phi_{\alpha||3}(\epsilon)(x) &= \frac{1}{\epsilon}\partial_3\phi_\alpha(\epsilon) - \epsilon\Gamma^\sigma_{3\alpha}(\epsilon)\phi_\sigma(\epsilon),\\
\phi_{3||\alpha}(\epsilon)(x) &= \frac{1}{\epsilon}\partial_\beta\phi_3(\epsilon) - \epsilon\Gamma^\sigma_{3\alpha}(\epsilon)\phi_\sigma(\epsilon),\\
\phi_{3||3}(\epsilon)(x) &= \frac{1}{\epsilon^2}\partial_3\phi_3(\epsilon).
\end{aligned}
\tag{4.30}
$$

Theorem 4.1 *Assume that the scaled initial data* $\{\phi_0(\epsilon), \phi_1(\epsilon)\}_{\epsilon>0} \in V(\Omega)\times[L^2(\Omega)]^3$ *of the problem* (3.9) *satisfy*

$$
E^{\phi(\epsilon)}(0) = \frac{1}{2}\{\|\epsilon\sqrt{\rho}\phi_{11}(\epsilon), \epsilon\sqrt{\rho}\phi_{12}(\epsilon), \sqrt{\rho}\phi_{13}(\epsilon)\|^2_{[L^2(\Omega)]^3} + a(\epsilon)(\phi_0(\epsilon), \phi_0(\epsilon))\} \le C_{11}. \tag{4.31}
$$

Let $\{\phi(\epsilon)\}_{\epsilon>0}$ *be the scaled* (*weak*) *solutions of* (3.9) *with initial data* $\{\phi_0(\epsilon), \phi_1(\epsilon)\}$*. Then there exists a subsequence* $\{\phi(\epsilon)\}_{\epsilon>0}$ (*still indexed by* ϵ *for notational convenience*) *satisfying the following.*

(i) *There exists* $\phi \in L^\infty(0,T;V(\Omega))\cap W^{1,\infty}(0,T;[L^2(\Omega)]^3)$ *such that, as* $\epsilon\to 0$,

$$
\begin{aligned}
\phi(\epsilon) &\to \phi && \text{weakly} * \text{ in } L^\infty(0,T;V(\Omega)),\\
\dot\phi_3(\epsilon) &\to \dot\phi_3 && \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\
\epsilon\dot\phi_\alpha(\epsilon) &\to 0 && \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\
e_{\alpha||\beta}(\phi(\epsilon)) &\to e_{\alpha||\beta}(\phi) && \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\
e_{\alpha||3}(\phi(\epsilon)) &\to 0 && \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)),\\
e_{3||3}(\phi(\epsilon)) &\to \frac{-\lambda}{\lambda+2\mu}e_{\alpha||\alpha}(\phi) && \text{weakly} * \text{ in } L^\infty(0,T;L^2(\Omega)).
\end{aligned}
\tag{4.32}
$$

(ii) *The limit function* $\phi = \{\phi_\alpha, \phi_3\}$ *is a Kirchhoff-Love displacement, that is,* ϕ_3 *is independent of* x_3,

$$
\phi_\alpha = \hat\phi_\alpha - x_3\partial_\alpha\phi_3, \quad \hat\phi_\alpha \text{ is independent of } x_3. \tag{4.33}
$$

Moreover, $\hat{\phi}_\alpha = (S(\phi_3))_\alpha$ where for a given $\phi_3 \in H_0^2(\omega), (S(\phi_3)) = (\hat{\phi}_\alpha, \phi_3)$ is uniquely determined by

$$\int_\omega \Big[\frac{4\lambda\mu}{\lambda+2\mu} e_{\sigma\sigma}(\hat{\phi}_\alpha)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\hat{\phi}_\alpha)\Big]\partial_\beta \eta_\alpha d\omega$$
$$= \int_\omega \Big[\frac{4\lambda\mu}{\lambda+2\mu}(\partial_{\alpha\beta}\varphi\phi_3)\delta_{\alpha\beta} + 4\mu(\partial_{\alpha\beta}\varphi\phi_3)\Big]\partial_\beta\eta_\alpha d\omega \quad \forall\, \eta_\alpha \in H_0^1(\omega) \tag{4.34}$$

and $\phi_3 \in C^0([0,T]; H_0^2(\omega)) \cap C^1([0,T]; L^2(\omega))$ is the unique solution of the 2D shell problem

$$\begin{aligned} &2\rho\ddot{\phi}_3 - \partial_{\alpha\beta} m_{\alpha\beta}(\phi_3) - (n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi) = 0 && \text{in } \omega\times(0,T),\\ &\phi_3 = \frac{\partial\phi_3}{\partial\nu} = 0 && \text{on } \partial\omega\times(0,T),\\ &\phi_3(0) = \frac{1}{2}\int_{-1}^{1}\phi_{03}dx_3, \quad \dot{\phi}_3(0) = \frac12\int_{-1}^{1}\phi_{13}dx_3 && \text{in } \omega, \end{aligned} \tag{4.35}$$

where

$$m_{\alpha\beta}(\zeta_3) = -\Big\{\frac{4\lambda\mu}{3\lambda+2\mu}\Delta\zeta_3\delta_{\alpha\beta} + \frac{4\mu}{3}\partial_{\alpha\beta}\zeta_3\Big\}, \tag{4.36}$$

$$n^{\varphi}_{\alpha\beta}(\zeta) = \frac{4\lambda\mu}{\lambda+2\mu}\tilde{e}_{\sigma\sigma}(\zeta)\delta_{\alpha\beta} + 4\mu\tilde{e}_{\alpha\beta}(\zeta) \tag{4.37}$$

and $\{\phi_{03}, \phi_{13}\}$ is the weak limit of $\{\phi_{03}(\epsilon), \phi_{13}(\epsilon)\}_{\epsilon>0}$ in $H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$.

Proof Letting $f^i(\epsilon) = 0$ in (4.26), we see that $\phi(\epsilon)$ satisfies

$$\int_\Omega \rho[\epsilon^2\ddot{\phi}_\alpha(\epsilon)v_\beta g^{\alpha\beta}(\epsilon) + \epsilon\ddot{\phi}_\alpha(\epsilon)v_3 g^{\alpha3}(\epsilon) + \epsilon\ddot{\phi}_3(\epsilon)v_\alpha g^{\alpha3}(\epsilon) + \ddot{\phi}_3(\epsilon)v_3 g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx$$
$$+ \int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi(\epsilon))e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}\,dx = 0, \quad \forall\, v\in V(\Omega). \tag{4.38}$$

Taking $v = \dot{\phi}(\epsilon)(x,t)$ in the above equation, we have

$$\frac12\frac{d}{dt}\int_\Omega \rho[\epsilon^2\dot{\phi}_\alpha(\epsilon)\dot{\phi}_\beta(\epsilon)g^{\alpha\beta}(\epsilon) + 2\epsilon\dot{\phi}_\alpha(\epsilon)\dot{\phi}_3(\epsilon)g^{\alpha3}(\epsilon) + \dot{\phi}_3(\epsilon)\dot{\phi}_3(\epsilon)g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dx$$
$$+ \frac12\frac{d}{dt}\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi(\epsilon))e_{i||j}(\epsilon)(\phi(\epsilon))\sqrt{g(\epsilon)}\,dx = 0. \tag{4.39}$$

Using the positive definiteness of $(g^{ij}(\epsilon))$ and integrating from 0 to t, $0 < t \le T$, we get

$$\begin{aligned} &\frac12\int_\Omega \rho(\epsilon\dot{\phi_\alpha}(\epsilon))^2\sqrt{g(\epsilon)}\,dx + \frac12\int_\Omega\rho(\dot{\phi}_3(\epsilon))^2\sqrt{g(\epsilon)}\,dx\\ &+\frac12\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi(\epsilon))e_{i||j}(\epsilon)(\phi(\epsilon))\sqrt{g(\epsilon)}\,dx\\ \le{}& \frac12\int_\Omega\rho(\epsilon\phi_{1\alpha}(\epsilon))^2\sqrt{g(\epsilon)}\,dx\\ &+\frac12\int_\Omega\rho(\phi_{13})^2\sqrt{g(\epsilon)}\,dx + \frac12\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi_0(\epsilon))e_{i||j}(\epsilon)(\phi_0(\epsilon))\sqrt{g(\epsilon)}\,dx. \end{aligned} \tag{4.40}$$

Using the generalized Korn's inequality (cf. [1, Lemma 4.2]),

$$\Big\{\sum_i \|v_i\|^2_{1,\Omega}\Big\} \le C\Big\{\sum_{i,j}\|\tilde{e}_{ij}(v)\|^2_{0,\Omega}\Big\} \quad \forall\, v\in V(\Omega). \tag{4.41}$$

It can be shown that there exists $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$,

$$\Big\{ \sum_i \|v_i\|^2_{1,\Omega} \Big\} \leq C_{13} \Big\{ \sum_{i,j} \|e_{i||j}(\epsilon)(v)\|^2_{0,\Omega} \Big\}, \quad \forall\, v \in V(\Omega). \tag{4.42}$$

Using the assumption (4.31) and the above inequality we have

$$\begin{aligned}
&\rho\|\epsilon\dot{\phi}_\alpha(\epsilon)\|^2_{0,\Omega} + \rho\|\dot{\phi}_3(\epsilon)\|^2_{0,\Omega} + \|\phi_i(\epsilon)\|^2_{1,\Omega} \\
&\leq \rho C_{14}\Big(\|\epsilon\dot{\phi}_\alpha(\epsilon)\|^2_{0,\Omega} + \|\dot{\phi}_3(\epsilon)\|^2_{0,\Omega} + \sum_i \|e_{i||j}(\epsilon)(\phi(\epsilon))\|^2_{0,\Omega} \Big) \\
&\leq \rho C_{15}\Big(\int_\Omega [(\epsilon\phi_{0\alpha}(\epsilon))^2 + (\phi_{o3}(\epsilon))^2 + A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi_0(\epsilon))e_{i||j}(\epsilon)(\phi_0(\epsilon))]\sqrt{g(\epsilon)}\, dx \Big) \\
&\leq C_{16}.
\end{aligned} \tag{4.43}$$

Hence

$$\|\epsilon\dot{\phi}_\alpha(\epsilon)\|_{0,\Omega} \leq C_{16}, \quad \|\dot{\phi}_3(\epsilon)\|_{0,\Omega} \leq C_{16}, \quad \|\phi_i(\epsilon)\|_{1,\Omega} \leq C_{16}, \quad \|e_{i||j}(\epsilon)(\phi(\epsilon))\|_{0,\Omega} \leq C_{16}. \tag{4.44}$$

From this, it can be shown, by using the same arguments as in [1], that for each fixed $t \in [0,T]$, the weak $*$ convergence (4.32) holds, ϕ is of the form (4.33) and

$$\begin{aligned}
\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi(\epsilon))e_{i||j}(\epsilon)(v)\sqrt{g(\epsilon)}\, dx \to &- \int_\omega m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega n^\varphi_{\alpha\beta}(\phi)\partial_{\alpha\beta}\varphi\eta_3 d\omega \\
&+ \int_\omega n^\varphi_{\alpha\beta}(\phi)\partial_\beta\eta_\alpha d\omega
\end{aligned} \tag{4.45}$$

for all $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$.

Since $(\epsilon\dot{\phi}_\alpha(\epsilon), \dot{\phi}_3(\epsilon)) \to (0, \dot{\phi}_3)$ weak $*$ in $L^\infty(0,T;L^2(\Omega))$, it follows that for fixed $v = (v_i) = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$,

$$\begin{aligned}
\int_\Omega \epsilon\dot{\phi}_\alpha(\epsilon)v_\alpha\sqrt{g(\epsilon)}\, dx &\to 0 && \text{weak} * \text{ in } L^\infty(0,T), \\
\int_\Omega \dot{\phi}_3(\epsilon)v_3\sqrt{g(\epsilon)}\, dx &\to \int_\Omega \dot{\phi}_3 v_3 dx && \text{weak} * \text{ in } L^\infty(0,T).
\end{aligned}$$

This implies that

$$\int_0^T \int_\Omega \epsilon\ddot{\phi}_\alpha(\epsilon)v_\alpha\zeta\sqrt{g(\epsilon)}\, dxdt = -\int_0^T \int_\Omega \epsilon\dot{\phi}_\alpha(\epsilon)v_\alpha\dot{\zeta}\sqrt{g(\epsilon)}\, dxdt \to 0, \quad \forall\, \zeta \in D(0,T) \tag{4.46}$$

and

$$\begin{aligned}
\int_0^T \int_\Omega \ddot{\phi}_3(\epsilon)v_3\zeta\sqrt{g(\epsilon)}\, dxdt &= -\int_0^T \int_\Omega \dot{\phi}_3(\epsilon)v_3\dot{\zeta}\sqrt{g(\epsilon)}\, dxdt \to -\int_\Omega \dot{\phi}_3 v_3\dot{\zeta}\, dxdt \\
&= \int_\Omega \ddot{\phi}_3 v_3\zeta\, dxdt, \quad \forall\, \zeta \in D(0,T),
\end{aligned} \tag{4.47}$$

i.e.,

$$\int_\Omega \epsilon\ddot{\phi}_\alpha(\epsilon)v_\alpha\sqrt{g(\epsilon)}\, dx \to 0 \quad \text{and} \quad \int_\Omega \ddot{\phi}_3(\epsilon)v_3\sqrt{g(\epsilon)}\, dx \to \int_\Omega \ddot{\phi}_3 v_3\, dx \quad \text{in } D'(0,T). \tag{4.48}$$

Hence passing to the limit in (4.38) by taking $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$, we get

$$2\rho\int_\omega \ddot{\phi}_3\eta_3 d\omega - \int_\omega m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega n^\varphi_{\alpha\beta}(\phi)\partial_{\alpha\beta}\varphi\eta_3 d\omega + \int_\omega n^\varphi_{\alpha\beta}(\phi)\partial_\beta\eta_\alpha d\omega = 0 \quad (4.49)$$

for all $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$. This is equivalent to

$$2\rho\int_\omega \ddot{\phi}_3\eta_3 d\omega - \int_\omega m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega n^\varphi_{\alpha\beta}(\phi)\partial_{\alpha\beta}\varphi\eta_3 d\omega = 0, \quad \forall\,\eta_3 \in H^2_0(\omega), \quad (4.50)$$

$$\int_\omega n^\varphi_{\alpha\beta}(\phi)\partial_\beta\eta_\alpha d\omega = 0, \quad \forall\,\eta_\alpha \in H^1_0(\omega). \quad (4.51)$$

Since ϕ is of the form (4.33), the equation (4.51) can be written as

$$\begin{aligned}&\int_\omega \Big[\frac{4\lambda\mu}{\lambda+2\mu}e_{\sigma\sigma}(\hat{\phi}_\alpha)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\hat{\phi}_\alpha)\Big]\partial_\beta\eta_\alpha d\omega \\ &= \int_\omega \Big[\frac{4\lambda\mu}{\lambda+2\mu}(\partial_{\alpha\beta}\varphi\phi_3)\delta_{\alpha\beta} + 4\mu(\partial_{\alpha\beta}\varphi\phi_3)\Big]\partial_\beta\eta_\alpha d\omega, \quad \forall\,\eta_\alpha \in H^1_0(\omega). \quad (4.52)\end{aligned}$$

The left-hand side of the above equation is elliptic over $H^1_0(\omega)$ and for a given $\phi_3 \in H^2_0(\omega)$, the right hand side defines a linear functional over $H^1_0(\omega)$ and hence by Lax-Milgram Theorem, $\hat{\phi}_\alpha$ can be uniquely determined in terms of ϕ_3.

Theorem 4.2 *Assume that the initial data* $\{\theta_0(\epsilon), \theta_1(\epsilon)\} \in V(\Omega)\times[L^2(\Omega)]^3$ *and the applied body forces* $\{f^i(\epsilon)\}$ *of the variational problem* (4.26) *satisfy the following.*

(i) $f^\alpha(\epsilon) \to f^\alpha \in X(\Omega),\ f^3(\epsilon) \to f^3 \in L^1(0,T;L^2(\Omega))$.

(ii) *The sequence* $\{\theta_0(\epsilon)\}$ *verifies*

$$\begin{aligned} \theta_0(\epsilon) &\to \theta_0 && \textit{strongly in } V(\Omega),\\ e_{\alpha||3}(\theta_0)) &\to 0 && \textit{strongly in } L^2(\Omega),\\ e_{3||3}(\theta_0(\epsilon)) &\to \frac{-\lambda}{\lambda+2\mu}e_{\alpha||\alpha}(\theta_0) && \textit{strongly in } L^2(\Omega).\end{aligned} \quad (4.53)$$

(iii) *The sequence* $\{\theta_1(\epsilon)\}$ *satisfies*

$$\begin{aligned} \epsilon\theta_{1\alpha}(\epsilon) &\to 0 && \textit{strongly in } L^2(\Omega),\\ \theta_{13}(\epsilon) &\to \theta_{13} && \textit{strongly in } L^2(\Omega),\ \theta_{13} \in L^2(\omega).\end{aligned} \quad (4.54)$$

Then the solutions $\{\theta(\epsilon)\}_{\epsilon>0}$ *of* (4.26) *satisfy the following.*

(i) *There exists a function* $\theta \in L^\infty(0,T;V(\Omega)) \cap H^1(0,T;[L^2(\Omega)]^3)$ *such that*

$$\begin{aligned} \theta(\epsilon) &\to \theta && \textit{strongly in } L^2(0,T;V(\Omega)),\\ \dot{\theta}_3(\epsilon) &\to \dot{\theta}_3 && \textit{strongly in } L^2(0,T;L^2(\Omega)),\\ \epsilon\dot{\theta}_\alpha(\epsilon) &\to 0 && \textit{strongly in } L^2(0,T;L^2(\Omega)),\\ e_{\alpha||\beta}(\theta(\epsilon)) &\to e_{\alpha||\beta}(\theta) && \textit{strongly in } L^2(0,T;L^2(\Omega)),\\ e_{\alpha||3}(\theta(\epsilon)) &\to 0 && \textit{strongly in } L^2(0,T;L^2(\Omega)),\\ e_{3||3}(\theta(\epsilon)) &\to \frac{-\lambda}{\lambda+2\mu}e_{\alpha||\alpha}(\theta) && \textit{strongly in } L^2(0,T;L^2(\Omega)).\end{aligned} \quad (4.55)$$

(ii) *The limit function $\theta = \{\theta_\alpha, \theta_3\}$ is a Kirchhoff-Love displacement, that is θ_3 is independent of x_3,*

$$\theta_\alpha = \hat{\theta}_\alpha - x_3 \partial_\alpha \theta_3, \quad \hat{\theta}_\alpha \text{ is independent of } x_3. \tag{4.56}$$

Moreover, $\hat{\theta}_\alpha = (S_{f^\alpha}(\theta_3))_\alpha$, where for a given $f^\alpha \in X(\Omega)$ and $\theta_3 \in H_0^2(\omega)$, $(S_{f^\alpha}(\theta_3)) = (\hat{\theta}_\alpha, \theta_3)$ is uniquely determined by

$$\begin{aligned} &\int_\omega \Big[\frac{4\lambda\mu}{\lambda+2\mu} e_{\sigma\sigma}(\hat{\theta}_\alpha)\delta_{\alpha\beta} + 4\mu e_{\alpha\beta}(\hat{\theta}_\alpha)\Big]\partial_\beta \eta_\alpha d\omega \\ = &\int_\omega \Big[\frac{4\lambda\mu}{\lambda+2\mu}(\partial_{\alpha\beta}\varphi\theta_3)\delta_{\alpha\beta} + 4\mu(\partial_{\alpha\beta}\varphi\theta_3)\Big]\partial_\beta\eta_\alpha d\omega \\ &+ \int_\omega \Big(\int_{-1}^1 f^\alpha dx_3\Big)\eta_\alpha d\omega \quad \forall\, \eta_\alpha \in H_0^1(\omega) \end{aligned} \tag{4.57}$$

and $\theta_3 \in C^0([0,T], H_0^2(\omega)) \cap C^1([0,T]; L^2(\omega))$ is the unique solution of the 2D shell problem

$$\begin{aligned} &2\rho\ddot{\theta}_3 - \partial_{\alpha\beta} m_{\alpha\beta}(\theta_3) - (n^\varphi_{\alpha\beta}(S_{f^\alpha}(\theta_3))\partial_{\alpha\beta}\varphi) = \int_{-1}^1 f^3 dx_3 + \partial_\alpha \int_{-1}^1 x_3 f^\alpha dx_3 \quad \text{in } \omega \times (0,T), \\ &\theta_3 = \frac{\partial\theta_3}{\partial\nu} = 0 \quad \text{on } \partial\omega \times (0,T), \\ &\theta_3(0) = \frac{1}{2}\int_{-1}^1 \theta_{03} dx_3, \quad \dot{\theta}_3(0) = \frac{1}{2}\int_{-1}^1 \theta_{13} dx_3 \quad \text{in } \omega, \end{aligned} \tag{4.58}$$

where $\{\theta_{03}, \theta_{13}\}$ is the weak limit of $\{\theta_{03}(\epsilon), \theta_{13}(\epsilon)\}_{\epsilon>0}$ in $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$.

Proof Using the boundedness of $f^\alpha(\epsilon) \in X(\Omega)$ and $f^3(\epsilon) \in L^1(0,T,L^2(\Omega))$, it can be shown by proceeding the same way as in Theorem 4.1 that the convergences (4.55) holds weak $*$ in $L^\infty(0,T,L^2(\Omega))$ (hence weakly in $L^2(0,T,L^2(\Omega))$), (θ_i) is of the form (4.56) and (θ_i) satisfies

$$\begin{aligned} &2\int_\omega \ddot{\theta}_3\eta_3 dx - \int_\omega m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\eta_3 d\omega - \int_\omega n^\varphi_{\alpha\beta}(\theta)\partial_{\alpha\beta}\eta_3 d\omega + \int_\omega n^\varphi_{\alpha\beta}(\theta)\partial_\beta\eta_\alpha d\omega \\ = &\int_\omega f^1\eta_i d\omega - \int_\omega \Big(\int_{-1}^1 x_3 f^\alpha dx_3\Big)\partial_\alpha\eta_3 d\omega \end{aligned} \tag{4.59}$$

for all $v = (\eta_\alpha - x_3\partial_\alpha\eta_3, \eta_3) \in V_{KL}(\Omega)$. This is equivalent to (4.57)–(4.58).

To show the strong convergence of $(\epsilon\dot{\theta}_\alpha(\epsilon), \dot{\theta}_3(\epsilon), e_{i||j}(\epsilon)(\theta(\epsilon)))$ in $(L^2(0,T;L^2(\Omega))^2 \times L^2(0,T;(L^2(\Omega))^9)$, it is enough to show that they converge in norm as we already know that they converge weakly.

For $\sigma_{ij}, \tau_{ij} \in (L^2(0,T;L^2(\Omega))^9$, we define

$$(\sigma_{ij}, \tau_{ij}) = \int_0^T \int_\Omega A^{ijkl}(0)\sigma_{kl}\tau_{ij} dx d\tau. \tag{4.60}$$

Using (4.26), we have

$$\begin{aligned} &\int_0^T \int_\Omega \rho(\epsilon\dot{\theta}_\alpha(\epsilon))^2 dx d\tau + \int_0^T \int_\Omega \rho(\dot{\theta}_3(\epsilon))^2 dx d\tau \\ &+ \int_0^T \int_\Omega A^{ijkl}(0) e_{k||l}(\epsilon)(\theta(\epsilon)) e_{i||j}(\epsilon)(\theta(\epsilon)) dx d\tau \\ = &\int_0^T \int_\Omega \rho\epsilon^2 \dot{\theta}_\alpha(\epsilon)\dot{\theta}_\beta(\epsilon) g^{\alpha\beta}(\epsilon)\sqrt{g(\epsilon)}\, dx d\tau + \int_0^T \int_\Omega \rho\epsilon\dot{\theta}_\alpha(\epsilon)\dot{\theta}_3(\epsilon) g^{\alpha 3}(\epsilon)\sqrt{g(\epsilon)}\, dx d\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T\int_\Omega \rho(\dot\theta_3(\epsilon))^2 g^{33}(\epsilon)dxd\tau+\int_0^T\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\theta(\epsilon))e_{i||j}(\epsilon)(\theta(\epsilon))\sqrt{g(\epsilon)}\,dxd\tau\\
&-\int_0^T\int_\Omega \rho\epsilon^2\dot\theta_\alpha(\epsilon)\dot\theta_\beta(\epsilon)(g^{\alpha\beta}(\epsilon)\sqrt{g(\epsilon)}-\delta_{\alpha\beta})dxd\tau-\int_0^T\int_\Omega \rho(\dot\theta_3(\epsilon))^2(g^{33}(\epsilon)\sqrt{g(\epsilon)}-1)\,dxd\tau\\
&-\int_0^T\int_\Omega \rho\epsilon\dot\theta_\alpha(\epsilon)\dot\theta_3(\epsilon)g^{\alpha3}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau\\
&-\int_0^T\int_\Omega (A^{ijkl}(\epsilon)\sqrt{g(\epsilon)}-A^{ijkl}(0))e_{k||l}(\epsilon)(\theta(\epsilon))e_{i||j}(\epsilon)(\theta(\epsilon))dxd\tau\\
=&\int_0^T\int_\Omega \rho\epsilon^2\theta_{1\alpha}(\epsilon)\theta_{1\beta}(\epsilon)g^{\alpha\beta}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau+\int_0^T\int_\Omega \rho\epsilon\theta_{1\alpha}(\epsilon)\theta_{13}(\epsilon)g^{\alpha3}(\epsilon)\sqrt{g(\epsilon)}dxd\tau\\
&+\int_0^T\int_\Omega \rho(\theta_{13}(\epsilon))^2 g^{33}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau+\int_0^T\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\theta_0(\epsilon))e_{i||j}(\epsilon)(\theta_0(\epsilon))\sqrt{g(\epsilon)}\,dxd\tau\\
&+2\int_0^T\int_0^t\int_\Omega f^i(\epsilon)\dot\theta_i(\epsilon)\sqrt{g(\epsilon)}\,dxdtd\tau-\int_0^T\int_\Omega \rho\epsilon^2\dot\theta_\alpha(\epsilon)\dot\theta_\beta(\epsilon)(g^{\alpha\beta}(\epsilon)\sqrt{g(\epsilon)}-\delta_{\alpha\beta})dxd\tau\\
&-\int_0^T\int_\Omega \rho(\dot\theta_3(\epsilon))^2(g^{33}(\epsilon)\sqrt{g(\epsilon)}-1)\,dxd\tau-\int_0^T\int_\Omega \rho\epsilon\dot\theta_\alpha(\epsilon)\dot\theta_3(\epsilon)g^{\alpha3}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau\\
&-\int_0^T\int_\Omega (A^{ijkl}(\epsilon)\sqrt{g(\epsilon)}-A^{ijkl}(0))e_{k||l}(\epsilon)(\theta(\epsilon))e_{i||j}(\epsilon)(\theta(\epsilon))dxd\tau\\
=&\int_0^T\int_\Omega \rho\epsilon^2\theta_{1\alpha}(\epsilon)\theta_{1\beta}(\epsilon)g^{\alpha\beta}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau+\int_0^T\int_\Omega \rho\epsilon\theta_{1\alpha}(\epsilon)\theta_{13}(\epsilon)g^{\alpha3}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau\\
&+\int_0^T\int_\Omega \rho(\theta_{13}(\epsilon))^2 g^{33}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau+\int_0^T\int_\Omega A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\theta_0(\epsilon))e_{i||j}(\epsilon)(\theta_0(\epsilon))\sqrt{g(\epsilon)}\,dxd\tau\\
&-2\int_0^T\int_\Omega f^\alpha(\epsilon)(0)\theta_{0\alpha}(\epsilon)\sqrt{g(\epsilon)}\,dxdtd\tau+2\int_0^T\int_\Omega f^\alpha(\epsilon)(t)\theta_\alpha(\epsilon)(t)\sqrt{g(\epsilon)}\,dxdtd\tau\\
&-2\int_0^T\int_0^t\int_\Omega \dot f^\alpha(\epsilon)\theta_\alpha(\epsilon)\sqrt{g(\epsilon)}\,dxdtd\tau+2\int_0^T\int_0^t\int_\Omega f^3(\epsilon)\dot\theta_3(\epsilon)\sqrt{g(\epsilon)}\,dxdtd\tau\\
&-\int_0^T\int_\Omega \rho\epsilon^2\dot\theta_\alpha(\epsilon)\dot\theta_\beta(\epsilon)(g^{\alpha\beta}(\epsilon)\sqrt{g(\epsilon)}-\delta_{\alpha\beta})dxd\tau-\int_0^T\int_\Omega (\dot\theta_3(\epsilon))^2(g^{33}(\epsilon)\sqrt{g(\epsilon)}-1)\,dxd\tau\\
&-\int_0^T\int_\Omega \rho\epsilon\dot\theta_\alpha(\epsilon)\dot\theta_3(\epsilon)g^{\alpha3}(\epsilon)\sqrt{g(\epsilon)}\,dxd\tau\\
&-\int_0^T\int_\Omega (A^{ijkl}(\epsilon)\sqrt{g(\epsilon)}-A^{ijkl}(0))e_{k||l}(\epsilon)(\theta(\epsilon))e_{i||j}(\epsilon)(\theta(\epsilon))dxd\tau. \qquad (4.61)
\end{aligned}
$$

Letting $\epsilon \to 0$ and using (4.18)–(4.20), (4.53), (4.54) and (4.55) the right hand side of the above equation goes to

$$
\begin{aligned}
&\int_0^T\int_\Omega \rho(\theta_{13})^2dxd\tau+\int_0^T a(\theta_0,\theta_0)d\tau-2\int_0^T\int_\Omega f^\alpha(0)\theta_\alpha(0)dxd\tau+2\int_0^T\int_\Omega f^\alpha(t)\theta_\alpha(t)dxd\tau\\
&-2\int_0^T\int_0^t\int_\Omega \dot f^\alpha\theta_\alpha dxdtd\tau+2\int_0^T\int_0^t\int_\Omega f^3\dot\theta_3dxdtd\tau, \qquad (4.62)
\end{aligned}
$$

where

$$a(\theta_0,\theta_0)=-\int_\omega m_{\alpha\beta}(\theta_0)\partial_{\alpha\beta}\theta_{03}d\omega-\int_\omega n^\varphi_{\alpha\beta}(\theta_0)\partial_{\alpha\beta}\varphi\theta_{03}d\omega+\int_\omega n^\varphi_{\alpha\beta}(\theta_0)\partial_\beta\theta_{0\alpha}d\omega. \qquad (4.63)$$

On the other hand,

$$
\begin{aligned}
&\int_0^T \int_\Omega \rho(\dot{\theta}_3)^2 dx d\tau + \int_0^T \int_\Omega A^{ijkl}(0) e_{i||j}(\theta) e_{l||k}(\theta) dx d\tau \\
&= \int_0^T \Big[\int_\Omega \rho(\theta_{13})^2 dx + a(\theta_0,\theta_0) + 2\int_0^T \int_\Omega f^i \dot{\theta}_i dx dt\Big] d\tau \\
&= \int_0^T \Big[\int_\Omega \rho(\theta_{13})^2 dx + a(\theta_0,\theta_0) - 2\int_\Omega f^\alpha(0)\theta_\alpha(0) dx + 2\int_\Omega f^\alpha(t)\theta_\alpha(t) dx\Big] d\tau \\
&\quad - 2\int_0^T \int_0^t \int_\Omega \dot{f}^\alpha \theta_\alpha dx dt d\tau + 2\int_0^T \int_0^t \int_\Omega f^3 \dot{\theta}_3 dx dt d\tau \\
&= \int_0^T \int_\Omega \rho(\theta_{13})^2 dx d\tau + \int_0^T a(\theta_0,\theta_0) d\tau - 2\int_0^T \int_\Omega f^\alpha(0)\theta_\alpha(0) dx d\tau + 2\int_0^T \int_\Omega f^\alpha(t)\theta_\alpha(t) dx d\tau \\
&\quad - 2\int_0^T \int_0^t \int_\Omega \dot{f}^\alpha \theta_\alpha dx dt d\tau + 2\int_0^T \int_0^t \int_\Omega f^3 \dot{\theta}_3 dx dt d\tau. \qquad (4.64)
\end{aligned}
$$

From (4.62) and (4.64) it follows that $\|(\epsilon\dot{\theta_\alpha}(\epsilon),\ \dot{\theta_3}(\epsilon), e_{i||j}(\epsilon)(\theta(\epsilon)))\| \to \|(0,\dot{\theta_3}, e_{i||j})\|$.

That $\theta_i(\epsilon) \to \theta_i$ in $V(\Omega)$ can be proved using the strong convergence of $e_{i||j}(\epsilon)(\theta(\epsilon))$ to $e_{i||j}$ in $(L^2(\Omega))^9$.

Lemma 4.2 *For $\eta_3, \zeta_3 \in L^2(0,T;H_0^2(\omega))$ define*

$$
b(\zeta,\eta) = \int_0^T \int_\omega n^\varphi_{\alpha\beta}(S(\zeta))\partial_{\alpha\beta}\varphi\eta d\omega dt. \qquad (4.65)
$$

Then $b(\cdot\,,\cdot)$ is symmetric.

Proof We first claim that for $\eta = (\eta_i),\ \zeta = (\zeta_i) \in L^2(0,T;(H_0^1(\omega))^2 \times L^2(0,T;H_0^2(\omega))$ the bilinear form $B(\cdot\,,\cdot)$ defined by

$$
B(\zeta_i,\eta_i) = \int_0^T \int_\omega n^\varphi_{\alpha\beta}(\zeta)[\partial_{\alpha\beta}\varphi\eta_3 + \partial_\beta\eta_\alpha] dx dt \qquad (4.66)
$$

is symmetric. We have

$$
\begin{aligned}
B(\zeta_i,\eta_i) &= \int_0^T \int_\omega n^\varphi_{\alpha\beta}(\zeta)[\partial_{\alpha\beta}\varphi\eta_3 + \partial_\beta\eta_\alpha] d\omega dt \\
&= \int_0^T \int_\omega \Big[\frac{2\lambda\mu}{\lambda+2\mu} e^\varphi_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 2\mu e^\varphi_{\alpha\beta}(\zeta)\Big]\partial_{\alpha\beta}\varphi\eta_3 d\omega dt \\
&\quad + \int_0^T \int_\omega \Big[\frac{2\lambda\mu}{\lambda+2\mu} e^\varphi_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 2\mu e^\varphi_{\alpha\beta}(\zeta)\Big]\partial_\beta\eta_\alpha d\omega dt \\
&= \int_0^T \int_\omega \Big[\frac{2\lambda\mu}{\lambda+2\mu}(e_{\rho\rho}(\zeta) + \partial_{\rho\rho}\varphi\eta_3)\delta_{\alpha\beta} + 2\mu(e_{\alpha\beta}(\zeta) + \partial_{\alpha\beta}\zeta_3)\Big]\partial_{\alpha\beta}\eta_3 d\omega dt \\
&\quad + \int_0^T \int_\omega \Big[\frac{2\lambda\mu}{\lambda+2\mu}(e_{\rho\rho}(\zeta) + \partial_{\rho\rho}\varphi\eta_3)\delta_{\alpha\beta} + 2\mu(e_{\alpha\beta}(\zeta) + \partial_{\alpha\beta}\zeta_3)\Big]\partial_\beta\eta_\alpha d\omega dt \\
&= \int_0^T \int_\omega \frac{2\lambda\mu}{\lambda+2\mu}[\partial_\rho\zeta_\rho + \partial_{\rho\rho}\varphi\zeta_3][\partial_{\alpha\alpha}\varphi\eta_3 + \partial_\alpha\eta_\alpha] d\omega dt
\end{aligned}
$$

$$+\int_0^T\int_\omega 2\mu[e_{\alpha\beta}(\zeta)+\partial_{\alpha\beta}\varphi\zeta_3][\partial_{\alpha\beta}\varphi\eta_3+\partial_\beta\eta_\alpha]d\omega dt$$
$$=\int_0^T\int_\omega\Big[\frac{2\lambda\mu}{\lambda+2\mu}e^\varphi_{\rho\rho}(\zeta)e^\varphi_{\alpha\alpha}(\eta)+2\mu e^\varphi_{\alpha\beta}(\zeta)e^\varphi_{\alpha\beta}(\eta)\Big]d\omega dt, \tag{4.67}$$

which is symmetric in (ζ_i) and (η_i).

Let $(\zeta_\alpha),(\eta_\alpha)\in L^2(0,T;H^1(\omega))$ be such that $(S(\zeta_3))=(\zeta_\alpha,\zeta_3),(S(\eta_3))=(\eta_\alpha,\eta_3)$. Then

$$\begin{aligned}
b(\zeta_3,\eta_3)&=\int_0^T\int_\omega n^\varphi_{\alpha\beta}(S(\zeta_3))\partial_{\alpha\beta}\varphi\eta_3 d\omega dt\\
&=\int_0^T\int_\omega n^\varphi_{\alpha\beta}(S(\zeta_3))\partial_{\alpha\beta}\varphi\eta_3 d\omega dt+\int_0^T\int_\omega n^\varphi_{\alpha\beta}(S(\zeta_3))\partial_\beta\eta_\alpha d\omega dt\\
&=\int_0^T\int_\omega n^\varphi_{\alpha\beta}(\zeta)[\partial_{\alpha\beta}\varphi\eta_3+\partial_\beta\eta_\alpha]d\omega dt\\
&=B(S\zeta_3,S\eta_3)=B(S\eta_3,S\zeta_3)=b(\eta_3,\zeta_3).
\end{aligned}\tag{4.68}$$

Hence $b(\cdot\,,\,\cdot)$ is symmetric.

Lemma 4.3 *Let θ_3 be the solution of* (4.58) *with $f^\alpha\in L^1(0,T;H^3_0(\omega))$, $f^3\in L^1(0,T;H^2_0(\omega))$ and $\{\theta_{03},\theta_{13}\}\in H^3(\omega)\cap H^2_0(\omega)$ and let ϕ_3 be the solution of* (4.35) *with $\{\phi_{03},\phi_{13}\}\in H^3(\omega)\cap H^2_0(\omega)$. Then the following identity holds*

$$\begin{aligned}
&-\int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\gamma dt\\
&=\Big[\int_\omega 2\rho(\dot\theta_3(q_\alpha\partial_\alpha\phi_3)+\dot\phi_3(q_3\partial_\alpha\theta_3))d\omega\Big]_0^T+2\int_{\omega\times(0,T)}2\rho\dot\theta_3\dot\phi_3 d\omega dt\\
&\quad-2\int_{\omega\times(0,T)}m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt+2\int_{\omega\times(0,T)}n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega\\
&\quad+\int_{\omega\times(0,T)}q_\zeta\partial_\zeta(n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega-\int_{\omega\times(0,T)}n^\varphi_{\alpha\beta}(S_{f^\alpha}(\theta_3)\partial_{\alpha\beta}\varphi)q_\zeta\partial_\zeta\phi_3 d\omega\\
&\quad-\int_{\omega\times(0,T)}\Big(\int_{-1}^1(f^3+x_3\partial_\alpha f^\alpha)dx_3\Big)q_\alpha\partial_\alpha\phi_3 d\omega dt.
\end{aligned}\tag{4.69}$$

Proof The above identity follows by multiplying the first equation of (4.58) by $q_\alpha\partial_\alpha\phi_3$, where ϕ_3 is a solution of (4.35) and the first equation of (4.35) by $q_\alpha\partial_\alpha\theta_3$, where θ_3 is a solution of (4.58) and integrating by parts (here we assume that the solutions ϕ_3 and θ_3 belong to $H^{\frac{5}{2}+\delta}(\omega)$ and refer the reader to [14]).

Lemma 4.4 *Suppose that the sequence $\{\phi_0(\epsilon),\phi_1(\epsilon)\}\in H^2(\Omega)\cap V(\Omega)$ satisfies the assumptions of Theorem* 4.1 *and let us denote by $\phi(\epsilon)$ the corresponding sequence of solutions satisfying* (4.32)–(4.35). *Assume also that the sequence $\{\theta_0(\epsilon),\theta_1(\epsilon)\}\in H^2(\Omega)\cap V(\Omega)$ and $f(\epsilon)\in L^1(0,T;V(\Omega))$ satisfy the hypothesis of Theorem* 4.2 *and let $\theta(\epsilon)$ be the corresponding sequence of solutions satisfying* (4.55)–(4.58). *Then*

$$\lim_{\epsilon\to0}\Big[\int_{\Sigma_0}(q(\epsilon).\nu)A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\theta(\epsilon))e_{i||j}(\epsilon)(\phi(\epsilon))\sqrt{g^\epsilon}\,dxdt$$
$$+\int_{\Sigma_\pm}(q(\epsilon).\nu)[\epsilon^2\dot\phi_\alpha(\epsilon)\dot\theta_\beta(\epsilon)g^{\alpha\beta}(\epsilon)+\epsilon\dot\phi_\alpha(\epsilon)\dot\theta_3(\epsilon)g^{\alpha3}(\epsilon)]\sqrt{g(\epsilon)}\,dxdt$$

$$
\begin{aligned}
&+\int_{\Sigma_\pm}(q(\epsilon).\nu)[\epsilon\dot{\phi}_3(\epsilon)\dot{\theta}_\alpha(\epsilon)g^{\alpha 3}(\epsilon)+\dot{\phi}_3(\epsilon)\dot{\theta}_3(\epsilon)g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dxdt\\
&-\int_{\Sigma_\pm}(q(\epsilon).\nu)A^{ijkl}(\epsilon)e_{k||l}(\epsilon)(\phi(\epsilon))e_{i||j}(\epsilon)(\theta(\epsilon))\sqrt{g^\epsilon}\,dxdt\Big]\\
=&\Big[\int_\omega 2\rho(\dot{\theta}_3(q_\alpha\partial_\alpha\phi_3)+\dot{\phi}_3(q_\alpha\partial_\alpha\theta_3))d\omega\Big]_0^T+3\int_{\omega\times(0,T)}2\rho\dot{\theta}_3\dot{\phi}_3d\omega dt\\
&+\int_{\omega\times(0,T)}m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3d\omega dt+\int_{\omega\times(0,T)}n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3d\omega dt\\
&-\int_{\omega\times(0,T)}\Big(\int_{-1}^1(f^3+x_3\partial_\alpha f^\alpha)dx_3\Big)q_\sigma\partial_\sigma\phi_3d\omega dt-\int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha q_\sigma dx_3\Big)\partial_\sigma(S(\phi_3))_\alpha d\omega dt\\
&+\int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3d\omega dt+\int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha q_\alpha\partial_{\alpha\sigma}\varphi dx_3\Big)\phi_3d\omega dt\\
&-\int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha q_\sigma dx_3\Big)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt. \qquad (4.70)
\end{aligned}
$$

Proof From equations (4.27)–(4.29), we notice that to compute the limit of the boundary integral in (4.70) it is enough to compute the limit on the righthand side of (4.29). Using (4.23), (4.32) and (4.55), we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\Big[\rho\int_\Omega\epsilon^2\dot{\theta}_\alpha(\epsilon)[q_j(\epsilon).\phi_{k||l}(\epsilon)(g^{\alpha k}(\epsilon))(g^l(\epsilon))_j]\\
&+\epsilon\dot{\theta}_3(\epsilon)[q_j(\epsilon).\phi_{k||l}(\epsilon)(g^{3k}(\epsilon))(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]_0^T\\
=&\lim_{\epsilon\to 0}\Big[\rho\int_\Omega\epsilon^2\dot{\theta}_\alpha(\epsilon)[q_j(\epsilon).\phi_{\beta||\gamma}(\epsilon)(g^{\alpha\beta}(\epsilon))(g^\gamma(\epsilon))_j]\\
&+\epsilon^2\dot{\theta}_\alpha(\epsilon)[q_j(\epsilon).\phi_{\beta||3}(\epsilon)(g^{\alpha\beta}(\epsilon))(g^3(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]_0^T\\
&+\lim_{\epsilon\to 0}\Big[\rho\int_\Omega\epsilon^2\dot{\theta}_\alpha(\epsilon)[q_j(\epsilon).\phi_{3||\beta}(\epsilon)(g^{\alpha 3}(\epsilon))(g^\beta(\epsilon))_j]\\
&+\epsilon^2\dot{\theta}_\alpha(\epsilon)[q_j(\epsilon).\phi_{3||3}(\epsilon)(g^3(\epsilon))_\alpha(g^3(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]_0^T\\
&+\lim_{\epsilon\to 0}\Big[\rho\int_\Omega\epsilon\dot{\theta}_3(\epsilon)[q_j(\epsilon).\phi_{\beta||\gamma}(\epsilon)(g^{3\beta}(\epsilon))(g^\gamma(\epsilon))_j]\\
&+\epsilon\dot{\theta}_3(\epsilon)[q_j(\epsilon).\phi_{\beta||3}(\epsilon)(g^{3\beta}(\epsilon))(g^3(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]\\
&+\lim_{\epsilon\to 0}\Big[\rho\int_\Omega\epsilon\dot{\theta}_3(\epsilon)[q_j(\epsilon).\phi_{3||\beta}(\epsilon)(g^{33}(\epsilon))(g^\beta(\epsilon))_j]\\
&+\epsilon\dot{\theta}_3(\epsilon)[q_j(\epsilon).\phi_{3||3}(\epsilon)(g^{33}(\epsilon))(g^3(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]\\
=&\Big[\int_\Omega\rho\dot{\theta}_3(q_\alpha\partial_\alpha\phi_3)dx\Big]_0^T. \qquad (4.71)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\Big[\rho\int_\Omega\epsilon^2\dot{\phi}_\alpha(\epsilon)[q_j(\epsilon).\theta_{k||l}(\epsilon)(g^{\alpha k}(\epsilon))(g^l(\epsilon))_j]\\
&+\epsilon\dot{\phi}_3(\epsilon)[q_j(x).\theta_{k||l}(\epsilon)(g^{3k}(\epsilon))(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]_0^T=\Big[\int_\Omega\rho\dot{\phi}_3(q_\alpha\partial_\alpha\theta_3)\,dx\Big]_0^T. \qquad (4.72)
\end{aligned}
$$

Also

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega 3\rho[\epsilon^2\dot\theta_\alpha(\epsilon)\dot\phi_\beta(\epsilon)g^{\alpha\beta}(\epsilon)+\epsilon\dot\theta_\alpha(\epsilon)\dot\phi_3(\epsilon)g^{\alpha 3}(\epsilon)$$
$$+\epsilon\dot\phi_\alpha(\epsilon)\dot\theta_3(\epsilon)g^{\alpha 3}(\epsilon)+\dot\theta_3(\epsilon)\dot\phi_3(\epsilon)g^{33}(\epsilon)]\sqrt{g(\epsilon)}\,dxdt$$
$$=3\int_{\omega\times(0,T)}2\rho\dot\theta_3\dot\phi_3 d\omega dt. \tag{4.73}$$

Since $q_3(\epsilon)=\epsilon\varphi+\epsilon x_3+O(\epsilon^2),\ \partial_3\phi_3=0$ and $\partial_3\phi_\alpha=-\partial_\alpha\phi_3$, we have

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega f^\alpha(\epsilon)[q_j(\epsilon).\phi_{\alpha||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dxdt$$
$$=\lim_{\epsilon\to 0}\int_0^T\int_\Omega f^\alpha(\epsilon)[q_j(\epsilon).\phi_{\alpha||\gamma}(\epsilon)(g^\gamma(\epsilon))_j+q_j(\epsilon).\phi_{\alpha||3}(\epsilon)(g^3(\epsilon))_j]\sqrt{g(\epsilon)}\,dxdt$$
$$=\int_0^T\int_\Omega f^\alpha(q_\sigma\partial_\sigma\phi_\alpha)dxdt-\int_0^T\int_\Omega f^\alpha(q_\sigma\partial_{\sigma\alpha}\varphi\phi_3)dxdt+\int_0^T\int_\Omega f^\alpha(q_\sigma\partial_\alpha\phi_3)\partial_\sigma\varphi dxdt$$
$$-\int_0^T\int_\Omega f^\alpha(\varphi\partial_\alpha\phi_3)dxdt-\int_0^T\int_\Omega f^\alpha(x_3\partial_\alpha\phi_3)dxdt. \tag{4.74}$$

Since $\phi_\alpha=\hat\phi_\alpha-x_3\partial_\alpha\phi_3=(S(\phi_3))_\alpha-x_3\partial_\alpha\phi_3$, we have $\partial_\sigma\phi_\alpha=\partial_\sigma(S(\phi_3))_\alpha-x_3\partial_{\alpha\sigma}\phi_3$. Hence

$$\int_0^T\int_\Omega f^\alpha(q_\sigma\partial_\sigma\phi_\alpha)dxdt-\int_0^T\int_\Omega f^\alpha(q_\sigma\partial_{\sigma\alpha}\varphi\phi_3)dxdt+\int_0^T\int_\Omega f^\alpha(q_\sigma\partial_\alpha\phi_3)\partial_\sigma\varphi dxdt$$
$$-\int_0^T\int_\Omega f^\alpha(\varphi\partial_3\phi_\alpha)dxdt-\int_0^T\int_\Omega f^\alpha(x_3\partial_\alpha\phi_3)dxdt$$
$$=\Big\langle\int_{-1}^1 f^\alpha q_\sigma dx_3,\partial_\sigma(S(\phi_3))_\alpha\Big\rangle-\Big\langle\int_{-1}^1 f^\alpha q_\sigma x_3 dx_3,\partial_{\sigma\alpha}\phi_3\Big\rangle-\Big\langle\int_{-1}^1 f^\alpha q_\sigma\partial_{\alpha\sigma}\varphi dx_3,\phi_3\Big\rangle$$
$$+\Big\langle\int_{-1}^1 f^\alpha q_\sigma dx_3,\partial_\alpha\phi_3\partial_\sigma\varphi\Big\rangle-\Big\langle\int_{-1}^1 f^\alpha dx_3,\varphi\partial_\alpha\phi_3\Big\rangle-\Big\langle\int_{-1}^1 f^\alpha x_3 dx_3,\partial_\alpha\phi_3\Big\rangle, \tag{4.75}$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the dual product between $L^1(0,T;L^2(\omega))$ and its dual.

Also,

$$\lim_{\epsilon\to 0}\int_0^T\int_\Omega \epsilon f^3(\epsilon)[q_j(\epsilon).\phi_{3||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dxdt$$
$$=\lim_{\epsilon\to 0}\Big[\int_0^T\int_\Omega \epsilon f^3(\epsilon)[q_j(\epsilon).\phi_{3||\gamma}(\epsilon)(g^\gamma(\epsilon))_j+q_j(\epsilon).\phi_{3||3}(\epsilon)(g^3(\epsilon))_j]\sqrt{g(\epsilon)}\,dxdt\Big]$$
$$=\int_0^T\int_\Omega f^3(q_\sigma\partial_\sigma\phi_3)dxdt. \tag{4.76}$$

Adding the above two equations, we get

$$\lim_{\epsilon\to 0}\Big[\int_0^T\int_\Omega f^\alpha(\epsilon)[q_j(\epsilon).\phi_{\alpha||l}(\epsilon)(g^l(\epsilon))_j+\epsilon f^3(\epsilon)(q_j(\epsilon).\phi_{3||l}(\epsilon)(g^l(\epsilon))_j]\sqrt{g(\epsilon)}\,dx\Big]$$
$$=\Big\langle f^3+x_3\partial_\alpha f^\alpha,q_\sigma\partial_\sigma\phi_3\Big\rangle+\Big\langle\int_{-1}^1 f^\alpha q_\sigma dx_3,\partial_\sigma(S(\phi_3))_\alpha\Big\rangle$$
$$-\Big\langle\int_{-1}^1 f^\alpha dx_3,\varphi\partial_\alpha\phi_3+\Big\langle\int_{-1}^1 f^\alpha q_\sigma dx_3,\partial_\alpha\phi_3\partial_\sigma\varphi\Big\rangle-\Big\langle\int_{-1}^1 f^\alpha q_\sigma\partial_{\alpha\sigma}\varphi dx_3,\phi_3\Big\rangle. \tag{4.77}$$

It can be shown as in [1] that for fixed $t \in [0, T]$,

$$\begin{aligned}
&\int_Q A^{ijkl}(\epsilon) e_{i||j}(\epsilon)(\theta(\epsilon)) e_{k||l}(\epsilon)(\phi(\epsilon)) \sqrt{g(\epsilon)}\, dxdt \to \\
&- \int_{\omega\times(0,T)} m_{\alpha\beta}(\phi_3)\partial_{\alpha\beta}\theta_3 dxdt - \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(\phi)\partial_{\alpha\beta}\varphi\theta_3 dxdt \\
&- \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(\phi)\partial_{\beta}\theta_{\alpha} dxdt.
\end{aligned} \tag{4.78}$$

The result follows from (4.71)–(4.73), (4.77) and (4.78).

Lemma 4.5 *Let $T > T(\epsilon)$ and $\psi(\epsilon)$ be the sequence of solutions of the problem* (4.28) *with initial data $\{\psi_0(\epsilon), \psi_1(\epsilon)\}$ satisfying*

$$\|\{\psi_0(\epsilon), \psi_1(\epsilon)\}\|_{[V(\Omega)]'\times[L^2(\Omega)]^3} \le C. \tag{4.79}$$

Assume also that the sequence of functions $\{\theta_0(\epsilon), \theta_1(\epsilon)\}$ and $\{f^{\alpha}(\epsilon), f^3(\epsilon)\}$ satisfy the assumptions of Theorem 4.1. *Then there exists $\psi_\alpha \in [X(\Omega)]'$ and $\psi_3 \in L^\infty(0,T; L^2(\Omega))$ such that*

$$\begin{aligned}
&\langle \{\rho\psi_{13}, -\rho\psi_{03}\}, \{\theta_{03}, \theta_{13}\}\rangle \\
&= \langle \psi_i, f^i\rangle + \int_{\omega\times(0,T)} [2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3] d\omega dt \\
&- \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta) m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt - \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt \\
&- \int_{\omega\times(0,T)} q_\zeta\partial_\zeta(n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega dt + \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(S_{f^\alpha}(\theta_3))\partial_{\alpha\beta}\varphi q_\zeta\partial_\zeta\phi_3 d\omega dt \\
&- \int_{\omega\times(0,T)} \int_{-1}^{1} (f^\alpha q_\sigma dx_3)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt + \int_{\omega\times(0,T)} \Big(\int_{-1}^{1} f^\alpha q_\sigma dx_3\Big)\partial_\sigma(S(\phi_3))_\alpha d\omega dt \\
&- \int_{\omega\times(0,T)} \Big(\int_{-1}^{1} f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt - \int_{\omega\times(0,T)} \Big(\int_{-1}^{1} f^\alpha q_\alpha\partial_{\alpha\sigma}\varphi dx_3\Big)\phi_3 d\omega dt,
\end{aligned} \tag{4.80}$$

where

- $-\{\psi_{13}, \psi_{03}\}$ *is the weak limit in $[V(\Omega)]' \times L^2(\Omega)$ of a subsequence of $\{\psi_{13}(\epsilon), \psi_{03}(\epsilon)\}_{\epsilon>0}$,*
- $-\psi_\alpha$ *is the weak limit of the subsequence of $\{\psi_\alpha(\epsilon)\}$ in $[X(\Omega)]'$, ψ_3 is the weak $*$ limit of the subsequence of $\{\psi_3(\epsilon)\}$ in $L^\infty(0,T; L^2(\Omega))$,*
- $-\theta_3$ *is the solution of* (4.58),
- $-\phi_3$ *is the solution of* (4.35) *with initial data*

$$\Big\{\frac{1}{2}\int_1^{-1} \phi_{03} dx_3, \frac{1}{2}\int_1^{-1} \phi_{13} dx_3\Big\}$$

and $\{\phi_{03}, \phi_{13}\}$ is the weak limit of $\{\phi_{03}(\epsilon), \phi_{13}(\epsilon)\}$ in $V(\Omega) \times L^2(\Omega)$ and

$$\{\phi_0(\epsilon), \phi_1(\epsilon)\} = (\Lambda^\epsilon)^{-1}(\{\psi_1(\epsilon), -\psi_0(\epsilon)\}).$$

Proof Note that the solution $\psi(\epsilon)$ of (4.28) with initial data $\{\psi_0(\epsilon), \psi_1(\epsilon)\}$ satisfies (4.29). We want to compute the limit as $\epsilon \to 0$ of (4.29).

Note that because of (4.79), there exists a subsequence of $\{\psi_0(\epsilon), \psi_1(\epsilon)\}_{\epsilon>0}$ (still denoted by ϵ for notaional convenience) that weakly converges in $[L^2(\Omega)]^3 \times [V(\Omega)]'$. Also since $\{\theta_0(\epsilon), \theta_1(\epsilon)\}_{\epsilon>0}$ converges strongly in $V(\Omega) \times [L^2(\Omega)]^3$, we have

$$\begin{aligned}
&\rho \int_\Omega ([\epsilon^2 \dot{\psi}_\alpha(\epsilon)(0)\theta_{0\beta}(\epsilon)g^{\alpha\beta}(\epsilon) + \epsilon\dot{\psi}_\alpha(\epsilon)(0)\theta_{03}(\epsilon)g^{\alpha 3}(\epsilon)])\sqrt{g(\epsilon)}\,dx \\
&+ \rho \int_\Omega ([\epsilon\dot{\psi}_3(\epsilon)(0)\theta_{0\alpha}(\epsilon)g^{3\alpha}(\epsilon) + \dot{\psi}_3(\epsilon)(0)\theta_{03}(\epsilon)g^{33}(\epsilon)])\sqrt{g(\epsilon)}\,dx \\
&- \rho \int_\Omega ([\epsilon^2 \psi_\alpha(\epsilon)(0)\theta_{1\alpha}(\epsilon)g^{\alpha\beta}(\epsilon) + \epsilon\psi_\alpha(\epsilon)(0)\theta_{13}(\epsilon)g^{\alpha 3}(\epsilon)]))\sqrt{g(\epsilon)}\,dx \\
&- \rho \int_\Omega ([\epsilon\psi_3(\epsilon)(0)\theta_{1\alpha}(\epsilon)g^{3\alpha}(\epsilon) + \psi_3(\epsilon)(0)\theta_{13}(\epsilon)g^{33}(\epsilon)])\sqrt{g(\epsilon)}\,dx \to \\
&\int_\Omega \rho(\psi_{13}\theta_{03} - \psi_{03}\theta_{13})dx \quad \text{as } \epsilon \to 0.
\end{aligned} \tag{4.81}$$

From the assumption (4.79) and estimate (3.22), it follows that the initial data $\{\phi_0(\epsilon), \phi_1(\epsilon)\} = \Lambda^{-1}(\epsilon)(\{\psi_1(\epsilon), -\psi_0(\epsilon)\}$ satisfy

$$\|(\epsilon\sqrt{\rho}\phi_{11}(\epsilon), \epsilon\sqrt{\rho}\phi_{12}(\epsilon), \sqrt{\rho}\phi_{13}(\epsilon))\|^2_{[L^2(\Omega)]^3} + a(\epsilon)(\phi_0(\epsilon), \phi_0(\epsilon)) \le C. \tag{4.82}$$

Hence $\{\phi_0(\epsilon), \phi_1(\epsilon)\}_{\epsilon>0}$ satisfy the assumption (4.31). Let $\{\phi_0, \phi_1\}$ be the weak limit of the subsequence of $\{\phi_0(\epsilon), \phi_1(\epsilon)\}_{\epsilon>0}$ in $V(\Omega) \times [L^2(\Omega)]^3$.

Choosing $\{\theta_0(\epsilon), \theta_1(\epsilon)\} = \{0, 0\}$ in (4.29) and using (4.82) it follows that

$$|\langle\psi_3(\epsilon), f^3(\epsilon)\rangle| \le C\int_0^T \|f^3(\epsilon)\|_{[L^2(\Omega)]^3}dt \quad \text{if } f^\beta(\epsilon) = 0,\ \beta = 1, 2, \tag{4.83}$$

$$|\langle\psi_\beta(\epsilon), f^\beta(\epsilon)\rangle| \le C\int_0^T \|f^\beta(\epsilon)\|_{X(\Omega)}dt \qquad \text{if } f^3(\epsilon) = 0,\ f^\alpha(\epsilon) = 0 \quad \text{for } \beta \ne \alpha. \tag{4.84}$$

Hence there exists $\psi_\alpha \in [X(\Omega)]', \alpha = 1, 2$ and $\psi_3 \in L^\infty(0, T; L^2(\Omega))$ such that

$$\sum_i \langle\psi_i(\epsilon), f^i(\epsilon)\rangle \to \sum_i \langle\psi_i, f^i\rangle. \tag{4.85}$$

The identity (4.80) follows from (4.81), (4.85), (4.69) and (4.70).

Lemma 4.6 *The limit displacement $\psi = (\psi_i)$ of the controlled displacement $\psi(\epsilon)$ is a Kirchhoff-Love displacement, that is,*

$$\begin{aligned}
&\psi_3 \textit{ is independent of } x_3, \\
&\psi_\alpha = \hat{\psi}_\alpha - x_3\partial_\alpha\psi_3, \quad \hat{\psi}_\alpha \textit{ is independent of } x_3 \textit{ and is a function of } \phi_3.
\end{aligned} \tag{4.86}$$

Proof (i) To show that ψ_3 is independent of x_3, it is enough to show that

$$\langle\psi_3, -\partial_3 f\rangle = 0, \quad \forall\, f \in D(\Omega \times (0, T)). \tag{4.87}$$

In (4.28), we consider sequences $\{\theta_0(\epsilon), \theta_1(\epsilon)\}_{\epsilon>0}, \{f(\epsilon)\}_{\epsilon>0}$ such that

$$\begin{aligned}
&\theta_0(\epsilon) \to 0 \quad \text{in } V(\Omega), \quad \theta_1(\epsilon) \to 0 \quad \text{in } [L^2(\Omega)]^3, \\
&(f^1(\epsilon), f^2(\epsilon), f^3(\epsilon)) \to (0, 0, -\partial_3 f) \quad \text{in } [X(\Omega)]^2 \times L^1(0, T; L^2(\Omega)).
\end{aligned}$$

It can be verified that the above sequences satisfy all the hypothesis of Theorem 4.2. Note that when $f^\alpha = 0$, we have $S_{f^\alpha} = S$. At the limit we obtain (cf. (4.80))

$$\begin{aligned} 0 = \langle \psi_3, -\partial_3 f\rangle &+ \int_{\omega\times(0,T)} [2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3]d\omega dt \\ &- \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\varphi\phi_3 d\omega dt - \int_{\omega\times(0,T)} n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt \\ &- \int_{\omega\times(0,T)} q_\zeta\partial_\zeta(n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega dt + \int_{\omega\times(0,T)} n^\varphi_{\alpha\beta}(S(\theta_3))\partial_{\alpha\beta}\varphi q_\zeta\partial_\zeta\phi_3 d\omega dt, \end{aligned} \tag{4.88}$$

where θ_3 is the solution of

$$\begin{aligned} &2\rho\ddot{\theta}_3 - \partial_{\alpha\beta}m_{\alpha\beta}(\theta_3) - n^\varphi_{\alpha\beta}(S(\theta_3))\partial_{\alpha\beta}\varphi = \int_{-1}^1 (-\partial_3 f)dx_3 && \text{in } \omega\times(0,T), \\ &\theta_3 = \partial_\nu\theta_3 = 0 && \text{on } \omega\times(0,T), \\ &\theta_3(0) = \dot{\theta}_3(0) = 0 && \text{in } \omega. \end{aligned} \tag{4.89}$$

Since $f^3 \in D(\Omega\times(0,T))$, we have $\int_{-1}^1(-\partial_3 f^3)dx_3 = 0$ and so $\theta_3 = 0$. Therefore it follows that $\langle\psi_3, -\partial_3 f\rangle = 0$.

(ii) To prove that $\psi_\alpha = \hat{\psi}_\alpha - x_3\partial_\alpha\psi_3,\ \hat{\psi}_\alpha$ independent of x_3, it is enough to prove that

$$\langle \partial_3\psi_\alpha + \partial_\alpha\psi_3, f^\alpha\rangle = 0, \quad \forall\, f^\alpha \in D(\Omega\times(0,T)). \tag{4.90}$$

That is,

$$\langle \psi_\alpha, -\partial_3 f^\alpha\rangle + \langle\psi_3, -\partial_\alpha f^\alpha\rangle = 0, \quad \forall\, f^\alpha \in D(\Omega\times(0,T)). \tag{4.91}$$

We consider sequences $\{\theta_0(\epsilon), \theta_1(\epsilon)\}, \{f(\epsilon)\}$ such that

$$\begin{aligned} &\theta_0(\epsilon) \to \theta_0 = (\theta_{0\alpha}, 0) && \text{in } V(\Omega), && \theta_1(\epsilon) \to \theta_1 = (\theta_{1\alpha}, 0) && \text{in } [L^2(\Omega)]^3, \\ &f^\alpha(\epsilon) \to -\partial_3 f^\alpha && \text{in } X(\Omega), && f^\beta(\epsilon) \to 0 && \text{in } X(\Omega),\ \beta\neq\alpha, \\ &f^3(\epsilon) \to -\partial_\alpha f^\alpha && \text{in } L^1(0,T;L^2(\Omega)). \end{aligned}$$

Again it can be verified that the above sequences satisfy all the hypothesis of Theorem 4.2. Note that $\int_{-1}^1 -\partial_3 f^\alpha dx_3 = 0$ when $f^\alpha \in D(\Omega\times(0,T))$ and hence $S_{f^\alpha} = S$. At the limit we obtain (cf. (4.80))

$$\begin{aligned} 0 = \langle\psi_\alpha, -\partial_3 f^\alpha\rangle &+ \langle\psi_3, -\partial_\alpha f^\alpha\rangle + \int_{\omega\times(0,T)} [2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3]d\omega dt \\ &- \int_{\partial\omega\times(0,T)} (q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt - \int_{\omega\times(0,T)} n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt \\ &- \int_{\omega\times(0,T)} q_\zeta\partial_\zeta(n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega dt + \int_{\omega\times(0,T)} n^\varphi_{\alpha\beta}(S(\theta_3))\partial_{\alpha\beta}\varphi q_\zeta\partial_\zeta\phi_3 d\omega dt \\ &+ \int_{\omega\times(0,T)}\Big(\int_{-1}^1 -\partial_3 f^\alpha q_\sigma dx_3\Big)\partial_\sigma(S(\phi_3))d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^1 -\partial_3 f^\alpha q_\sigma dx_3\Big)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt \\ &- \int_{\omega\times(0,T)}\Big(\int_{-1}^1 -\partial_3 f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^1 -\partial_3 f^\alpha dx_3\Big)q_\sigma\partial_{\sigma\alpha}\varphi d\omega dt, \end{aligned} \tag{4.92}$$

where θ_3 is the solution of

$$
\begin{aligned}
&2\rho\ddot{\theta}_3 - \partial_{\alpha\beta}m_{\alpha\beta}(\theta_3) - n^{\varphi}_{\alpha\beta}(S(\theta_3))\partial_{\alpha\beta}\varphi = \int_{-1}^{1}[(-\partial_\alpha f^\alpha) - \partial_\alpha(x_3(\partial_3 f^\alpha))]dx_3 \quad \text{in } \omega\times(0,T),\\
&\theta_3 = \partial_\nu\theta_3 = 0 \quad \text{on } \omega\times(0,T), \qquad\qquad (4.93)\\
&\theta_3(0) = \dot{\theta}_3(0) = 0 \quad \text{in } \omega.
\end{aligned}
$$

Since $f^\alpha \in D(\Omega\times(0,T))$, we have

$$
\int_{-1}^{1}[(-\partial_\alpha f^\alpha) - \partial_\alpha(x_3(\partial f^\alpha))]dx_3 = -\int_{-1}^{1}\partial_\alpha f^\alpha dx_3 + \partial_\alpha\Big(\int_{-1}^{1} f^\alpha dx_3\Big) = 0. \qquad (4.94)
$$

Hence $\theta_3 = 0$ is the unique solution of (4.93). Therefore (4.92) becomes

$$
\begin{aligned}
0 = &\langle\psi_\alpha, -\partial_3 f^\alpha\rangle + \langle\psi_3, -\partial_\alpha f^\alpha\rangle + \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha q_\sigma dx_3\Big)\partial_\sigma(S(\phi_3))d\omega dt\\
&- \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha dx_3\Big)q_\sigma\partial_{\sigma\alpha}\varphi d\omega dt\\
&- \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha q_\sigma dx_3\Big)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt. \qquad (4.95)
\end{aligned}
$$

Since $f^\alpha \in D(\Omega\times(0,T))$, we have

$$
\begin{aligned}
&\int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha q_\sigma dx_3)\partial_\sigma(S(\phi_3)\Big)d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt\\
&- \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha dx_3\Big)q_\sigma\partial_{\sigma\alpha}\varphi d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} -\partial_3 f^\alpha q_\sigma dx_3\Big)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt\\
&= 0. \qquad (4.96)
\end{aligned}
$$

Hence

$$
0 = \langle\psi_\alpha, -\partial_3 f^\alpha\rangle + \langle\psi_3, -\partial_\alpha f^\alpha\rangle. \qquad (4.97)
$$

(iii) To prove that $\hat{\psi}_\alpha = Z(\phi_3)$, let us consider in (4.80) $\theta_{03} = \theta_{13} = 0, f^3 = 0$ and $f^\alpha \neq 0$, and $f^\alpha \in D(\omega\times(0,T))$. Then we have

$$
\begin{aligned}
0 = &\langle\hat{\psi}_\alpha - x_3\partial_\alpha\psi_3, f^\alpha\rangle + \int_{\omega\times(0,T)}[2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3]d\omega dt\\
&- \int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt - \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt\\
&- \int_{\omega\times(0,T)} q_\zeta\partial_\zeta(n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega dt + \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(S_{f^\alpha}(\theta_3))\partial_{\alpha\beta}\varphi q_\zeta\partial_\zeta\phi_3 d\omega dt\\
&+ \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} f^\alpha q_\sigma dx_3\Big)\partial_\sigma(S(\phi_3)_\alpha)d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} f^\alpha q_\sigma dx_3\Big)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt\\
&- \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt - \int_{\omega\times(0,T)}\Big(\int_{-1}^{1} f^\alpha q_\sigma\partial_{\alpha\sigma} dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt. \qquad (4.98)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\langle\hat{\psi}_\alpha, f^\alpha\rangle \\
&= -\int_{\omega\times(0,T)}[2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3]d\omega dt \\
&\quad + \int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt + \int_{\omega\times(0,T)} n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt \\
&\quad + \int_{\omega\times(0,T)} q_\zeta\partial_\zeta(n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega dt - \int_{\omega\times(0,T)} n^\varphi_{\alpha\beta}(S_{f^\alpha}(\theta_3))\partial_{\alpha\beta}\varphi q_\zeta\partial_\zeta\phi_3 d\omega dt \\
&\quad - \int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha q_\sigma dx_3\Big)\partial_\sigma(S(\phi_3)_\alpha)d\omega dt + \int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha q_\sigma dx_3\Big)\partial_\alpha\phi_3\partial_\sigma\varphi d\omega dt \\
&\quad + \int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt + \int_{\omega\times(0,T)}\Big(\int_{-1}^1 f^\alpha q_\sigma\partial_{\alpha\sigma} dx_3\Big)\varphi\partial_\alpha\phi_3 d\omega dt. \qquad (4.99)
\end{aligned}
$$

For a given $\phi_3 \in H^2_0(\omega)$, the right-hand side of the above equation defines a bounded linear functional on $H^1_0(\omega)$. Hence

$$\langle\hat{\psi}_\alpha, f^\alpha\rangle = \langle Z(\phi_3), f^\alpha\rangle, \quad \forall\, f^\alpha \in H^1_0(\omega) \tag{4.100}$$

and therefore $\hat{\psi}_\alpha = Z(\phi_3)$.

We can now identify the limit problem corresponding to (4.80) using the symmetry of the bilinear form $b(\cdot\,,\,\cdot)$ defined in (4.65):

$$
\begin{aligned}
&2\rho\ddot{y}_3 + \frac{8\mu(\lambda+2\mu)}{3(\lambda+2\mu)}\Delta^2 y_3 - n^\varphi_{\alpha\beta}(S(y_3))\partial_{\alpha\beta}\varphi \\
&= 2\rho\ddot{\phi}_3 + \frac{8\mu(\lambda+2\mu)}{(\lambda+2\mu)}\Delta^2\phi_3 + n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi - q_\zeta\partial_\zeta(n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi) \\
&\quad + n^\varphi_{\alpha\beta}(S(q_\zeta\partial_\zeta\phi_3))\partial_{\alpha\beta}\varphi \quad \text{in } \omega\times(0,T), \\
&y_3 = 0 \quad \text{on } \partial\omega\times(0,T), \\
&\partial_\nu y_3 = (q_\alpha\nu_\alpha)\Delta\phi_3 \quad \text{on } \partial\omega\times(0,T), \\
&y_3(T) = \dot{y}_3(T) = 0 \quad \text{in } \omega,
\end{aligned} \tag{4.101}
$$

where ϕ_3 is the unique solution of the homogeneous 2D problem

$$
\begin{aligned}
&2\rho\ddot{\phi}_3 + \frac{8\mu(\lambda+2\mu)}{3(\lambda+2\mu)}\Delta^2\phi_3 - n^\varphi_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi = 0 && \text{in } \omega\times(0,T), \\
&\phi_3 = \partial_\nu\phi_3 = 0 && \text{on } \partial\omega\times(0,T), \\
&\phi_3(0) = \phi_{03}, \quad \dot{\phi}_3(0) = \phi_{13} && \text{in } \omega.
\end{aligned} \tag{4.102}
$$

Definition 4.1 *The function y_3 is a solution of the 2D problem (4.101) in the transposition sense if $\phi_3 \in L^\infty(0,T;L^2(\omega))$, the traces $\{y_3(0), \dot{y}_3(0)\}$ makes sense in $L^2(\omega)\times H^{-2}(\omega)$ and y_3 satisfies*

$$
\begin{aligned}
&\langle\{\rho\dot{y}_3(0), -\rho y_3(0)\}, \{\theta_{03}, \theta_{13}\}\rangle \\
&= \langle y_3, g_3\rangle + \int_{\omega\times(0,T)}[2\rho\dot{\theta}_3\dot{\phi}_3 + 3m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3]d\omega dt - \int_{\partial\omega\times(0,T)}(q_\zeta\nu_\zeta)m_{\alpha\beta}(\theta_3)\partial_{\alpha\beta}\phi_3 d\omega dt
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt - \int_{\omega\times(0,T)} q_\zeta\partial_\zeta(n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\theta_3 d\omega dt \\
&+ \int_{\omega\times(0,T)} n^{\varphi}_{\alpha\beta}(S(q_\zeta\partial_\zeta\phi_3))\partial_{\alpha\beta}\varphi\theta_3 d\omega dt
\end{aligned}
\tag{4.103}
$$

for any $g_3 \in L^1(0,T;L^2(\omega))$ *and any* $\{\theta_{03},\theta_{13}\} \in H^2_0(\omega)\times L^2(\omega)$ *with* θ_3 *the solution of*

$$
\begin{aligned}
&2\rho\ddot{\theta}_3 + \frac{8\mu(\lambda+2\mu)}{3(\lambda+2\mu)}\Delta^2\theta_3 - n^{\varphi}_{\alpha\beta}(S(\theta_3))\partial_{\alpha\beta}\varphi = g_3 && \text{in } \omega\times(0,T),\\
&\theta_3 = \partial_\nu\theta_3 = 0 && \text{on } \partial\omega\times(0,T),\\
&\theta_3(0)=\theta_{03}, \quad \dot{\theta}_3(0)=\theta_{13} && \text{in } \omega.
\end{aligned}
\tag{4.104}
$$

Theorem 4.3 *There exists a unique solution to the problem* (4.101) *in the transposition sense.*

Proof Multiplying the first equation of (4.101) by θ_3, a solution of (4.104), and integrate by parts, it can be shown using duality arguments the existence of a unique solution $y_3 \in L^\infty(0,T;L^2(\omega))$.

Theorem 4.4 *Let* $T > T(\epsilon)$ *and* $\psi(\epsilon)$ *be the scaled* (*weak*) *solutions of problem* (2.7) *with controls* (3.23). *Suppose that the scaled initial data* $\{\psi_0(\epsilon),\psi_1(\epsilon)\}$ *verifies*

$$
\|\{\psi_1(\epsilon),-\psi_0(\epsilon)\}\|_{[V(\Omega)]'\times[L^2(\Omega)]^3} \le C.
\tag{4.105}
$$

Then there exists a subsequence of $\psi(\epsilon)$ (*still indexed by* ϵ) *and functions* $\{\psi_\alpha,\psi_3\}$ *in* $[X(\Omega)]'2\times L^\infty(0,T;L^2(\Omega))$ *such that, for any* $\{f_1,f_2,f_3\}\in[X(\Omega)]^2\times L^1(0,T;L^2(\Omega))$

$$
\langle\psi_i(\epsilon),f^i\rangle \to \langle\psi_i,f^i\rangle \quad \text{as } \epsilon\to 0.
$$

Moreover, the limit function $\psi=(\psi_i)$ *satisfies*

(i) $\psi=(\psi_i)$ *is a Kirchhoff-Love displacement, that is,* ψ_3 *is independent of* x_3 *and* $\psi_\alpha = \hat{\psi}_\alpha - x_3\partial_\alpha\psi_3$, *where* $\hat{\psi}_\alpha$ *is independent of* x_3.

(ii) ψ_3 *is the solution* (*in the transposition sense*) *of the following* 2D *problem*

$$
\begin{aligned}
&2\rho\ddot{\psi}_3 + \frac{8\mu(\lambda+2\mu)}{3(\lambda+2\mu)}\Delta^2\psi_3 - n^{\varphi}_{\alpha\beta}(S(\psi_3))\partial_{\alpha\beta}\varphi\\
&= 2\rho\ddot{\phi}_3 + \frac{8\mu(\lambda+2\mu)}{(\lambda+2\mu)}\Delta^2\phi_3 - n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi - q_\zeta\partial_\zeta(n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi)\\
&\quad + n^{\varphi}_{\alpha\beta}(S(q_\zeta\partial_\zeta\phi_3))\partial_{\alpha\beta}\varphi \quad \text{in } \omega\times(0,T),\\
&\psi_3 = 0 \quad \text{on } \partial\omega\times(0,T)\\
&\partial_\nu\psi_3 = (q_\alpha\nu_\alpha)\Delta\phi_3 \quad \text{on } \partial\omega\times(0,T),\\
&\psi_3(0) = \frac{1}{2}\int_{-1}^{1}\psi_{03}dx_3, \quad \dot{\psi}_3(0) = \frac{1}{2}\int_{-1}^{1}\psi_{13}dx_3 \quad \text{in } \omega,\\
&\psi_3(T) = \dot{\psi}_3(T) = 0 \quad \text{in } \omega.
\end{aligned}
\tag{4.106}
$$

The pair $\{\psi_{03},\psi_{13}\}$ *is the weak limit in the space* $L^2(\Omega)\times[V(\Omega)]'$ *of the sequence* $\{\psi_{03}(\epsilon),$

$\psi_{13}(\epsilon)\}_{\epsilon>0}$ *and* ϕ_3 *is the unique solution of the homogeneous* $2D$ *problem*

$$\begin{aligned} & 2\rho\ddot{\phi}_3 + \frac{8\mu(\lambda+2\mu)}{3(\lambda+2\mu)}\Delta^2\phi_3 - n^{\varphi}_{\alpha\beta}(S(\phi_3))\partial_{\alpha\beta}\varphi = 0 && \text{in } \omega\times(0,T), \\ & \phi_3 = 0, \quad \partial_\nu\phi_3 = 0 && \text{on } \partial\omega\times(0,T), \\ & \phi_3(0) = \phi_{03}, \quad \dot{\phi}_3(0) = \phi_{13} && \text{in } \omega, \end{aligned} \tag{4.107}$$

where $\{\phi_{03},\phi_{13}\}$ *is the weak limit in the space* $[V(\Omega)]\times L^2(\Omega)$ *of the sequence* $\{\phi_0(\epsilon),\phi_1(\epsilon)\} = \Lambda^{-1}(\epsilon)(\{\psi_1(\epsilon),-\psi_0(\epsilon)\})$.

(iii) $\hat{\psi}_\alpha = Z(\phi_3)$.

Proof To prove the theorem, it is enough to prove that $\psi_3 = y_3$, where y_3 satisfies (4.103). Note that (4.80) will coincide with (4.103) if we are able to prove that (4.80) is valid for $f^\alpha = 0$, any pair $\{\theta_{03},\theta_{13}\}\in H^2_0(\omega)\times L^2(\omega)$ and any $g_3\in L^1(0,T;L^2(\omega))$. Choosing

$$\begin{aligned} f(\epsilon) &= (0,0,g_3), \quad \theta_1(\epsilon) = (0,0,\theta_{13}), \\ \theta_0(\epsilon) &= (-x_3\partial_1\theta_{03}, -x_3\partial_2\theta_{03}, \theta_{03}+\epsilon^2 v(\epsilon)) \end{aligned}$$

such that

$$\begin{aligned} \epsilon v(\epsilon) &\to 0 && \text{in } H^1(\Omega), \\ \partial_3 v(\epsilon) &\to \frac{\lambda}{\lambda+2\mu}x_3\Delta\theta_3 && \text{in } L^2(\Omega) \end{aligned}$$

(note that this is possible because $H^1_{\Gamma_0}(\Omega)$ is dense in $L^2(\Omega)$), it follows that

$$\begin{aligned} & f(\epsilon)\to(0,0,g_3) \quad \text{strongly in } [X(\Omega)]^2\times L^1(0,T;L^2(\Omega)), \\ & \theta_0(\epsilon)\to(-x_3\partial_1\theta_{03},-x_3\partial_2\theta_{03},\theta_{03}) \quad \text{strongly in } V(\Omega), \\ & e_{\alpha||3}(\theta_0(\epsilon))\to 0, \quad e_{3||3}(\theta_0(\epsilon))\to\frac{\lambda}{\lambda+2\mu}x_3\Delta\theta_{03} \quad \text{strongly in } L^2(\Omega), \\ & \epsilon\theta_{1\alpha}(\epsilon)\to 0, \quad \theta_{13}(\epsilon)\to\theta_{13} \quad \text{strongly in } L^2(\Omega). \end{aligned} \tag{4.108}$$

Hence we conclude that (4.80) is valid for any pair $\{\theta_{03},\theta_{13}\}\in H^2_0(\omega)\times L^2(\omega)$ and any $g_3\in L^1(0,T;L^2(\omega))$.

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