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The Limiting Behavior for Observations That Change with Time**

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Abstract Consider a system where units have random magnitude entering according to a homogeneous or nonhomogeneous Poisson process, while in the system, a unit's magnitude may change with time. In this paper, the authors obtain some results for the limiting behavior of the sum process of all unit magnitudes present in the system at time t.

Keywords Stochastic system, Strong laws of large numbers, Random weighted sums, Poisson process
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1 Introduction

Consider a system, where units have random magnitude entering according to a homogeneous or nonhomogeneous Poisson process, staying for a random period of time, and then departing, while in the system, a unit's magnitude may change with time. Moreover, a unit's length of stay (lifetime) in the system may depend on its initial magnitude.

Let $T_k, k = 1, 2, \cdots$ be the arrival time sequence. Suppose that the kth entering unit at time T_k, has a lifetime L_k, and a magnitude $X_k(s)$ at time $T_k + s$, for $s \geq 0$. We call the kth unit active (present in the system) at time t if $T_k \leq t < T_k + L_k$. Suppose that

(A1) the unit's arrival time sequence $\{T_k, k \geq 1\}$ is generated by a nonhomogenous Poisson process $\{N(t), t \geq 0\}$, which has continuous intensity function $\lambda(t)$;

(A2) $(X(\cdot), L), (X_1(\cdot), L_1), (X_2(\cdot), L_2), \cdots$ is an i.i.d. sequence of random pairs, and independent of the process $\{N(t), t \geq 0\}$, where $\{X(t), t \geq 0\}$ is a monotone (nonincreasing or nondecreasing) process;

(A3) $P(0 \leq L \leq \infty) = 1$ and $P(L > 0) > 0$.

At time $t \geq 0$, the size of the active population is

$$S(t) = \sum_{k=1}^{N(t)} I(T_k \leq t < T_k + L_k),$$

and the proportion of the active population with magnitude exceeding y is

$$\theta(y; t) = \Big(\frac{1}{S(t)}\Big) \sum_{k=1}^{N(t)} I(T_k \leq t < T_k + L_k, X_k(t - T_k) > y).$$

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When $P(L=\infty)=1$ (units never leave the system) Anisimov [2] proved some weak limit theorems for evolving accumulation process, which are sums of the form $\sum_{k\geq 1} X_k(t-T_k)I(T_k\leq t)$ in the above notations. In Rothmann and Russo [5,6], assuming that a unit's magnitude remains constant for the duration of its life, and that the arrival time sequence is fixed or the interarrival time sequence is i.i.d., they obtained some specific limits about the mean magnitude and percentiles within the active population. In Rothmann and Russo [7,8], they discussed these limits further when the arrival time sequence is generated by a Poisson process.

Define the lower function $\lambda_L(s)=\inf_{t\geq s}\lambda(t)$ for $s>0$. Rothmann and Russo [8] proved the following theorems.

Theorem A *Suppose that the conditions* (A1), (A2), (A3) *hold with X a nonincreasing process. If $\frac{\lambda(s)}{\log s}\to\infty$, and $\frac{\lambda_L(s)}{\lambda(s)}\to 1$, then*

$$\frac{S(t)}{ES(t)}\longrightarrow 1 \quad a.s.$$

$$\left|\theta(y;t)-\frac{\int_0^t P(L>s,X(s)>y)\lambda(t-s)ds}{\int_0^t P(L>s)\lambda(t-s)ds}\right|\longrightarrow 0 \quad a.s.$$

Theorem B *Suppose that the conditions* (A1), (A2), (A3) *hold with X a nonincreasing process. If for some positive α, we have $0\leq\lambda(t)\leq\alpha+t^{\alpha}$ for all $t>0$, $\frac{\lambda_L(s)}{\lambda(s)}\to 1$, and*

$$\Big(\frac{\lambda(t)}{\log t}\Big)\int_0^t P(L>s)ds\to\infty,$$

then the conclusions of Theorem A *remain true.*

However, in storage of counters, insurance risk theory, reliability theory, counter models, etc, it is also of interest to study the limiting behavior of the sum process $\Big\{\sum_{k=1}^{N(t)} X_k(t-T_k),\ t\geq 0\Big\}$ of all unit magnitudes present in the system at time t (cf. Karlin and Taylor [3], or Rothmann and Russo [8]). In this paper, we investigate a version of such system and show some laws of large numbers about the sum process. We assume:

(C1) The unit's arrival time sequence $\{T_k,k\geq 1\}$ is generated by a homogenous or nonhomogenous Poisson process $\{N(t),t\geq 0\}$, which has continuous intensity function $\lambda(t)$.

(C2) $(X,X_1,X_2,\cdots)$ is an i.i.d. sequence of nonnegative random variables, and independent of the process $\{N(t),t\geq 0\}$.

(C3) The kth entering unit has a magnitude X_k at time T_k, and then shrinks with time at the rate $h(t-T_k)$, a function of the time staying in the system. Here $h(t)$ is a nonnegative nonincreasing function for $t\geq 0$ with $h(0)=1$, and $h(t)=0$ for $t<0$.

Under (C1)–(C3), the process that keeps track of the total magnitude of all units present in the system at time t is

$$S_{N(t)}(t)=\sum_{k=1}^{N(t)} h(t-T_k)X_k.$$

Now, we give our results as follows.

Let $\mu = EX$, $\mu_r = EX^r$ for $r > 0$. Define for all $t \geq 0$ and $0 < r < \infty$

$$m(t) = \int_0^t \lambda(s)ds, \quad L_r(t) = \int_0^t h^r(t-s)\lambda(s)ds.$$

Then we have

Theorem 1.1 *Suppose that conditions* (C1), (C2), (C3) *are satisfied, and that* $\mu_r < \infty$ *for some* $r > 1$. *If*

$$L_1(t) \to \infty \quad \text{as } t \to \infty, \tag{1.1}$$

$$\liminf_{\substack{t_2\to\infty \\ t_1-t_2\to\infty}} \frac{L_1(t_1)}{L_1(t_2)} \geq 1, \tag{1.2}$$

$$\liminf_{t\to\infty} \frac{L_1(t)^{r'}}{m(t)} > 0 \tag{1.3}$$

for some $1 \leq r' < r \leq 2$ *or* $1 \leq r' < 2 \leq r$, *then*

$$\frac{S_{N(t)}(t)}{L_1(t)} \to \mu \quad a.s. \quad \text{as } t \to \infty. \tag{1.4}$$

Remark 1.1 From the proof of the theorem we shall know that conditions $1 \leq r' < r \leq 2$ or $1 \leq r' < 2 \leq r$ can be replaced by $1 \leq r' \leq r \leq 2$ or $1 \leq r' \leq 2 \leq r$ respectively, if we replace (1.3) by $\liminf\limits_{t\to\infty} \frac{L_1(t)^{r'}}{m(t)\log^\beta m(t)} > 0$ with β satisfying $\beta > \frac{r'}{r-1}$.

Remark 1.2 We can see that conditions (1.2) and (1.3) of Theorem 1.1 is reasonable from the following two examples.

Example 1.1 Let $\lambda(t) = \lambda t^{\alpha_1}, t \geq 0$ for some $\alpha_1 > -1$, and

$$h(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq e, \\ \left(\frac{t}{e}\right)^{\alpha_2} & \text{for } t \geq e \end{cases}$$

for some $-1 < \alpha_2 < 0$. Moreover, we assume $\alpha_1 + \alpha_2 + 1 > 0$. Then

$$m(t) = \frac{\lambda t^{\alpha_1+1}}{\alpha_1+1} \to \infty \quad \text{as } t \to \infty,$$
$$L_1(t) = \lambda e^{-\alpha_2}\beta(\alpha_1+1, \alpha_2+1)t^{\alpha_1+\alpha_2+1} + o(t^{\alpha_1+\alpha_2+1}) \quad \text{as } t \to \infty,$$

and therefore

$$\liminf_{t\to\infty} \frac{L_1^{(\alpha_1+1)/(\alpha_1+\alpha_2+1)}(t)}{m(t)} = \frac{(\lambda\beta(\alpha_1+1,\alpha_2+1))^{(\alpha_1+1)/(\alpha_1+\alpha_2+1)}}{\frac{\lambda}{\alpha_1+1}} > 0,$$
$$\liminf_{\substack{t_2\to\infty \\ t_1-t_2\to\infty}} \frac{L_1(t_1)}{L_1(t_2)} \geq 1,$$

which implies that conditions (1.1)–(1.3) in Theorem 1.1 are satisfied. Hence (1.4) holds true.

Example 1.2 If take

$$\lambda(t) = \begin{cases} \lambda & \text{for } 0 \leq t \leq e, \\ \lambda(\log t)^{\alpha_1} & \text{for } t \geq e \end{cases}$$

for some $\alpha_1 > -1$ instead of $\lambda(t) = \lambda t^{\alpha_1},\ t \geq 0$ for some $\alpha_1 > -1$, we have also the conclusion as in Example 1.1

Theorem 1.2 *Suppose that conditions* (C1), (C2), (C3) *and* (1.2) *are satisfied, and that there exists a constant* M *such that* $Ee^{\alpha X} \leq M$ *for some* $\alpha > 0$. *If*

$$m(t) \to \infty \quad \text{as } t \to \infty$$

and

$$\liminf_{t\to\infty} \frac{L_1^{r''}(t)}{\log m(t)} > 0 \tag{1.5}$$

for some $0 < r'' < 1$, *then*

$$\frac{S_{N(t)}(t)}{L_1(t)} \to \mu \quad a.s. \quad \text{as } t \to \infty.$$

The following is a result about moment convergence.

Theorem 1.3 *Suppose that conditions* (C1), (C2), (C3) *and* (1.1), (1.2) *are satisfied. If* $\mu_r < \infty$ *for* $1 < r \leq 2$, *then*

$$\frac{S_{N(t)}(t)}{L_1(t)} \xrightarrow{L^r} \mu \quad \text{as } t \to \infty.$$

If, in addition,

$$\frac{(m(t))^{(r-2)/(r-1)}}{L_1^2(t)} \to 0 \quad \text{as } t \to \infty, \tag{1.6}$$

then the conclusion is also true for $r > 2$.

We are also interested in the case $0 < r \leq 1$, and conjecture that the similar results are true under suitable conditions. As an example, we give a result, which can easily be obtained from [1, Theorem 2.3].

Theorem 1.4 *Suppose that conditions* (C1), (C2), (C3) *are satisfied. If* $\mu_r < \infty$ *for some* $0 < r < 1$ *and* $m(t) \to \infty$ *as* $t \to \infty$, *then*

$$\frac{S_{N(t)}(t)}{(N(t))^{1/r}} \to 0 \quad a.s. \quad \text{as } t \to \infty \quad \text{and} \quad \frac{S_{N(t)}(t)}{(m(t))^{1/r}} \to 0 \quad a.s. \quad \text{as } t \to \infty.$$

Throughout the paper, C denotes an absolute constant; c_r denotes a constant depending only on r. The values C and c_r may vary from line to line.

2 Proofs

We begin with some lemmas before the proofs of theorems.

Lemma 2.1 *Suppose that conditions* (C1), (C2), (C3) *are satisfied. Then for any* $t \geq 0$, $S_{N(t)}(t)$ *and* $\sum_{k=1}^{N(t)} h(t - U_k(t))X_k$ *have the identical distribution, where* $\{U(t), U_k(t), k \geq 1\}$ *is a* *sequence of* i.i.d. *random variables with common density function* $f_U(s) = \frac{\lambda(s)}{m(t)}$, $0 < s < t$, *and independent of* $N(t)$ *and* $\{X_k, k \geq 1\}$.

Proof By the definition of T_k, for given $t \geq 0$, we have

$$\{T_k < t\} = \{N(t) \geq k\}$$

and thus by the increment independence of a Poisson process, for all $0 < s < t, k \leq n$, we have

$$\begin{aligned} P\{T_k < s \mid N(t) = n\} &= \frac{P\{N(s) \geq k, N(t) = n\}}{P\{N(t) = n\}} \\ &= \frac{\sum\limits_{i=k}^{n} P\{N(s) = i, N(t) - N(s) = n - i\}}{P\{N(t) = n\}} \\ &= \frac{\sum\limits_{i=k}^{n} C_n^i (m(s))^i (m(t) - m(s))^{n-i}}{(m(t))^n}. \end{aligned}$$

Hence the conditional density function of T_k under the condition $\{N(t) = n\}$ is given by

$$\begin{aligned} &f_{T_k|\{N(t)=n\}}(s \mid t) \\ &= \frac{\sum\limits_{i=k}^{n} (iC_n^i \lambda(s)(m(s))^{i-1}(m(t) - m(s))^{n-i} - (n-i)C_n^i \lambda(s)(m(s))^i (m(t) - m(s))^{n-i-1})}{(m(t))^n} \\ &= \frac{n!}{(k-1)!(n-k)!} \Big(\frac{m(s)}{m(t)}\Big)^{k-1} \Big(1 - \frac{m(s)}{m(t)}\Big)^{n-k} \frac{\lambda(s)}{m(t)}, \quad 0 < s < t. \end{aligned} \tag{2.1}$$

Let $U(t),\ U_k(t),\ k \geq 1$ be as in the lemma. Then from well known results on order statistics and (2.1) we have that the conditional distribution of $T_1, T_2, \cdots, T_n$ under the condition $\{N(t) = n\}$ is just the same as the distribution of sequential order statistics $U_{(1)}(t), U_{(2)}(t), \cdots, U_{(n)}(t)$ generated by $U_1(t), U_2(t), \cdots, U_n(t)$. So we have

$$\begin{aligned} P\{S_{N(t)}(t) < x\} &= \sum_{n=1}^{\infty} P\{S_{N(t)}(t) < x \mid N(t) = n\} \cdot P\{N(t) = n\} \\ &= \sum_{n=1}^{\infty} P\Big\{\sum_{k=1}^{n} h(t - T_k)X_k < x \mid N(t) = n\Big\} \cdot P\{N(t) = n\} \\ &= \sum_{n=1}^{\infty} P\Big\{\sum_{k=1}^{n} h(t - U_{(k)}(t))X_k < x \mid N(t) = n\Big\} \cdot P\{N(t) = n\} \\ &= \sum_{n=1}^{\infty} P\Big\{\sum_{k=1}^{n} h(t - U_k(t))X_k < x \mid N(t) = n\Big\} \cdot P\{N(t) = n\} \\ &= P\Big\{\sum_{k=1}^{N(t)} h(t - U_k(t))X_k < x\Big\}. \end{aligned} \tag{2.2}$$

Since $N(t)$ is a Poisson process, it is easy to show the following lemma.

Lemma 2.2 *Suppose that* $m(t) \to \infty$ *as* $t \to \infty$. *Then*

$$\frac{N(t)}{m(t)} \to 1 \quad a.s. \quad as\ t \to \infty.$$

Lemma 2.3 *Fix $t>0$. If $m(t)\ge 1$, then $E(N(t)^r)\le c_r(m(t))^r$ for all $r>0$.*

Proof First let r be a positive integer n. The characteristic function of the random variable $N(t)$ is $G(s)=\exp\{m(t)(\mathrm{e}^{\mathrm{i}s}-1)\}$. Then we can conclude that for any integer $n\ge 1$, the nth derivative of $G(s)$

$$G^{(n)}(s)=\mathrm{i}m(t)\mathrm{e}^{\mathrm{i}s}\sum_{k=0}^{n-1}C_{n-1}^{k}\mathrm{i}^kG^{(n-k-1)}(s). \tag{2.3}$$

In fact, (2.3) is obvious for $n=1,2$. Assuming that it holds for all integers not larger than $n-1$, we have

$$\begin{aligned}G^{(n)}(s)=(G^{(n-1)}(s))'&=\Big(\mathrm{i}m(t)\mathrm{e}^{\mathrm{i}s}\sum_{k=0}^{n-2}C_{n-2}^{k}\mathrm{i}^kG^{(n-k-2)}(s)\Big)'\\&=\mathrm{i}m(t)\Big(\mathrm{i}\mathrm{e}^{\mathrm{i}s}\sum_{k=0}^{n-2}C_{n-2}^{k}\mathrm{i}^kG^{(n-k-2)}(s)+\mathrm{e}^{\mathrm{i}s}\sum_{k=0}^{n-2}C_{n-2}^{k}\mathrm{i}^kG^{(n-k-1)}(s)\Big)\\&=\mathrm{i}m(t)\mathrm{e}^{\mathrm{i}s}\Big(\sum_{k=0}^{n-1}C_{n-2}^{k-1}\mathrm{i}^kG^{(n-k-1)}(s)+\sum_{k=0}^{n-2}C_{n-2}^{k}\mathrm{i}^kG^{(n-k-1)}(s)\Big)\\&=\mathrm{i}m(t)\mathrm{e}^{\mathrm{i}s}\sum_{k=0}^{n-1}C_{n-1}^{k}\mathrm{i}^kG^{(n-k-1)}(s).\end{aligned}$$

Then (2.3) is proved.

Now from (2.3) we can show that

$$E(N(t))^n=|G^{(n)}(0)|\le (m(t))^n+2^{n^2}(m(t))^{n-1}.$$

This is obvious for $n=1,2,3$. We assume that it holds for integers which are not larger than $n-1$ ($n\ge 4$). Then by (2.3) and the induction hypothesis, we have

$$\begin{aligned}E(N(t))^n&\le m(t)\sum_{k=0}^{n-1}C_{n-1}^{k}((m(t))^{n-k-1}+2^{(n-k-1)^2}(m(t))^{n-k-2})\\&\le((m(t))^n+2^{(n-1)^2}(m(t))^{n-1})+2^{n-1}\cdot 2^{(n-2)^2}(m(t))^{n-1}\\&\le (m(t))^n+2^{n^2}(m(t))^{n-1}.\end{aligned}$$

Choosing $c_n=2^{n^2}+1$, we obtain the conclusion when r is a positive integer.

For any non-integer $r>0$, there exists a positive integer n such that $n-1<r\le n$. It is clear that

$$E(N(t))^r\le \big(E(N(t))^n\big)^{r/n}\le (2^{(r+1)^2}+1)(m(t))^r.$$

This completes the proof of Lemma 2.3.

Applying Lemmas 2.1–2.3, we can obtain some moment estimates of the random weighted sum $S_{N(t)}(t)$ in terms of the moments of X and $L_r(t)$. By the definition

$$L_r(t)=EN(t)h^r(t-U(t))=m(t)Eh^r(t-U(t)).$$

Lemma 2.4 *Suppose that conditions* (C1), (C2), (C3) *are satisfied and that* $\mu_r > 0$. *Then we have*

(i) $E|S_{N(t)}(t)|^r \le \mu_r L_r(t)$ *for* $0 < r \le 1$;

(ii) $E|S_{N(t)}(t) - \mu N(t) Eh(t-U(t))|^r \le c_r \mu_r L_r(t)$ *for* $1 < r \le 2$.

Further, if $m(t) \ge 1$, *then*

(iii) $E|S_{N(t)}(t) - \mu N(t) Eh(t-U(t))|^r \le c_r \mu_r (m(t))^{r/2-1} L_r(t)$ *for* $r \ge 2$.

Proof We prove only (ii) and (iii), the proof of (i) is similar. By Lemma 2.1 we have

$$\begin{aligned}
&E|S_{N(t)}(t) - \mu N(t) Eh(t-U(t))|^r \\
&= \sum_{n=1}^{\infty} E\{|S_{N(t)}(t) - N(t) Eh(t-U(t))|^r X \mid N(t)=n\} \cdot P\{N(t)=n\} \\
&= \sum_{n=1}^{\infty} E\Big\{\Big|\sum_{k=1}^{n} (h(t-U_k(t))X_k - Eh(t-U_k(t))X_k\Big|^r\Big\} \cdot P\{N(t)=n\}.
\end{aligned}$$

If $1 \le r \le 2$, then it follows by the moment inequality for sums of independent random variables that

$$\begin{aligned}
&E|S_{N(t)}(t) - \mu N(t) Eh(t-U(t))|^r \\
&\le c_r \sum_{n=1}^{\infty} nE|h(t-U(t))X|^r \cdot P\{N(t)=n\} = c_r \mu_r L_r(t).
\end{aligned}$$

Similarly, for $r \ge 2$, by Lemmas 2.1 and 2.3 and the moment inequality (see [4, p.62]) we have

$$\begin{aligned}
&E|S_{N(t)}(t) - \mu N(t) Eh(t-U(t))|^r \\
&= \sum_{n=1}^{\infty} E\Big|\sum_{k=1}^{n} (h(t-U_k(t))X_k - Eh(t-U_k(t))X_k\Big|^r \cdot P\{N(t)=n\} \\
&\le c_r \sum_{n=1}^{\infty} n^{r/2} E|h(t-U(t))X|^r \cdot P\{N(t)=n\} \\
&= \frac{c_r \mu_r L_r(t) E(N(t))^{r/2}}{m(t)} \\
&\le c_r \mu_r (m(t))^{r/2-1} L_r(t).
\end{aligned}$$

Lemma 2.5 (See [4]) *Let* $X_1, \cdots, X_n$ *be independent random variables and write* $S_n = \sum_{k=1}^{n} X_k$. *Suppose there exist positive constants* $g_1, \cdots, g_n$ *and* T *such that*

$$Ee^{tX_k} \le e^{\frac{1}{2} g_k t^2}, \quad k = 1, \cdots, n$$

for $0 \le t \le T$. *Let* $G = \sum_{k=1}^{n} g_k$. *Then for* $0 \le x \le GT$

$$P(|S_n| \ge x) \le 2e^{-x^2/2G}.$$

Lemma 2.6 *Let $X_1, \cdots, X_n$ be independent random variables with means zero and write $S_n = \sum\limits_{k=1}^{n} X_k$. If there exist constants $b_k > 0$ and $a > 0$ such that $\frac{E|X_k|^n \leq b_k n! a^{n-2}}{2}$ for all $n \geq 2$, then for all $x > 0$, we have*

$$P\{|S_n| \geq x\sqrt{n}\} \leq 2\exp\left\{-\frac{x^2 n}{2\sum\limits_{k=1}^{n} b_k + 2ax\sqrt{n}}\right\}.$$

Proof Fix $x > 0$, and let $T = \frac{x\sqrt{n}}{\sum\limits_{k=1}^{n} b_k + ax\sqrt{n}}$. Then for any $0 \leq t \leq T < \frac{1}{a}$, by the conditions, we have

$$\begin{aligned}
Ee^{tX_k} &= \sum_{n=0}^{\infty} \frac{E(tX_k)^n}{n!} \leq 1 + \sum_{n=2}^{\infty} \frac{E(t|X_k|)^n}{n!} \leq 1 + \frac{b_k t^2}{2(1-at)} \\
&\leq 1 + \frac{b_k t^2 \Big(\sum\limits_{k=1}^{n} b_k + ax\sqrt{n}\Big)}{2\sum\limits_{k=1}^{n} b_k}, \quad k = 1, \cdots, n.
\end{aligned}$$

Write

$$g_k = \frac{b_k \Big(\sum\limits_{k=1}^{n} b_k + ax\sqrt{n}\Big)}{\sum\limits_{k=1}^{n} b_k}, \quad G = \sum_{k=1}^{n} g_k = \sum_{k=1}^{n} b_k + ax\sqrt{n}.$$

By Lemma 2.5, we have

$$P\{|S_n| \geq x\sqrt{n}\} \leq 2\exp\Big\{-\frac{(x\sqrt{n})^2}{2G}\Big\} = 2\exp\left\{-\frac{x^2 n}{2\sum\limits_{k=1}^{n} b_k + 2ax\sqrt{n}}\right\}.$$

Proof of Theorem 1.1 Case (i) $1 < r \leq 2$.

By the Markov inequality and Lemma 2.4, we have

$$\begin{aligned}
P\{|S_{N(t)}(t) - \mu N(t)Eh(t-U(t))| \geq \varepsilon\mu L_1(t)\} &\leq \frac{E|S_{N(t)}(t) - \mu N(t)Eh(t-U(t))|^r}{(\varepsilon\mu L_1(t))^r} \\
&\leq \frac{c_r \mu_r L_r(t)}{(\varepsilon\mu L_1(t))^r} \leq \frac{c_r}{(L_1(t))^{r-1}}.
\end{aligned} \tag{2.4}$$

Choose a nondecreasing sequence $\{t_n, n \geq 1\}$ satisfying

$$m(t_n) = n^{r/(r-1)}, \tag{2.5}$$

by (2.5) and condition (1.3)

$$L_1(t_n) \geq Cn^{r/(r-1)r'}. \tag{2.6}$$

Then, by (2.4), (2.6), Lemma 2.2 and the Borel-Cantelli lemma, we obtain

$$\frac{S_{N(t_n)}(t_n)}{N(t_n)Eh(t_n - U(t_n))X_k} \to \mu \quad \text{a.s.} \quad \text{as } n \to \infty. \tag{2.7}$$

From (2.5) and (2.6), we have

$$\frac{m(t_n)-m(t_{n-1})}{L_1(t_n)} \leq \frac{n^{r/(r-1)}-(n-1)^{r/(r-1)}}{Cn^{r/((r-1)r')}} \to 0 \quad \text{as } n\to\infty. \tag{2.8}$$

Moreover, write

$$\begin{aligned} L_1(t_n)-L_1(t_{n-1}) &= \int_{t_{n-1}}^{t_n} h(t_n-s)\lambda(s)ds + \int_0^{t_{n-1}} (h(t_n-s)-h(t_{n-1}-s))\lambda(s)ds \\ &=: l_n + l'_n. \end{aligned} \tag{2.9}$$

Noting the following facts:

(a) for any $\varepsilon>0$, there exists $N\geq 1$ such that $L_1(t_n)\geq(1-\varepsilon)L_1(t_{n-1})$ for all $n\geq N$ by condition (1.2);

(b) $\frac{l_n}{L_1(t_n)} \leq \frac{\int_{t_{n-1}}^{t_n}\lambda(s)ds}{L_1(t_n)} = \frac{m(t_n)-m(t_{n-1})}{L_1(t_n)} \to 0$ by (2.8);

(c) $l'_n \leq 0$;

we have

$$\frac{L_1(t_n)}{L_1(t_{n-1})} \to 1 \quad \text{as } n\to\infty. \tag{2.10}$$

Further, when $t_{n-1}<t\leq t_n$, noting that $h(t)$ is nonincreasing for all $t\geq 0$, we have

$$\begin{aligned} S_{N(t)}(t) &\leq \sum_{k=1}^{N(t_n)} h(t-T_k)X_k \leq \sum_{k=1}^{N(t_{n-1})} h(t_{n-1}-T_k)X_k + \sum_{k=N(t_{n-1})+1}^{N(t_n)} X_k, \\ S_{N(t)}(t) &\geq \sum_{k=1}^{N(t_{n-1})} h(t-T_k)X_k \geq \sum_{k=1}^{N(t_n)} h(t_n-T_k)X_k - \sum_{k=N(t_{n-1})+1}^{N(t_n)} X_k; \end{aligned} \tag{2.11}$$

$$\begin{aligned} L_1(t) &\leq \int_0^{t_n} h(t-s)\lambda(s)ds = \int_0^{t_{n-1}} h(t-s)\lambda(s)ds + \int_{t_{n-1}}^{t_n} h(t-s)\lambda(s)ds \\ &\leq \int_0^{t_{n-1}} h(t_{n-1}-s)\lambda(s)ds + \int_{t_{n-1}}^{t_n} \lambda(s)ds = L_1(t_{n-1}) + (m(t_n)-m(t_{n-1})), \\ L_1(t) &\geq \int_0^{t_{n-1}} h(t-s)\lambda(s)ds = \int_0^{t_n} h(t-s)\lambda(s)ds - \int_{t_{n-1}}^{t_n} h(t-s)\lambda(s)ds \\ &\geq \int_0^{t_{n-1}} h(t_n-s)\lambda(s)ds - \int_{t_{n-1}}^{t_n} \lambda(s)ds = L_1(t_n) - (m(t_n)-m(t_{n-1})). \end{aligned} \tag{2.12}$$

And by the properties of the Poisson process and the independence between $\{X_k, k\geq 1\}$ and $\{N(t), t\geq 0\}$, we have

$$\begin{aligned} P\Big\{\Big|\sum_{k=N(t_{n-1})+1}^{N(t_n)} (X_k-EX_k)\Big| \geq \varepsilon L_1(t_n)\Big\} &= P\Big\{\Big|\sum_{k=1}^{N(t_n)-N(t_{n-1})} (X_k-EX_k)\Big| \geq \varepsilon L_1(t_n)\Big\} \\ &\leq \frac{E\Big|\sum_{k=1}^{N(t_n)-N(t_{n-1})} (X_k-EX_k)\Big|^r}{(\varepsilon L_1(t_n))^r}. \end{aligned} \tag{2.13}$$

In the same way as the proof of Lemma 2.4, we have

$$E\Big|\sum_{k=N(t_{n-1})+1}^{N(t_n)}(X_k-EX_k)\Big|^r \le c_r\mu_r E(N(t_n)-N(t_{n-1})) = c_r\mu_r(m(t_n)-m(t_{n-1})). \quad (2.14)$$

And then by (2.5), (2.6), (2.8), (2.13), (2.14) and the Borel-Cantelli lemma, we have

$$\sum_{k=N(t_{n-1})+1}^{N(t_n)}\frac{X_k}{L_1(t_n)} = \sum_{k=N(t_{n-1})+1}^{N(t_n)}\frac{X_k-EX_k}{L_1(t_n)} + \mu\cdot\frac{N(t_n)-N(t_{n-1})}{m(t_n)-(t_{n-1})}\cdot\frac{m(t_n)-m(t_{n-1})}{L_1(t_n)} \to 0 \quad \text{a.s.} \quad \text{as } n\to\infty. \quad (2.15)$$

Therefore, it follows by (2.7), (2.11), (2.12) and (2.15) that

$$\frac{S_{N(t)}(t)}{L_1(t)} \le \frac{\sum\limits_{k=1}^{N(t_{n-1})} h(t_{n-1}-T_k)X_k + \sum\limits_{k=N(t_{n-1})+1}^{N(t_n)} X_k}{L_1(t_n)-(m(t_n)-m(t_{n-1}))} \xrightarrow{\text{a.s.}} \mu \quad \text{as } n\to\infty,$$

$$\frac{S_{N(t)}(t)}{L_1(t)} \ge \frac{\sum\limits_{k=1}^{N(t_n)} h(t_{n-1}-T_k)X_k - \sum\limits_{k=N(t_{n-1})+1}^{N(t_n)} X_k}{L_1(t_n)+(m(t_n)-m(t_{n-1}))} \xrightarrow{\text{a.s.}} \mu \quad \text{as } n\to\infty,$$

which completes the proof of Theorem 1.1 in the case of $1<r\le 2$.

Case (ii) $r>2$.

By Lemma 2.4, we have

$$\begin{aligned} &P\{|S_{N(t)}(t)-\mu N(t)Eh(t-U(t))|\ge \varepsilon\mu L_1(t)\}\\ \le\ &\frac{E|S_{N(t)}(t)-\mu N(t)Eh(t-U(t))|^r}{(\varepsilon\mu L_1(t))^r}\\ \le\ &\frac{c_r(m(t))^{r/2-1}L_r(t)}{(L_1(t))^r} \le \frac{c_r(m(t))^{r/2-1}}{(L_1(t))^{r-1}}. \end{aligned} \quad (2.16)$$

Choose a nondecreasing sequence $\{t_n\}$ satisfying

$$m(t_n)=n^2. \quad (2.17)$$

By (2.17) and condition (1.3), we get

$$L_1(t_n)\ge Cn^{2/r'}. \quad (2.18)$$

Then we just follow the same procedure as the proof in the case (i) with (2.4)–(2.6) replaced by (2.16)–(2.18) respectively, and the proof is completed.

Proof of Theorem 1.2 Let $Y_k(t)=h(t-U_k(t))X_k$, $Z_k(t)=Y_k(t)-EY_k(t)$. Then by the c_r-inequality we have

$$E|Z_k(t)|^n \le 2^{n-1}(E(Y_k(t))^n+(EY_k(t))^n) \le 2^nE(Y_k(t))^n \le \frac{2^nEX_k^n\cdot L_1(t)}{m(t)}, \quad (2.19)$$

and by the condition $Ee^{\alpha X_k} \le M$ we have

$$EX_k^n = n\int_0^\infty x^{n-1}P\{X_k \ge x\}dx \le nM\int_0^\infty x^{n-1}e^{-\alpha x}dx = \frac{Mn!}{\alpha^n}. \tag{2.20}$$

So from (2.19) and (2.20), by Lemma 2.6 with $b_k = \frac{8ML_1(t)}{m(t)\alpha^2}$ and $a = \frac{2}{\alpha}$, we have

$$\begin{aligned} P\Big\{\Big|\sum_{k=1}^n Z_k\Big| \ge \frac{\varepsilon\mu nL_1(t)}{m(t)}\Big\} &\le 2\exp\Big\{-\frac{\big(\frac{\varepsilon\mu\sqrt{n}L_1(t)}{m(t)}\big)^2}{\frac{16ML_1(t)}{m(t)\alpha^2}+\frac{4\varepsilon\mu L_1(t)}{m(t)\alpha}}\Big\} \\ &\le 2\exp\Big\{-\frac{(\varepsilon\mu)^2}{\frac{16M}{\alpha^2}+\frac{4\varepsilon\mu}{\alpha}}\cdot n\frac{L_1(t)}{m(t)}\Big\} \\ &\le 2\exp\Big\{-\frac{C\varepsilon^2 nL_1(t)}{m(t)}\Big\}. \end{aligned} \tag{2.21}$$

Since there exists a constant c such that $1-\exp(-x) \ge cx$ for $0 \le x \le x_0$, given any $0 < \varepsilon < 1$, it follows from Lemma 2.1 and (2.21) that

$$\begin{aligned} & P\Big\{\Big|\frac{S_{N(t)}(t)}{\mu N(t)Eh(t-U(t))}-1\Big| \ge \varepsilon\Big\} \\ &= \sum_{n=1}^\infty P\Big\{\Big|\sum_{k=1}^n\big(h(t-U_k(t))X_k - Eh(t-U_k(t))X_k\big)\Big| \ge \frac{\varepsilon\mu nL_1(t)}{m(t)}\Big\}\cdot P\{N(t)=n\} \\ &\le 2\sum_{n=1}^\infty \exp\Big\{-\frac{C\varepsilon^2 nL_1(t)}{m(t)}\Big\}\cdot\frac{(m(t))^n}{n!}\cdot\exp\{-m(t)\} \\ &\le 2\exp\Big\{-m(t)\Big(1-\exp\Big(-\frac{C\varepsilon^2L_1(t)}{m(t)}\Big)\Big)\Big\} \le 2\exp\{-C\varepsilon^2L_1(t)\}. \end{aligned} \tag{2.22}$$

Now we choose a nondecreasing sequence $\{t_n, n \ge 1\}$ such that

$$m(t_n) = n. \tag{2.23}$$

Then from condition (1.5), we have

$$L_1(t_n) \ge C(\log n)^{1/r''}. \tag{2.24}$$

Further, noting that $N(t_n)-N(t_{n-1})$ has a Poisson distribution with mean one and $Ee^{\alpha X_k} \le M$ for all $k \ge 1$, we have

$$\begin{aligned} P\Big\{\sum_{k=N(t_{n-1})+1}^{N(t_n)}\frac{X_k}{L_1(t_n)} \ge \varepsilon\Big\} &\le E\exp\Big\{\alpha\sum_{k=N(t_{n-1})+1}^{N(t_n)}X_k\Big\}\cdot\exp\{-\alpha\varepsilon L_1(t_n)\} \\ &\le M\exp\{-C\alpha\varepsilon(\log n)^{1/r''}\}, \end{aligned}$$

which implies

$$\sum_{k=N(t_{n-1})+1}^{N(t_n)}\frac{X_k}{L_1(t_n)} \to 0 \quad \text{a.s.} \quad \text{as } n\to\infty. \tag{2.25}$$

Then we just follow the same procedure as the proof of Theorem 1.1(i) with (2.4), (2.6), (2.15) replaced by (2.22)–(2.25) respectively, and the proof is completed.

Proof of Theorem 1.3 (i) Case $1 < r \le 2$.

By Lemma 2.4, as $t \to \infty$, we have

$$\frac{E|S_{N(t)}(t) - \mu N(t) Eh(t - U(t))|^r}{L_1^r(t)} \le \frac{c_r \mu_r L_r(t)}{L_1^r(t)} \to 0. \tag{2.26}$$

Moreover, we have

$$\begin{aligned}&\frac{E|\mu N(t) Eh(t-U(t)) - \mu m(t) Eh(t-U(t))|^r}{L_1^r(t)}\\ &= \frac{\mu^r E|N(t) - m(t)|^r}{m^r(t)} \le \frac{\mu^r (E|N(t) - m(t)|^2)^{r/2}}{m^r(t)} = \mu^r m^{-r/2}(t) \to 0 \quad \text{as } t \to \infty.\end{aligned} \tag{2.27}$$

By (2.26) and (2.27), c_r-inequality and Lemma 2.2, the proof is completed.

(ii) Case $r > 2$.

By Lemma 2.4 and condition (1.6), as $t \to \infty$, we have

$$\frac{E|S_{N(t)}(t) - \mu N(t) Eh(t - U(t))|^r}{L_1^r(t)} \le \frac{c_r \mu_r (m(t))^{r/2-1} L_r(t)}{L_1^r(t)} \to 0. \tag{2.28}$$

Moreover, by the Hölder inequality, Lemma 2.3 and c_r-inequality, we have

$$\begin{aligned}&\frac{E|\mu N(t) Eh(t-U(t)) - \mu m(t) Eh(t-U(t))|^r}{L_1^r(t)}\\ &= \frac{\mu^r E|N(t) - m(t)|^r}{m^r(t)} \le \frac{\mu^r (E|N(t) - m(t)|^2)^{1/2} (E|N(t) - m(t)|^{2(r-1)})^{1/2}}{m^r(t)}\\ &= c_r \mu^r m^{-1/2}(t) \to 0 \quad \text{as } t \to \infty.\end{aligned} \tag{2.29}$$

By (2.28), (2.29), c_r-inequality and Lemma 2.2, the proof is completed.

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