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Bifurcations of Invariant Tori and Subharmonic Solutions of Singularly Perturbed System***

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Abstract This paper deals with bifurcations of subharmonic solutions and invariant tori generated from limit cycles in the fast dynamics for a nonautonomous singularly perturbed system. Based on Poincaré map, a series of blow-up transformations and the theory of integral manifold, the conditions for the existence of invariant tori are obtained, and the saddle-node bifurcations of subharmonic solutions are studied.

Keywords Singular perturbation, Subharmonic solution, Saddle-Node, Invariant torus

2000 MR Subject Classification 34A26, 34E15, 34C25, 34C29, 34C45

1 Introduction

In the paper, we investigate the periodic orbits and invariant tori of a three dimensional singularly perturbed system of the general form

$$\begin{cases} \dot{x} = f(x,y) + \varepsilon h(t,x,y,\delta,\varepsilon) \in R^2, \\ \dot{y} = \varepsilon g(t,x,y,\delta,\varepsilon) \in R, \quad 0 \le \varepsilon \ll 1, \end{cases} \tag{1.1}$$

where $h(t,x,y,\delta,\varepsilon)$ and $g(t,x,y,\delta,\varepsilon)$ are T-periodic in t. For simplicity, the functions f, g and h are assumed to be sufficiently smooth with respect to the arguments throughout the paper.

There are many three dimensional singularly perturbed models taking such a form. For example, the Van den Pol -Duffing oscillator and the Lorenz model for high Rayleigh numbers can be put into system (1.1) (see [1–3]). Especially, in [4] Wiggins and Holmes called system (1.1) slowly varying oscillator and pointed out that the systems taking the form occur as models of simple nonlinear elastic structures subject to feedback control when there is a nonnegligible time constant in the control process.

The above three dimensional singularly perturbed system with a slow variable has been discussed by many people and various results have been obtained (see [1–7]). For example, in [2, 7], Stiefenhofer studied the bifurcations of a three dimensional autonomous systems with singular Hopf points and Bogdanov points, and obtained the existence of local closed orbits and

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invariant tori near the origin by means of the direct method and Naimark-Sacker bifurcation theorem. In [4], Wiggins and Holmes studied the bifurcations of periodic orbits and subharmonic solutions under the main condition that the fast system $\dot{x} = f(x, y)$ is a planar Hamilitonian system.

In this paper, we suppose that in the fast dynamics $\dot{x} = f(x, y_0)$ of system (1.1) at $\varepsilon = 0$, there exists a hyperbolic periodic orbit L: $x = u(\theta)$, $0 \le \theta \le T_0$. Without loss of generality, let $y_0 = 0$. Applying the methods of the bifurcation theory of periodic orbits and invariant tori used in [8–17] and some new technique, we investigate some global behavior for system (1.1). Using the curvilinear coordinate transformation and the succession function, the sufficient conditions and necessary conditions for a hyperbolic limit cycle in the fast x-dynamics to generate the saddle-node bifurcations of subharmonic solutions are obtained. Finally, the bifurcations of invariant tori are discussed by means of the Floquet theory, the method of averaging and the integral manifold theory (see [8, 10, 13]).

This paper is organized as follows. In Section 2, we discuss the saddle-node bifurcations of subharmonic solutions generated from a hyperbolic limit cycle L in the fast x-dynamics. In Section 3, the invariant torus bifurcations near a hyperbolic limit cycle L in the fast x-dynamics are discussed.

2 Saddle-Node Bifurcation of Subharmonic Solutions

In this section, we assume that the limit cycle L in the fast dynamics $\dot{x} = f(x, 0)$ is hyperbolic. This means that

$$I_0 = \frac{1}{T_0} \oint_{L_0} \mathrm{tr} f_x(x, 0) dt \ne 0. \tag{2.1}$$

Also we suppose that the periods T_0 and T satisfy

$$\frac{T_0}{T} = \frac{m}{k}, \quad (m, k) = 1. \tag{2.2}$$

This means that $\frac{T_0}{T}$ is rational. In order to obtain the saddle-node bifurcations of subharmonic solutions of system (1.1) for sufficiently small $\varepsilon \ne 0$, we perform curvilinear coordinate transformation

$$x = u(\theta) + Z(\theta)r = G(\theta, r), \quad 0 \le \theta \le T_0, \tag{2.3}$$

where $Z(\theta) = (-v_2(\theta), v_1(\theta))^T$, $V(\theta) = \frac{u'(\theta)}{|u'(\theta)|} = \frac{f(u(\theta),0)}{|f(u(\theta),0)|} = (v_1(\theta), v_2(\theta))^T$.

Following the similar arguments to [8, 11], we have the following result.

Lemma 2.1 *The periodic transformation*

$$\begin{cases} x = G(\theta, r), \\ y = y, \end{cases} \qquad 0 \le \theta \le T_0$$

transforms system (1.1) *into the following bi-periodic system:*

$$\begin{aligned} \dot{\theta} &= 1 + f_1(\theta, r) + E(\theta, r)F(t, u + Zr, y, \delta, \varepsilon), \\ \dot{r} &= A(\theta)r + f_2(\theta, r) + Z^T(\theta)F(t, u + Zr, y, \delta, \varepsilon), \\ \dot{y} &= \varepsilon g(t, u(\theta) + Z(\theta)r, y, \delta, \varepsilon), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
A(\theta) &= \mathrm{tr} f_x(u,0) - \frac{d}{d\theta}\ln|f(u,0)|,\\
E(\theta,r) &= (|f(u,0)| + V^T Z'(\theta) r)^{-1} V^T(\theta),\\
f_1(\theta,r) &= E(\theta,r)[f(u+Zr,0) - f(u,0) - Z'(\theta)r],\\
f_2(\theta,r) &= Z^T[f(u+Zr,0) - f(u,0) - f_x(u,0)Zr],\\
F(t,x,y,\delta,\varepsilon) &= f(x,y) - f(x,0) + \varepsilon h(t,x,y,\delta,\varepsilon).
\end{aligned}$$

Performing the scaling transformation $r = \varepsilon\xi,\ y = \varepsilon\eta$, we have

$$\begin{aligned}
\dot\theta &= 1 + \varepsilon s_1(t,\theta,\xi,\eta,\delta) + O(\varepsilon^2),\\
\dot\xi &= A(\theta)\xi + s_2(t,\theta,\xi,\eta,\delta) + O(\varepsilon),\\
\dot\eta &= g(t,u(\theta),0,\delta,0) + \varepsilon s_3(t,\theta,\xi,\eta,\delta) + O(\varepsilon^2),
\end{aligned}\tag{2.5}$$

where

$$\begin{aligned}
s_1(t,\theta,\xi,\eta,\delta) &= |f(u(\theta),0)|^{-1} V^T(\theta)[(f_x(u(\theta),0)Z(\theta) - Z'(\theta))\xi\\
&\quad + f_y(u(\theta),0)\eta + h(t,u(\theta),0,\delta,0)],\\
s_2(t,\theta,\xi,\eta,\delta) &= Z^T(\theta)[f_y(u(\theta),0)\eta + h(t,u(\theta),0,\delta,0)],\\
s_3(t,\theta,\xi,\eta,\delta) &= g_x(t,u(\theta),0,\delta,0)Z(\theta)\xi + g_y(t,u(\theta),0,\delta,0)\eta + g_\varepsilon(t,u(\theta),0,\delta,0).
\end{aligned}$$

From (2.5), we may suppose that the solutions of system (2.5) with the vector (θ_0, ξ_0, η_0) as its initial value have the expansions of the form

$$\begin{aligned}
\theta(t,\delta,\varepsilon,\theta_0,\xi_0,\eta_0) &= \theta_0 + t + \varepsilon\theta_1(t,\delta,\theta_0,\xi_0,\eta_0) + O(\varepsilon^2),\\
\xi(t,\delta,\varepsilon,\theta_0,\xi_0,\eta_0) &= \xi_1(t,\delta,\theta_0,\xi_0,\eta_0) + O(\varepsilon),\\
\eta(t,\delta,\varepsilon,\theta_0,\xi_0,\eta_0) &= \eta_1(t,\delta,\theta_0,\xi_0,\eta_0) + \varepsilon\eta_2(t,\delta,\theta_0,\xi_0,\eta_0) + O(\varepsilon^2),
\end{aligned}\tag{2.6}$$

where

$$\begin{aligned}
&\theta_1(0,\delta,\theta_0,\xi_0,\eta_0) = 0, \quad && \xi_1(0,\delta,\theta_0,\xi_0,\eta_0) = \xi_0,\\
&\eta_1(0,\delta,\theta_0,\xi_0,\eta_0) = \eta_0, \quad && \eta_2(0,\delta,\theta_0,\xi_0,\eta_0) = 0.
\end{aligned}$$

From (2.5)–(2.6), we can obtain

$$\begin{aligned}
\dot\theta_1 &= s_1(t,\theta_0+t,\xi_1,\eta_1,\delta),\\
\dot\xi_1 &= A(\theta_0+t)\xi_1 + s_2(t,\theta_0+t,\xi_1,\eta_1,\delta),\\
\dot\eta_1 &= g(t,u(\theta_0+t),0,\delta,0),\\
\dot\eta_2 &= g_x(t,u(\theta_0+t),0,\delta,0)u'(\theta_0+t)\theta_1 + s_3(t,\theta_0+t,\xi_1,\eta_1,\delta).
\end{aligned}\tag{2.7}$$

Therefore,

$$\begin{aligned}
\theta_1(t,\delta,\theta_0,\xi_0,\eta_0) &= \int_0^t s_1(s,\theta_0+s,\xi_1,\eta_1,\delta)ds,\\
\xi_1(t,\delta,\theta_0,\xi_0,\eta_0) &= \frac{1}{|f(u(\theta_0+t),0)|}\exp\Big\{\int_0^t \mathrm{tr} f_x(u(\theta_0+s),0)ds\Big\}\Big[|f(u(\theta_0),0)|\xi_0
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \int_0^t \exp\Big(-\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big) f(u(\theta_0+s),0)\\
&\qquad\qquad \wedge (f_y(u(\theta_0+s),0)\eta_1 + h(s,u(\theta_0+s),0,\delta,0))ds\Big],\\
&\eta_1(t,\delta,\theta_0,\xi_0,\eta_0) = \eta_0 + \int_0^t g(s,u(\theta_0+s),0,\delta,0)ds,\\
&\eta_2(t,\delta,\theta_0,\xi_0,\eta_0) = \int_0^t [g_x(s,u(\theta_0+s),0,\delta,0)u'(\theta_0+s)\theta_1 + s_3(s,\theta_0+s,\xi_1,\eta_1,\delta)]ds.
\end{aligned} \tag{2.8}
$$

Let $P(\theta_0,\xi_0,\eta_0,\delta,\varepsilon)$ denote Poincaré map, its m-th iteration is

$$
\begin{aligned}
P^m(\theta_0,\xi_0,\eta_0,\delta,\varepsilon) &= (\theta(mT,\delta,\varepsilon,\theta_0,\xi_0,\eta_0),\xi(mT,\delta,\varepsilon,\theta_0,\xi_0,\eta_0),\eta(mT,\delta,\varepsilon,\theta_0,\xi_0,\eta_0))\\
&= (P_1^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0),P_2^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0),P_3^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)).
\end{aligned} \tag{2.9}
$$

From (2.6)–(2.9), we can obtain the following result.

Theorem 2.1 *A necessary condition for the limit cycle L to generate a subharmonic solutions for ε sufficiently small is that there exists a θ_0 $(0 \le \theta_0 \le T_0)$ such that $\int_0^{mT} g(s,u(\theta_0+s),0,\delta,0)ds = 0$.*

Proof From (2.6) and (2.8), if $\int_0^{mT} g(s,u(\theta_0+s),0,\delta,0)ds \neq 0$, then the following inequality

$$P_3^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)) - \eta_0 \neq 0$$

holds for ε sufficiently small. This means that the limit cycle L can not generate a subharmonic solutions for ε sufficiently small.

So in the section we always suppose that $\int_0^{mT} g(s,u(\theta_0+s),0,\delta,0)ds = 0$.

Let $N_0 = e^{kT_0I_0}$. By (2.6) we have

$$P_2^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0) - \xi_0 = (N_0-1)\xi_0 + \frac{N_0R_1(mT,\theta_0)}{|f(u(\theta_0),0)|}\eta_0 + \frac{N_0R_2(mT,\theta_0,\delta)}{|f(u(\theta_0),0)|} + O(\varepsilon)$$

where

$$
\begin{aligned}
R_1(mT,\theta_0) &= \int_0^{mT} \exp\Big(-\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big) f(u(\theta_0+s),0)\wedge f_y(u(\theta_0+s),0)ds,\\
R_2(mT,\theta_0,\delta) &= \int_0^{mT} \exp\Big(-\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big) f(u(\theta_0+s),0)\\
&\quad \wedge \Big(f_y(u(\theta_0+s),0)\int_0^s g(z,u(\theta_0+z),0,\delta,0)dz + h(s,u(\theta_0+s),0,\delta,0)\Big)ds.
\end{aligned}
$$

By means of Implicit Function Theorem, we have

$$\xi_0 = \frac{N_0R_1(mT,\theta_0)}{(1-N_0)|f(u(\theta_0),0)|}\eta_0 + \frac{N_0R_2(mT,\theta_0,\delta)}{(1-N_0)|f(u(\theta_0),0)|} + O(\varepsilon) \doteq \xi_0^*(\theta_0,\eta_0,\delta,\varepsilon). \tag{2.10}$$

Let $Q(t) = |f(u(t),0)|^{-2}V^T(t)(f_x(u(t),0)Z(t) - Z'(t))$. From (2.5), (2.8) and (2.10), a computation yields

$$
\begin{aligned}
\theta_1(mT,\delta,\theta_0,\xi_0^*,\eta_0)|_{\varepsilon=0} &= K_1(mT,\theta_0)\eta_0 + W_1(mT,\delta,\theta_0),\\
\eta_2(mT,\delta,\theta_0,\xi_0^*,\eta_0)|_{\varepsilon=0} &= K_2(mT,\delta,\theta_0)\eta_0 + W_2(mT,\delta,\theta_0),
\end{aligned} \tag{2.11}
$$

where

$$
\begin{aligned}
K_1(mT,\theta_0) &= \int_0^{mT} Q(\theta_0+t)\exp\Big(\int_0^t \mathrm{tr} f_x(u(\theta_0+s),0)ds\Big)dt\Big[\frac{N_0R_1(mT,\theta_0)}{1-N_0}\\
&+\int_0^t \exp\Big(-\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big)f(u(\theta_0+s),0)\wedge f_y(u(\theta_0+s),0)ds\Big]\\
&+\int_0^{mT} |f(u(\theta_0+s),0)|^{-1}V^T(\theta_0+s)f_y(u(\theta_0+s),0)ds,\\
W_1(mT,\delta,\theta_0) &= \int_0^{mT} Q(\theta_0+t)\exp\Big(\int_0^t \mathrm{tr} f_x(u(\theta_0+s),0)ds\Big)dt\Big[\frac{N_0R_2(mT,\theta_0,\delta)}{1-N_0}\\
&+\int_0^t \exp\Big(-\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big)f(u(\theta_0+s),0)\\
&\wedge\Big(f_y(u(\theta_0+s),0)\int_0^s g(z,u(\theta_0+z),0,\delta,0)dz+h(s,u(\theta_0+s),0,\delta,0)\Big)ds\Big]\\
&+\int_0^{mT} |f(u(\theta_0+s),0)|^{-1}V^T(\theta_0+s)\Big[f_y(u(\theta_0+s),0)\\
&\cdot\int_0^s g(z,u(\theta_0+z),0,\delta,0)dz+h(s,u(\theta_0+s),0,\delta,0)\Big]ds,\\
K_2(mT,\delta,\theta_0) &= \int_0^{mT}\Big[g_x(s,u(\theta_0+s),0,\delta,0)u'(\theta_0+s)K_1(s,\theta_0)\\
&+\frac{Z(\theta_0+s)g_x(s,u(\theta_0+s),0,\delta,0)}{|f(u(\theta_0+s),0)|}\exp\Big(\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0\Big)dz\\
&\cdot\Big(\frac{N_0R_1(mT,\theta_0)}{1-N_0}+\int_0^s\exp\Big(-\int_0^z \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big)f(u(\theta_0+z),0)\\
&\wedge f_y(u(\theta_0+z),0)dz\Big)+g_y(s,u(\theta_0+s),0,\delta,0)\Big]ds,\\
W_2(mT,\delta,\theta_0) &= \int_0^{mT}\Big[g_x(s,u(\theta_0+s),0,\delta,0)u'(\theta_0+s)W_1(s,\delta,\theta_0)\\
&+\frac{Z(\theta_0+s)g_x(s,u(\theta_0+s),0,\delta,0)}{|f(u(\theta_0+s),0)|}\exp\Big(\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big)\\
&\cdot\Big(\frac{N_0R_2(mT,\theta_0,\delta)}{1-N_0}+\int_0^s\exp\Big(-\int_0^s \mathrm{tr} f_x(u(\theta_0+z),0)dz\Big)f(u(\theta_0+z),0)\\
&\wedge\Big(f_y(u(\theta_0+z),0)\int_0^z g(s,u(\theta_0+s),0,\delta,0)ds+h(z,u(\theta_0+z),0,\delta,0)\Big)dz\Big)\\
&+g_y(s,u(\theta_0+s),0,\delta,0)\int_0^s g(z,u(\theta_0+z),0,\delta,0)dz\\
&+g_\varepsilon(s,u(\theta_0+s),0,\delta,0)\Big]ds.
\end{aligned}
$$

Let

$$Q_i(mT,,\theta_0,\delta,)=\frac{W_2K_1-W_1K_2}{K_i},\quad i=1,2.$$

For the simplicity of notations, let

(H_1)

$$\int_0^{mT} g(s,u(\overline{\theta}_0+s),0,\delta_0,0)ds=0,$$

$$K_1(mT,\overline{\theta}_0)\overline{\eta}_0+W_1(mT,\delta_0,\overline{\theta}_0)=0,$$
$$K_2(mT,\delta_0,\overline{\theta}_0)\overline{\eta}_0+W_2(mT,\delta_0,\overline{\theta}_0)=0;$$

(H_2)

$$\begin{vmatrix} W'_{1\theta}(mT,\delta,\overline{\theta}_0)+K'_{1\theta}(mT,\overline{\theta}_0)\overline{\eta}_0 & K_1(mT,\overline{\theta}_0) \\ W'_{2\theta}(mT,\delta,\overline{\theta}_0)+K'_{2\theta}(mT,\delta,\overline{\theta}_0)\overline{\eta}_0 & K_2(mT,\delta,\overline{\theta}_0) \end{vmatrix}\neq 0;$$

(H_3)

$$K_i(mT,\overline{\theta}_0,\delta_0)\neq 0,\quad Q_{i\theta_0}(mT,\overline{\theta}_0,\delta_0)=0,$$
$$Q_{i\theta_0\theta_0}(mT,\overline{\theta}_0,\delta_0)\neq 0,\quad Q_{i\delta}(mT,\overline{\theta}_0,\delta_0)\neq 0.$$

Now we can obtain the following main result in the section.

Theorem 2.2 *Suppose* (2.1) *and* (2.2) *hold. If there exist* $\overline{\theta}_0\in[0,\ T_0)$, $\overline{\eta}_0$ *and* δ_0 *such that*

(i) (H_1) *and* (H_2) *hold, then system* (1.1) *has a subharmonic solution near* L *for* ε *sufficiently small;*

(ii) (H_1) *and* (H_3) *hold, then system* (1.1) *has a saddle-node bifurcation curve* $\delta=\delta_0+O(\varepsilon)$ *of subharmonic solutions for* ε *sufficiently small.*

Proof Using (2.9) and (2.11), we have

$$\begin{aligned} P_1^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)&=\theta_0+mT+\varepsilon[K_1(mT,\theta_0)\eta_0+W_1(mT,\delta,\theta_0)+O(\varepsilon)],\\ P_3^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)&=\eta_0+\varepsilon[K_2(mT,\delta,\theta_0)\eta_0+W_2(mT,\delta,\theta_0)+O(\varepsilon)]. \end{aligned}\tag{2.12}$$

In the following, we discuss (i) and (ii) separately.

(i) By means of Implicit Function Theorem, there exist functions $\theta_0=\overline{\theta}_0+O(\varepsilon)$ and $\eta_0=\overline{\eta}_0+O(\varepsilon)$ such that

$$P_1^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)=\theta_0+mT,\quad P_3^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)=\eta_0.$$

Substituting $\theta_0=\overline{\theta}_0+O(\varepsilon)$ and $\eta_0=\overline{\eta}_0+O(\varepsilon)$ into (2.10), we have

$$P_2^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)=\xi_0.$$

This means that system (1.1) has a subharmonic solution near L for ε sufficiently small.

(ii) Considering the case $K_1(mT,\overline{\theta}_0)\neq 0$, following the similar arguments to (2.10), we have

$$\eta_0=-\frac{W_1(mT,\delta,\theta_0)}{K_1(mT,\theta_0)}+O(\varepsilon)\tag{2.13}$$

such that $P_1^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)=\theta_0+mT$. Substituting (2.13) into the second equation of system (2.12) yields

$$P_3^m(\delta,\varepsilon,\theta_0,\xi_0,\eta_0)=\eta_0+\varepsilon[Q_1(mT,,\theta_0,\delta)+O(\varepsilon)]\doteq\eta_0+\varepsilon\widetilde{F}(\theta_0,\delta,\varepsilon).$$

Because (H_1) holds, it is not difficult to prove that

$$Q_1(mT,,\overline{\theta}_0,\delta_0)=0.$$

From (H_3), there exists a function $\theta_0 = a(\delta, \varepsilon)$ such that

$$\frac{\partial \widetilde{F}}{\partial \theta_0}(a(\delta, \varepsilon), \delta, \varepsilon) = 0.$$

The Taylor expansion of the function $\widetilde{F}(\theta_0, \delta, \varepsilon)$ at $\theta_0 = a(\delta, \varepsilon)$ is

$$\widetilde{F}(\theta_0, \delta, \varepsilon) = b_1(\delta, \varepsilon) + b_2(\delta, \varepsilon)(\theta_0 - a(\delta, \varepsilon))^2(1 + o(1)), \tag{2.14}$$

where

$$\begin{aligned} b_1(\delta, \varepsilon) &= \widetilde{F}(a(\delta, \varepsilon), \delta, \varepsilon) = Q_{1\delta}(mT, \overline{\theta}_0, \delta_0)(\delta - \delta_0) + O(\varepsilon), \\ b_2(\delta, \varepsilon) &= \frac{1}{2}\frac{\partial^2 \widetilde{F}}{\partial \theta_0^2}(a(\delta, \varepsilon), \delta, \varepsilon) = \frac{1}{2}Q_{1\theta_0\theta_0}(mT, \overline{\theta}_0, \delta_0) + O(\epsilon, \delta - \delta_0). \end{aligned}$$

So we obtian that if

$$b_1(\delta, \varepsilon)Q_{1\theta_0\theta_0}(mT, \overline{\theta}_0, \delta_0) > (=, <)0,$$

system (2.14) has no (a, two) zero root(s) for $\delta - \delta_0$ and ε sufficienly small. Because $Q_{1\delta}(mT, \overline{\theta}_0, \delta_0) \neq 0$, there exists a curve $\delta = \delta(\varepsilon) = \delta_0 + O(\varepsilon)$ such that $b_1(\delta(\varepsilon), \varepsilon) = 0$. This means that the curve $\delta = \delta_0 + O(\varepsilon)$ is a saddle-node bifurcation curve of subharmonic solutions of system (1.1) for ε sufficiently small.

For example, we consider the three dimensional singularly perturbed system under the periodic perturbation of the form

$$\begin{aligned} \dot{x_1} &= \frac{1}{m}x_2 + x_1(x_1^2 + x_2^2 - 1)(y + 1) + \varepsilon x_2(x_1^m \cos t + \delta), \\ \dot{x_2} &= -\frac{1}{m}x_1 + x_2(x_1^2 + x_2^2 - 1)(y + 1) - \varepsilon x_1(x_1^m \cos t + \delta), \\ \dot{y} &= \varepsilon(x_1^2 + x_2^2 + y - 1), \end{aligned} \tag{2.15}$$

where m is a positive integer. If $y = 0$ and $\varepsilon = 0$, the fast system

$$\begin{aligned} \dot{x_1} &= \frac{1}{m}x_2 + x_1(x_1^2 + x_2^2 - 1), \\ \dot{x_2} &= -\frac{1}{m}x_1 + x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

has a hyperbolic limit cycle

$$L_0 : (x_1, x_2) = \Big(\sin\frac{t}{m}, \cos\frac{t}{m}\Big), \quad 0 \le t \le 2m\pi = T_0.$$

Obviously,

$$\int_0^{mT} g(s, u(\theta_0 + s), 0, \delta, 0)ds = 0.$$

From (2.11), we have

$$\begin{aligned} &K_1(2m\pi, \theta_0) = 0, \quad K_2(2m\pi, \delta, \theta_0) = 2m\pi, \quad W_2 = 0, \\ &W_1(2m\pi, \delta, \theta_0) = 2\pi m^2\delta + m\int_0^{2m\pi} \sin^m\frac{\theta_0 + t}{m}\cos t\, dt. \end{aligned} \tag{2.16}$$

We only investigate the periodic orbits of the system (2.15) in the case $m = 1$ and $m = 2$. If $m = 1$ or $m = 2$, then

$$W_1(2m\pi, \delta, \theta_0) = \begin{cases} \pi(2\delta + \sin\theta_0), & m = 1, \\ 8\pi\delta - 2\pi\cos\theta_0, & m = 2. \end{cases}$$

From (2.16), the system

$$\begin{aligned} K_1(mT, \overline{\theta}_0)\overline{\eta}_0 + W_1(mT, \delta_0, \overline{\theta}_0) = 0, \\ K_2(mT, \delta_0, \overline{\theta}_0)\overline{\eta}_0 + W_2(mT, \delta_0, \overline{\theta}_0) = 0 \end{aligned}$$

is equivalent to the system

$$\begin{aligned} &\overline{\eta}_0 = 0, \\ &W_1(2m\pi, \delta_0, \overline{\theta}_0) = 0. \end{aligned}$$

This means that if there exist δ_0 and $\overline{\theta}_0$ such that $W_1(2m\pi, \delta_0, \overline{\theta}_0) = 0$, then hypothesis ($\mathrm{H}_1$) holds.

Due to

$$Q_2(2m\pi, \delta, \theta_0) = \frac{W_2K_1 - W_1K_2}{K_2} = -W_1(2m\pi, \delta, \theta_0),$$

we have

$$\begin{aligned} &Q_{2\delta}(2m\pi, \delta, \theta_0) \neq 0, \\ &Q_{2\theta_0}(2m\pi, \delta, \theta_0) = \begin{cases} -\pi\cos\theta_0, & m = 1, \\ -2\pi\sin\theta_0, & m = 2. \end{cases} \end{aligned}$$

If $m = 1$, by Theorem 2.1, there exist two saddle-node bifurcation curves of harmonic solutions, i.e.,

$$\delta = \delta_i(\varepsilon) = \frac{1}{2}(-1)^i + O(\varepsilon), \quad i = 1, 2,$$

such that system (2.15) has two harmonic solutions if $\delta_1(\varepsilon) < \delta < \delta_2(\varepsilon)$, one harmonic solution if $\delta = \delta_i(\varepsilon)$, and no harmonic solution if $\delta > \delta_2(\varepsilon)$ or $\delta < \delta_1(\varepsilon)$ near L_0 for small $\varepsilon \neq 0$.

Similarly, if $m = 2$, there exist four saddle-node bifurcation curves of subharmonic solutions, i.e.,

$$\delta = \delta_{ij}(\varepsilon) = \frac{1}{4}(-1)^i + O(\varepsilon), \quad i, j = 1, 2.$$

The subharmonic solutions of saddle-node type are

$$\begin{aligned} x_{1ij}(t, \varepsilon) &= \sin\frac{\theta_{ij}}{2} + O(\varepsilon), \\ x_{2ij}(t, \varepsilon) &= \cos\frac{\theta_{ij}}{2} + O(\varepsilon), \\ y_{ij} &= O(\varepsilon). \end{aligned}$$

Noting

$$\theta_{1j}(t, \varepsilon) = (2 + (-1)^j)\pi + t, \quad \theta_{2j}(t, \varepsilon) = (1 + (-1)^j)\pi + t,$$

we have

$$\theta_{i2} - \theta_{i1} = 2\pi, \quad i = 1, 2.$$

Therefore,

$$\Big(\sin\frac{\theta_{i1}}{2},\cos\frac{\theta_{i1}}{2}\Big)=-\Big(\sin\frac{\theta_{i2}}{2},\cos\frac{\theta_{i2}}{2}\Big).$$

System (2.15) is symmetric to the plane: $\begin{cases} x_1=0 \\ y=0 \end{cases}$ and the plane: $\begin{cases} x_2=0 \\ y=0 \end{cases}$. Therefore, we have

$$(x_{1i1},x_{2i1},y_{i1})=(-x_{1i2},-x_{2i2},y_{i2}).$$

This means that $\delta_{i1}(\varepsilon)=\delta_{i2}(\varepsilon)=\delta_i$. Thus, if $\delta_1(\varepsilon)<\delta<\delta_2(\varepsilon)$, system (2.15) has four subharmonic solutions. If $\delta<\delta_1(\varepsilon)$ or $\delta>\delta_2(\varepsilon)$, system (2.15) has no subharmonic solution for small $\varepsilon\neq 0$.

3 Invariant Torus Bifurcations

For simplicity, in this section we discuss invariant torus bifurcations of the the following system:

$$\begin{cases}\dot{x}=f(x,y)+\varepsilon h(t,x,y,\varepsilon)\in R^2,\\ \dot{y}=\varepsilon g(t,x,y,\varepsilon)\in R,\quad 0\le\varepsilon\ll 1,\end{cases}\tag{3.1}$$

where $h(t,x,y,\varepsilon)$ and $g(t,x,y,\varepsilon)\in R$ are T-periodic in t. In fact, system (3.1) is the form of system (1.1) without the parameter δ. In this section, we still assume that the fast system $\dot{x}=f(x,0)$ of system (3.1) has a hyperbolic limit cycle $L_0:\ x=u(\theta)\ (0\le\theta\le T_0)$. Different from Section 2, we suppose that noresonant condition holds, i.e., $\frac{T_0}{T}$ is irrational.

Performing transformation (2.3), we have the following result.

Lemma 3.1 *Under transformation* (2.3)*, system* (3.1) *can be written as*

$$\begin{aligned}\frac{dt}{d\theta}&=1+c_1(\theta)r+c_2(\theta)y+O(|r^2|+|ry|+|y^2|+|\varepsilon|),\\ \frac{dr}{d\theta}&=A(\theta)r+a(\theta)y+O(|r^2|+|ry|+|y^2|+|\varepsilon|),\\ \frac{dy}{d\theta}&=\varepsilon[g(t,u(\theta),0,0)+b_1(t,\theta)r+b_2(t,\theta)y+O(|r^2|+|ry|+|y^2|+|\varepsilon|)],\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}c_1(\theta)&=-|f(u(\theta),0)|^{-1}v^T(\theta)(f_x(u(\theta),0)Z(\theta)-Z'(\theta)),\\ c_2(\theta)&=-|f(u(\theta),0)|^{-1}v^T(\theta)f_y(u(\theta),0),\\ a(\theta)&=Z^T(\theta)f_y(u(\theta),0),\\ b_1(t,\theta)&=g_x(t,u(\theta),0,0)Z(\theta)-E(\theta,0)(f_x(u(\theta),0)Z(\theta)-Z'(\theta))g(t,u(\theta),0,0),\\ b_2(t,\theta)&=g_y(t,u(\theta),0,0)-E(\theta,0)f_y(u(\theta),0)g(t,u(\theta),0,0).\end{aligned}$$

Letting $y\to\varepsilon^{\frac{1}{2}}y$ and $r\to\varepsilon^{\frac{1}{2}}r$, we see that this blow-up transformation drives system (3.2) into the following bi-periodic system:

$$\begin{aligned}\frac{dt}{d\theta}&=1+(c_1(\theta)r+c_2(\theta)y)\varepsilon^{\frac{1}{2}}+O(\varepsilon),\\ \frac{dr}{d\theta}&=A(\theta)r+a(\theta)y+O(\varepsilon^{\frac{1}{2}}),\\ \frac{dy}{d\theta}&=\varepsilon^{\frac{1}{2}}[g(t,u(\theta),0,0)+(b_1(t,\theta)r+b_2(t,\theta)y)\varepsilon^{\frac{1}{2}}+O(\varepsilon)].\end{aligned}\tag{3.3}$$

We often use the method of averaging and the theory of integral manifold to obtain the existence of invariant tori. Noting that system (3.3) is not in a standard form for the method of averaging, so we can not apply the method of averaing directly. Following the Floquet theory, we obtain the periodic transformation

$$\begin{aligned} r &= e^{\int_0^\theta A(s)ds - \frac{\theta}{T_0}\int_0^{T_0} A(s)ds}\xi + \Big(ke^{-\frac{\theta}{T_0}\int_0^{T_0} A(s)ds} - k + \int_0^\theta e^{-\int_0^s A(s)ds}a(s)ds\Big)e^{\int_0^\theta A(s)ds}\eta, \\ y &= \eta, \end{aligned} \tag{3.4}$$

where $k = (e^{\int_0^{T_0} A(s)ds} - 1)^{-1}e^{\int_0^{T_0} A(s)ds}\int_0^{T_0} e^{-\int_0^s A(s)ds}a(s)ds$. Transformation (3.4) carries the system

$$\frac{dr}{d\theta} = A(\theta)r + a(\theta)y,$$
$$\frac{dy}{d\theta} = 0$$

into the system

$$\frac{d\xi}{d\theta} = \frac{1}{T_0}\int_0^{T_0} A(s)ds\xi + \frac{k}{T_0}\int_0^{T_0} A(s)ds\eta,$$
$$\frac{d\eta}{d\theta} = 0.$$

From the above, it is not difficult to prove Lemma 3.2.

Lemma 3.2 *Under transformation* (3.4), *system* (3.3) *can be carried into the following bi-periodic system:*

$$\begin{aligned} \frac{dt}{d\theta} &= 1 + (d_1(\theta)\xi + d_2(\theta)\eta)\varepsilon^{\frac{1}{2}} + O(\varepsilon), \\ \frac{d\xi}{d\theta} &= \frac{1}{T_0}\int_0^{T_0} A(s)ds\xi + \frac{k}{T_0}\int_0^{T_0} A(s)ds\eta + O(\varepsilon^{\frac{1}{2}}), \\ \frac{d\eta}{d\theta} &= \varepsilon^{\frac{1}{2}}[g(t,u(\theta),0,0) + (d_3(t,\theta)\xi + d_4(t,\theta)\eta)\varepsilon^{\frac{1}{2}} + O(\varepsilon)], \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} d_1(\theta) &= c_1(\theta)e^{\int_0^\theta A(s)ds - \frac{\theta}{T_0}\int_0^{T_0} A(s)ds}, \\ d_2(\theta) &= c_1(\theta)\Big(ke^{-\frac{\theta}{T_0}\int_0^{T_0} A(s)ds} - k + \int_0^\theta e^{-\int_0^s A(s)ds}a(s)ds\Big)e^{\int_0^\theta A(s)ds} + c_2(\theta), \\ d_3(t,\theta) &= b_1(t,\theta)e^{\int_0^\theta A(s)ds - \frac{\theta}{T_0}\int_0^{T_0} A(s)ds}, \\ d_4(t,\theta) &= b_1(t,\theta)\Big(ke^{-\frac{\theta}{T_0}\int_0^{T_0} A(s)ds} - k + \int_0^\theta e^{-\int_0^s A(s)ds}a(s)ds\Big)e^{\int_0^\theta A(s)ds} + b_2(t,\theta). \end{aligned}$$

In order to obtain invariant tori of system (3.1) near L_0, we apply the idea of the averaging method to the first and third equation of the system (3.5). Letting

$$\begin{aligned} \mu &= \varepsilon^{\frac{1}{2}}, \\ \overline{d}_1 &= \frac{1}{T_0}\int_0^{T_0} d_1(\theta)d\theta, \\ B(\theta,\xi,\eta) &= \int_0^\theta [(d_1(\theta) - m(\theta)\overline{A})\xi + (d_2(\theta) - \overline{A}km(\theta))\eta]d\theta - (\overline{d}_1 - \overline{m}\overline{A})\xi\theta - (\overline{d}_2 - \overline{A}k\overline{m})\eta\theta, \end{aligned}$$

where

$$m(\theta) = e^{-\overline{A}\theta} \int_0^\theta e^{\overline{A}s} (d_1(s) - \overline{d}_1 + \overline{m}\overline{A}) ds,$$
$$\overline{m} = \overline{A}(1 - e^{-\overline{A}T_0})^{-1} \int_0^{T_0} e^{-\overline{A}\theta} d\theta \int_0^\theta e^{\overline{A}s} (d_1(s) - \overline{d}_1) ds,$$

we have the following lemma.

Lemma 3.3 *Suppose that the nonresonant condition holds.*

(i) *If system* (3.1) *has invariant tori near* L_0 *for sufficiently small* ε, *then*

$$\int_0^{T_0} \int_0^T g(t, u(\theta), 0, 0) dt d\theta = 0.$$

(ii) *If* (1) $\overline{d}_1 = \overline{m}\overline{A}$, $\int_0^{T_0} \int_0^T g(t, u(\theta), 0, 0) dt d\theta = 0$ *and* $d_3(t, \theta) = 0$; (2) *the functions* $g(t, u(\theta), 0, 0)$, $g_t(t, u(\theta), 0, 0) B(\theta, \xi, \eta)$, *and* $d_4(t, \theta)$ *are trigonometric polynomials in* $\frac{2\pi}{T} t$ *and* $\frac{2\pi}{T_0} \theta$, *then there exist smooth transformations* $t = t(\phi, \theta, \rho, \mu)$ *and* $\eta = \eta(\phi, \theta, \rho, \mu)$ *such that system* (3.5) *can be carried into the following bi-periodic system:*

$$\begin{aligned} \frac{d\phi}{d\theta} &= 1 + \mu[(\overline{d}_2 - \overline{A}k\overline{m})\rho + O(\mu)], \\ \frac{d\xi}{d\theta} &= \overline{A}\xi + \overline{A}k\rho + O(\mu), \\ \frac{d\rho}{d\theta} &= \mu^2[\overline{d}_4 \rho + O(\mu)], \end{aligned} \tag{3.6}$$

where

$$\overline{d}_2 = \frac{1}{T_0} \int_0^{T_0} d_2(\theta) d\theta,$$
$$\overline{A} = \frac{1}{T_0} \int_0^{T_0} A(\theta) d\theta,$$
$$\overline{d}_4 = \frac{1}{T_0 T} \int_0^T \int_0^{T_0} d_4(\phi, \theta) d\phi d\theta.$$

Proof Following the averaging method, we look for the following transformation

$$\begin{aligned} t &= U(\phi, \theta, \xi, \rho, \mu) = \phi + u_1(\theta, \xi, \rho)\mu + O(\mu), \\ \eta &= V(\phi, \theta, \xi, \rho, \mu) = \rho + v_1(\phi, \theta)\mu + v_2(\phi, \theta, \xi, \rho)\mu^2 + O(\mu^3), \end{aligned} \tag{3.7}$$

which drives system (3.5) into the system of the form

$$\begin{aligned} \dot{\phi} &= 1 + \mu P(\phi, \theta, \xi, \rho, \mu), \\ \dot{\xi} &= Q_1(\phi, \theta, \xi, \rho, \mu), \\ \dot{\rho} &= \mu Q_2(\phi, \theta, \xi, \rho, \mu), \end{aligned} \tag{3.8}$$

where

$$P(\phi, \theta, \xi, \rho, \mu) = P_0(\xi, \rho) + O(\mu),$$
$$Q_1(\phi, \theta, \xi, \rho, \mu) = Q_{10}(\xi, \rho) + O(\mu),$$
$$Q_2(\phi, \theta, \xi, \rho, \mu) = Q_{20} + Q_{21}(\xi, \rho)\mu + O(\mu).$$

In fact, this is the method of averaging. It is clear that $Q_{10}(\xi,\rho)=\overline{A}\xi+\overline{A}k\rho$. From (3.5), (3.7) and (3.8), using the asymptomatic expansions and comparing the coefficients of terms with same order, we have

$$d_1(\theta)\xi+d_2(\theta)\rho=P_0(\xi,\rho)+\frac{\partial u_1}{\partial\theta}+\frac{\partial u_1}{\partial\xi}(\overline{A}\xi+\overline{A}k\rho). \tag{3.9}$$

If $P_0(\xi,\rho)=(\overline{d}_1-\overline{m}\overline{A})\xi+(\overline{d}_2-\overline{A}k\overline{m})\rho$, since the function $m(\theta)$ satisfies

$$\dot{m}(\theta)=-Am(\theta)+d_1(\theta)-\overline{d}_1+\overline{m}\overline{A},$$

a computation yields that the periodic function $u_1(\theta,\xi,\rho)=B(\theta,\xi,\rho)$ is a solution of equation (3.9). Similar to (3.9), we have

$$\frac{\partial v_1}{\partial\theta}+\frac{\partial v_1}{\partial\phi}=g(\phi,u(\theta),0,0)-Q_{20}. \tag{3.10}$$

From the theory of almost periodic function, noticing that $g(t,u(\theta),0,0)$ are trigonometric polynomial in $\frac{2\pi}{T}t$ and $\frac{2\pi}{T_0}\theta$, using Lemmas 4.1 and 4.2 of Chapter 12 in [8], we obtain that equation (3.10) has a bi-periodic solution and

$$Q_{20}=\frac{1}{T_0T}\int_0^{T_0}\int_0^{T}g(\phi,u(\theta),0,0)d\phi\, d\theta. \tag{3.11}$$

If $Q_{20}=\frac{1}{T_0T}\int_0^{T_0}\int_0^{T}g(\phi,u(\theta),0,0)d\phi\, d\theta\neq 0$, it is obvious that system (3.8) has no invariant torus near L. This means that system (3.1) has no invariant torus near L for small $\varepsilon\neq 0$. If we only prove that Lemma 3.3(i) holds, the condition that the function $g(t,u(\theta),0,0)$ are trigonometric polynomial in $\frac{2\pi}{T}t$ and $\frac{2\pi}{T_0}\theta$ is not necessary.

Under the conditions of Lemma 3.3(ii), from (3.5), (3.7) and (3.8), we may assume that $v_2(\phi,\theta,\xi,\rho)=v_2(\phi,\theta,\rho)$. Similar to (3.9), we have

$$\frac{\partial v_2}{\partial\theta}+\frac{\partial v_2}{\partial\phi}=D(\phi,\theta)-Q_{21}, \tag{3.12}$$

where $D(\phi,\theta)=g_t(\phi,u(\theta),0,0)u_1+d_4(\phi,\theta)\rho-\frac{\partial v_1}{\partial\phi}P_0$. It is clear that the function $D(\phi,\theta)$ is trigonometric polynomial in $\frac{2\pi}{T}\phi$ and $\frac{2\pi}{T_0}\theta$. By Lemmas 4.1 and 4.2 in [8] again, it is not difficult to see that equation (3.12) has a bi-periodic solution. Noticing that $\int_0^{T_0}\int_0^{T}g_t(\phi,u(\theta),0,0)d\phi d\theta$ $=0$ and $\int_0^{T_0}\int_0^{T}\frac{\partial v_1}{\partial\phi}d\phi d\theta=0$, we have

$$Q_{21}=\frac{1}{T_0T}\int_0^{T_0}\int_0^{T}d_4(\phi,\theta)d\phi d\theta. \tag{3.13}$$

This completes the proof of Lemma 3.3.

Theorem 3.1 *Assume that the hypotheses of Lemma* 3.3 *hold. If* $\overline{A}\,\overline{d}_4\neq 0$*, then system* (3.1) *has invariant tori near* L_0 *for sufficiently small* $\varepsilon\neq 0$*.*

Proof The matrix

$$\begin{pmatrix}\overline{A} & \overline{A}k\\ 0 & \mu^2\overline{d}_4\end{pmatrix}$$

has two eigenvalues $\lambda_1 = \overline{A}$ and $\lambda_2 = \varepsilon\overline{d}_4$. When $\overline{A}\overline{d}_4 \neq 0$, it is obvious that $\lambda_1 \neq \lambda_2$ for small $\varepsilon \neq 0$. Therefore, there exists a matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{23} \end{pmatrix},$$

such that

$$\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{23} \end{pmatrix}^{-1} \begin{pmatrix} \overline{A} & \overline{A}k \\ 0 & \mu^2\overline{d}_4 \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{23} \end{pmatrix} = \begin{pmatrix} \overline{A} & 0 \\ 0 & \mu^2\overline{d}_4 \end{pmatrix}. \tag{3.14}$$

In fact,

$$M = \begin{pmatrix} 1 & \dfrac{-\overline{A}k}{\overline{\overline{A} - \mu^2\overline{d_4}}} \\ 0 & 1 \end{pmatrix}.$$

Perform the linear transformation

$$\begin{aligned} \xi &= \zeta + \frac{-\overline{A}k}{\overline{A} - \mu^2\overline{d}_4}\tau, \\ \rho &= \tau. \end{aligned} \tag{3.15}$$

Therefore, system (3.6) can be written as

$$\begin{aligned} \frac{d\phi}{d\theta} &= 1 + h_1(\phi, \theta, \zeta, \tau, \mu), \\ \frac{d\zeta}{d\theta} &= \overline{A}\zeta + h_2(\phi, \theta, \zeta, \tau, \mu), \\ \frac{d\tau}{d\theta} &= \mu^2[\overline{d}_4\tau + h_3(\phi, \theta, \zeta, \tau, \mu)], \end{aligned} \tag{3.16}$$

where $h_i(\phi, \theta, \zeta, \tau, \mu)$ $(i = 1, 2, 3)$ are bi-periodic in $\frac{2\pi}{T_0}\theta$ and$\frac{2\pi}{T}t$, and $h_i(\phi, \theta, \zeta, \tau, \mu) = O(\mu)$. By [13, Theorem 2.1], system (3.16) has an invariant torus near L_0. Thus we completes the proof of Theorem 3.1.

For example, we consider the following example:

$$\begin{aligned} \dot{x_1} &= x_2 + x_1(x_1^2 + x_2^2 - 1)(y + 1) + \varepsilon x_2(x_1 \cos \pi t + \delta), \\ \dot{x_2} &= -x_1 + x_2(x_1^2 + x_2^2 - 1)(y + 1) - \varepsilon x_1(x_1 \cos \pi t + \delta), \\ \dot{y} &= \varepsilon(-3x_1 - 3x_2 + y + x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3). \end{aligned} \tag{3.17}$$

Similar to (2.15), if $y = 0$ and $\varepsilon = 0$, the fast system

$$\begin{aligned} \dot{x_1} &= x_2 + x_1(x_1^2 + x_2^2 - 1), \\ \dot{x_2} &= -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

has a hyperbolic limit cycle $L_0 : x_1 = \sin\theta,\ x_2 = \cos\theta\ (0 \le t \le 2\pi)$. Because of $\frac{T_0}{T} = \pi$, the noresonant condition are valid. First,

$$\begin{aligned} &\int_0^2 \int_0^{2\pi} g(t, u(\theta), 0, 0)dtd\theta \\ = {} & 2\int_0^{2\pi} (-3\sin\theta - 3\cos\theta + \sin^3\theta + sin^2\theta\cos\theta + \sin\theta\cos^2\theta + \cos^3\theta) = 0, \end{aligned}$$

this means the necessary condition for the existence of invariant tori of system (3.17) holds. Noting that $Z(\theta) = (\sin\theta, \cos\theta)^T$, and $v(\theta) = (\cos\theta, -\sin\theta)^T$, from (3.2), we have

$$\begin{aligned} &c_1(\theta) = 0, \quad c_2(\theta) = 0, \quad a(\theta) = 0, \\ &b_1(t,\theta) = 0, \quad b_2(t,\theta) = 1. \end{aligned} \tag{3.18}$$

Substituting (3.18) into (3.5) yields

$$d_3(t,\theta) = 0, \quad d_4(t,\theta) = 1. \tag{3.19}$$

Since $d_1(\theta) = 0$, it is clear that $\overline{m} = 0$. Therefore, $\overline{d}_1 = \overline{m}\overline{A}$. This means that the hypothesis of Lemma 3.4 holds. Noting that $\overline{A} = 2$ and $\overline{d}_4 = 1$, by Theorem 3.1, we see that system (3.17) has an invariant torus for small $\varepsilon \neq 0$.

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