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A New Approach to Synchronization Analysis of Linearly Coupled Map Lattices***

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Abstract In this paper, a new approach to analyze synchronization of linearly coupled map lattices (LCMLs) is presented. A reference vector $\widehat{x}(t)$ is introduced as the projection of the trajectory of the coupled system on the synchronization manifold. The stability analysis of the synchronization manifold can be regarded as investigating the difference between the trajectory and the projection. By this method, some criteria are given for both local and global synchronization. These criteria indicate that the left and right eigenvectors corresponding to the eigenvalue "0" of the coupling matrix play key roles in the stability of synchronization manifold for the coupled system. Moreover, it is revealed that the stability of synchronization manifold for the coupled system is different from the stability for dynamical system in usual sense. That is, the solution of the coupled system does not converge to a certain knowable $s(t)$ satisfying $s(t+1) = f(s(t))$ but to the reference vector on the synchronization manifold, which in fact is a certain weighted average of each $x^i(t)$ for $i = 1, \cdots, m$, but not a solution $s(t)$ satisfying $s(t+1) = f(s(t))$.

Keywords Linearly coupled map lattices, Synchronization, Synchronization manifold, Local stability of synchronization manifold, Global stability of synchronization manifold
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1 Introduction

Word "synchronization" comes from Greek, which means "share time". Today, in science and technology, it comes to be considered as "time coherence of different processes". This phenomenon also appears in wide range of real systems, such as in biology (see [1]), neural networks (see [2]), physiological process (see [3]) and others. In applications, there are many kinds of concepts of synchronization. For example, phase synchronization, imperfect synchronization, lag synchronization, and almost synchronization etc. In this paper, we consider complete synchronization, which can be described as $\lim\limits_{t\to\infty} x^i(t) - x^j(t) = 0$, where $x^i(t)$ denotes the state variable of node i, for all $i = 1, \cdots, m$.

Synchronization technique is applied to many fields, such as communication, seismology, and neural networks. In [4] and other papers, authors presented transmitter-receiver discrete-time systems with chaos synchronization for communication purpose. In this cryptographic

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scheme, the message is masked by some chaotic signal, which is transmitted to a receiver system. With synchronization between transmitter and receiver, the message can be recovered. In [5], authors observed existence of synchronized chaos in three-block Burridge-Knopoff model for earthquakes. By this technique, the dimensionality of the chaotic attractor decreases. It makes the analysis of the system much easier. In [6], we proposed a new model to recognize image by synchronization. As we showed that this method has strong robustness in recognition. It is clear that for applications in various research fields, theoretical analysis of synchronization is an important and necessary step.

Linearly coupled map lattices (LCMLs) is a large class of dynamical systems with discrete space and time, as well as continuous state. This class of dynamical systems has been investigated as theoretical models of spatiotemporal phenomena in a variety of problems in nonlinear systems and computation studies (for example, see [11, 12]). Coupled oscillator and chaotic systems were studied in [7–9]. In general, the coupled system can be described as

$$x^i(t+1) = f(x^i(t)) + \sum_{j=1}^{m} b_{ij} f(x^j(t)), \quad i = 1, \cdots, m, \tag{1.1}$$

where $x^i(t) = (x_1^i(t), x_2^i(t), \cdots, x_n^i(t))^\top \in R^n$ is the state variable of the i-th node, $t \in N$ is the discrete time, $f : R^n \to R^n$ is a continuous map, $B = (b_{ij}) \in R^{m,m}$ is the coupling matrix connecting the lattices, which is determined by the topological structure of the LCMLs and its entries satisfy $b_{ij} \geq 0$, for all $i \neq j$, and $\sum_{j=1}^{m} b_{ij} = 0$.

In many studies, the coupling scheme is assumed as:

$$x^i(t+1) = f(x^i(t)) + \frac{\varepsilon}{k_i} \sum_{j=1}^{m} a_{ij} f(x^j(t)), \quad i = 1, \cdots, m, \tag{1.2}$$

where $a_{ij} = a_{ji} = 1$, if there is a connection between node i and j; otherwise, $a_{ij} = a_{ji} = 0$, $k_i = \sum_{j \neq i} a_{ij}$ is the number of connection incidents of node i, $\varepsilon > 0$ is the coupling strength. Or more general

$$x^i(t+1) = f(x^i(t)) + c_i \sum_{j=1}^{m} a_{ij} f(x^j(t)), \quad i = 1, \cdots, m, \tag{1.3}$$

where $A = (a_{ij})$ is a symmetric coupling matrix, and c_i is coupling strength at node i.

Synchronization of chaotic systems has been an active topic for applications in many research fields (see [10]). There also are several papers in literatures, in which LCMLs with various coupling schemes were investigated. For example, in [13], local connected network, random network, global coupling network etc., were investigated. Spectral properties of various kinds of LCMLs, such as small-world lattice and scale-free network were discussed in [14, 15]. In [16], authors presented a thorough and theoretical analysis for synchronization of LCMLs.

Different from the approaches proposed in literature, in this paper, we present a new approach to analyze local (global) synchronization of LCMLs. A reference vector $\widehat{x}(t) = [\overline{x}(t), \cdots, \overline{x}(t)]^\top$ on the synchronization manifold, where $\overline{x}(t) = \sum_{k=1}^{m} \xi_k x^k(t)$ is some weighted average of the states of the coupled system, is introduced. This reference vector plays a key role in the investigation of stability of synchronization manifold. We derive synchronization by proving $x(t) - \widehat{x}(t) \to 0$ when $t \to \infty$. Moreover, it must be emphasized that generally $\overline{x}(t)$ is not a solution of the coupled system (1.1).

Furthermore, with the help of the reference vector, criteria for both local and global synchronization are given. These criteria indicate that the left and right eigenvectors corresponding to eigenvalue "0" of the coupling matrix play key roles in the stability of synchronization manifold for coupled system.

2 Some Preliminaries and Lemmas

In this section, we give some definitions and lemmas, which are used throughout this paper.

Definition 2.1 $\mathcal{S} = \{x = [x^1, \cdots, x^m],\ x^i \in R^n,\ x^i = x^j,\ i, j = 1, 2 \cdots, m\}$ *is said to be the synchronization manifold.*

It is easy to see that synchronization manifold $\mathcal{S}$ is an invariant manifold for the coupled system (1.1).

Definition 2.2 $x(t, x_0) = [x^1(t, x_0^1), \cdots, x^m(t, x_0^m)]$ *is defined as the solution of equations* (1.1) *with initial values*

$$x^i(0) = x_0^i \quad \text{for all } i = 1, 2, \cdots, m, \tag{2.1}$$

where $x_0^i = (x_{0,1}^i, \cdots, x_{0,n}^i)^\top \in R^n$ *and* $x_0 = (x_0^1, \cdots, x_0^m)$*. For simplicity, we denote* $x(t, x_0)$ *by* $x(t)$*.*

Definition 2.3 *We say the synchronization manifold* $\mathcal{S}$ *is locally exponentially stable for the coupled system* (1.1)*, or the coupled system* (1.1) *is locally exponentially synchronized, if there exist* $x_0 \in R^{n\times m}$*,* $\delta > 0$*,* $M > 0$*, and* $0 < \gamma < 1$ *such that for each* $\|x^i(0) - x_0\| \le \delta$*,*

$$\|x^i(t) - x^j(t)\| \le M\gamma^t$$

holds for all $i, j = 1, \cdots, m$ *and* $t \ge 0$*.*

Definition 2.4 *We say the synchronization manifold* $\mathcal{S}$ *is globally exponentially stable for the coupled system* (1.1)*, or the coupled system* (1.1) *is locally exponentially synchronized, if for each* $x^i(0) \in R^{n\times m}$*, there exist* $M > 0$*, and* $0 < \gamma < 1$ *such that*

$$\|x^i(t) - x^j(t)\| \le M\gamma^t$$

holds for all $i, j = 1, \cdots, m$ *and* $t \ge 0$*.*

Definition 2.5 *Suppose* $\kappa > 0$*,* $f : R^n \to R^n$*. If*

$$[f(x) - f(y)]^\top [f(x) - f(y)] \le \kappa^2 (x - y)^\top (x - y) \tag{2.2}$$

holds for all $x \ne y \in R^n$*, then we say* $f(x) \in F(\kappa)$*.*

Definition 2.6 *A matrix of order* m $A=(a_{ij})_{i,j=1}^{m}$ *is said* $A\in\mathbf{A1}$, *if*

(1)

$$a_{ij}\geq 0,\quad i\neq j,\quad a_{ii}=-\sum_{j=1,j\neq i}^{m}a_{ij},\quad i=1,2,\cdots,m. \tag{2.3}$$

(2) *Real part of eigenvalues of* A *are all negative except an eigenvalues* 0 *with multiplicity one.*

Definition 2.7 *A matrix of order* m $A=(a_{ij})_{i,j=1}^{m}$ *is said* $A\in\mathbf{A2}$, *if* A *is irreducible and*

$$a_{ij}=a_{ji}\geq 0,\quad i\neq j,\quad a_{ii}=-\sum_{j=1,j\neq i}^{m}a_{ij},\quad i=1,2,\cdots,m. \tag{2.4}$$

Furthermore, if

$$a_{ij}=a_{ji}=1,\quad i\neq j,\quad a_{ii}=-\sum_{j=1,j\neq i}^{m}a_{ij},\quad i=1,2,\cdots,m, \tag{2.5}$$

then it is said that $A\in\mathbf{A3}$.

The following two lemmas play key roles in the discussion of stability of synchronization manifold. Proofs for them will be given in Appendix.

Lemma 2.1 *If the matrix* $B\in\mathbf{A1}$, *then*

(1) $\mathbf{1}=[1,1,\cdots,1]^{\top}$ *is the right eigenvector of* B *corresponding to eigenvalue* 0 *with multiplicity* 1.

(2) *The left eigenvector of* $B:\xi=[\xi_1,\xi_2,\cdots,\xi_m]^{\top}\in R^m$ $\Big($*without loss of generality, we assume* $\sum_{i=1}^{m}\xi_i=1\Big)$ *corresponding to eigenvalue* 0 *has the following properties: It is non-zero and its multiplicity is* 1; *all* $\xi_i\geq 0$, $i=1,\cdots,m$. *More precisely,*

(a) B *is irreducible if and only if all* $\xi_i>0$, $i=1,\cdots,m$.

(b) B *is reducible if and only if for some* i, $\xi_i=0$. *In such case, by suitable rearrangement, we can assume that* $\xi^{\top}=[\xi_+^{\top},\xi_0^{\top}]$, *where* $\xi_+=[\xi_1,\xi_2,\cdots,\xi_p]^{\top}\in R^p$, *with all* $\xi_i>0$, $i=1,\cdots,p$; $\xi_0=[\xi_{p+1},\xi_{p+2},\cdots,\xi_m]^{\top}\in R^{m-p}$ *with all* $\xi_j=0$, $p+1\leq j\leq m$, *and* B *can be rewritten as* $\begin{bmatrix}B_{11}&B_{12}\\B_{21}&B_{22}\end{bmatrix}$, *where* $B_{11}\in R^{p,p}$ *is irreducible and* $B_{12}=0$.

Based on Lemma 2.1, we define the following transverse subspace $\mathcal{L}$ of synchronization manifold $\mathcal{S}$.

Definition 2.8 $\mathcal{L}=\Big\{x=[x^1,\cdots,x^m]:x^i\in R^n,\ i=1,\cdots,m,\ \textit{and}\ \sum_{k=1}^{m}\xi_k x^k=0\Big\}$.

In the following, we define

$$\overline{x}(t)=\sum_{k=1}^{m}\xi_k x^k(t),\qquad \delta x^i(t)=x^i(t)-\overline{x}(t),$$
$$\widehat{x}(t)=[\overline{x}(t),\cdots,\overline{x}(t)],\quad \delta x(t)=x(t)-\widehat{x}(t).$$

Geometrically, we decompose any $x(t)=\widehat{x}(t)\bigoplus\delta x(t)$, where $\widehat{x}(t)\in\mathcal{S}$, $\delta x(t)\in\mathcal{L}$. We derive synchronization by proving that δx converges to zero.

3 Local Stability of Synchronization Manifold

In this section, we investigate local stability of synchronization manifold.

Because $\sum\limits_{k=1}^{m} \xi_k b_{kj} = 0$, it is easy to see that

$$\overline{x}(t+1) = \sum_{k=1}^{m} \xi_k x_k(t+1) = \sum_{k=1}^{m} \xi_k \Big[f(x^k(t)) + \sum_{j=1}^{m} b_{kj} f(x^j(t)) \Big] = \sum_{k=1}^{m} \xi_k f(x^k(t)), \tag{3.1}$$

$$\begin{aligned} \delta x^i(t+1) &= x^i(t+1) - \overline{x}(t+1) = f(x^i(t)) - f(\overline{x}(t)) \\ &\quad - \sum_{k=1}^{m} \xi_k [f(x^k(t)) - f(\overline{x}(t))] + \sum_{j=1}^{m} b_{ij} [f(x^j(t)) - f(\overline{x}(t))]. \end{aligned} \tag{3.2}$$

By first order approximation, we have the following variation equations

$$\delta x^i(t+1) = Df(\overline{x}(t)) \Big[\delta x^i(t) + \sum_{j=1}^{m} b_{ij} \delta x^j(t) \Big].$$

Denote $\delta X(t) = [\delta x^1(t), \delta x^2(t), \cdots, \delta x^m(t)] \in R^{n,m}$, we have

$$\delta X(t+1) = Df(\overline{x}(t)) \delta X(t) (I_m + B^\top).$$

Let $B^\top = SJS^{-1}$ be the Jordanian decomposition of $B^\top$, where

$$J = \begin{bmatrix} 0 & & & & \\ & \lambda_2 & e_2 & & \\ & & \ddots & & \\ & & & \lambda_{m-1} & e_{m-1} \\ & & & & \lambda_m \end{bmatrix}$$

is its Jordan block matrix and $e_i = 0$ or 1, $i = 2, \cdots, m$ with $0 = \lambda_1 > \lambda_2 \geq \lambda_2 \geq \cdots \geq \lambda_m$ are eigenvalues of B. By Lemma 2.1, the first column of S is ξ and the first row of S^{-1} is $\mathbf{1}^\top$.

Let $\delta Y(t) = [\delta y^1(t), \cdots, \delta y^m(t)] = \delta X(t) S$. Then, we have

$$\delta Y(t+1) = Df(\overline{x}(t)) \delta Y (I + J),$$

because $\delta X(t) \xi = 0$. Thus

$$\delta y^1(t) = 0 \quad \text{for all } t > 0 \tag{3.3}$$

and

$$\delta y^k(t+1) = Df(\overline{x}(t)) [\delta y^k(t)(1 + \lambda_k) + e_{k-1} \delta y^{k-1}(t)], \quad k = 2, \cdots, m. \tag{3.4}$$

Theorem 3.1 *Suppose $B \in$ **A1**. If there exist constants $0 < \gamma_0 < \gamma < 1$ and an integer t_0 such that*

$$\|Df(\overline{x}(t))\|_2 \max_{k=2,m} |1 + \lambda_k| \leq \gamma_0, \quad t > t_0, \tag{3.5}$$

then the synchronization manifold is locally exponentially stable for system (1.1); *moreover, the convergence rate can be estimated by $O(\gamma^t)$.*

Proof Firstly, we consider the equation in system (3.4) with $e_{k-1} = 0$. In this case, we have

$$\|\delta y^k(t+1)\|_2 \le \|Df(\overline{x}(t))\|_2 \|\delta y^k\| |1+\lambda_k| \le \gamma_0 \|\delta y^{k,1}(t)\|_2 \quad \text{for all } t > t_0.$$

Therefore,

$$\delta y^k(t) = O(\gamma_0^{-t}), \quad k = 1, \cdots, m.$$

In the case that $e_{k-1} = 1$ in the equation in system (3.4), let

$$\delta y^k(t+1) = z^k(t+1) + \varepsilon^k(t+1), \quad k = 1, \cdots, m,$$

where

$$\begin{aligned} z^k(t+1) &= Df(\overline{x}(t))[\delta y^k(t)(1+\lambda_k)], \quad k = 1, \cdots, m, \\ \varepsilon^k(t+1) &= Df(\overline{x}(t))[e_{k-1}\delta y^{k-1}(t)], \quad\ \ k = 1, \cdots, m. \end{aligned}$$

By induction and structure of Jordan block matrix, we can assume that

$$\delta y^{k-1}(t) = O(\gamma^{-t}).$$

Therefore,

$$\delta y^k(t+1) = z^k(t+1) + O(\gamma^{-t}).$$

With previous arguments, we have

$$\|z^k(t+1)\|_2 \le \gamma_0 \|\delta y^k(t)\|_2.$$

Thus

$$\|\gamma^{t+1}\delta y^k(t+1)\|_2 = \gamma_0\gamma\|\gamma^t \delta y^k(t)\|_2 + O(1),$$

which means that $\|\gamma^t \delta y^k(t)\|_2$ is bounded and

$$\delta y^k(t) = O(\gamma^{-t}) \quad \text{for } k = 2, \cdots, m.$$

Combining with $\delta y^1(t) = 0$, we have

$$\delta X(t) = O(\gamma^t).$$

Theorem 3.1 is proved.

Remark 3.1 In the papers [7, 9, 11, 14–16], authors all assumed that all $x^i(t)$ are some small perturbations of a solution of the uncoupled system

$$s(t+1) = f(s(t)) \tag{3.6}$$

and used first order approximation around trajectory $s(t)$ to obtain variation equations. By these variation equations, under some conditions, stability of synchronization manifold is obtained. It is well known that a prerequisite requirement for this approach is that every trajectory $x_i(t)$ of the coupled system must be near the trajectory $s(t)$.

The following proposition can be used to investigate local stability for the coupled system.

Proposition 3.1 (See [17]) *Suppose the uncoupled system $s(t+1) = f(t)$ has an attractor set A and an attraction basin $W \supseteq A$. If for all $i = 1, \cdots, m$, $x_i(0) \in W$, $x_i(0) - s(0)$ are small, and all the Lyapunov exponents in the transverse directions on $s(t)$ for the coupled system are negative, then $x_i(t)$, $i = 1, \cdots, m$, can be synchronized.*

Although this proposition applies to some chaotic system, it is not yet known that if all systems satisfy conditions required in the previous proposition. In particular, $s(t)$ is a repeller.

Instead, in our approach, we do not impose these requirements. Theorem 3.1 concludes that if $(x^1(0), \cdots, x^m(0))$ is close to the synchronization manifold $\mathcal{S}$, inequalities (3.5) are satisfied (or all the Lyapunov exponents in the transverse directions on $\overline{x}(t)$ for the coupled system are negative), then $x_i(t)$, $i = 1, \cdots, m$, approach to synchronized trajectory.

The following example verifies our assertion. Consider system

$$s(t+1) = f(s(t)) \mod 2\pi, \tag{3.7}$$

where

$$f(\rho) = \begin{cases} 2\rho, & |\rho| \geq 1, \\ (1+\rho^2)\rho, & |\rho| \leq 1. \end{cases}$$

The coupled system is

$$\begin{aligned} x_\varepsilon^1(t+1) &= f(x_\varepsilon^1(t)) + \varepsilon[f(x_\varepsilon^2(t)) - f(x_\varepsilon^1(t))], \\ x_\varepsilon^2(t+1) &= f(x_\varepsilon^2(t)) + \varepsilon[f(x_\varepsilon^1(t)) - f(x_\varepsilon^2(t))]. \end{aligned} \tag{3.8}$$

It is clear that $s(t) = 0$ is a solution of the system (3.7), which is unstable and has no attractor. The variational equation in transverse direction (corresponding to the eigenvalue $\lambda_2 = -2$) on $s(t) = 0$ is

$$\delta y(t+1) = Df(s(t))(1-2\varepsilon)\delta y(t),$$

where $Df(s(t)) = 1$ for all t. Therefore, for any $0 < \varepsilon < 1$, by variational equations technique (or calculating Lyapunov exponents) on $s(t)$, the coupled system should be synchronized.

On the other hand, it is clear that

$$x_\varepsilon^1(t+1) - x_\varepsilon^2(t+1) = (1-2\varepsilon)[f(x_\varepsilon^1(t)) - f(x_\varepsilon^2(t))]. \tag{3.9}$$

Pick $|x^i(0)| < 0.0001$ and let ε vary from 0 to 1. For each ε, define the following quantity to measure the synchronization error

$$\mathrm{Gap}(\varepsilon) = \langle |x_\varepsilon^1(t) - x_\varepsilon^2(t)| \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes time average. Figure 1 (red line) indicates that approximately for $\varepsilon \in [0.1, 0.9]$, the coupled system (3.8) is synchronized. Instead, if ε is not in this interval, (3.8) can not be synchronized, no matter how small the initial values are. It means that the variational equation in transverse direction on $s(t)$ converges to zero (or negative Lyapunov exponent in the transverse direction on $s(t)$) does not necessarily implies synchronization.

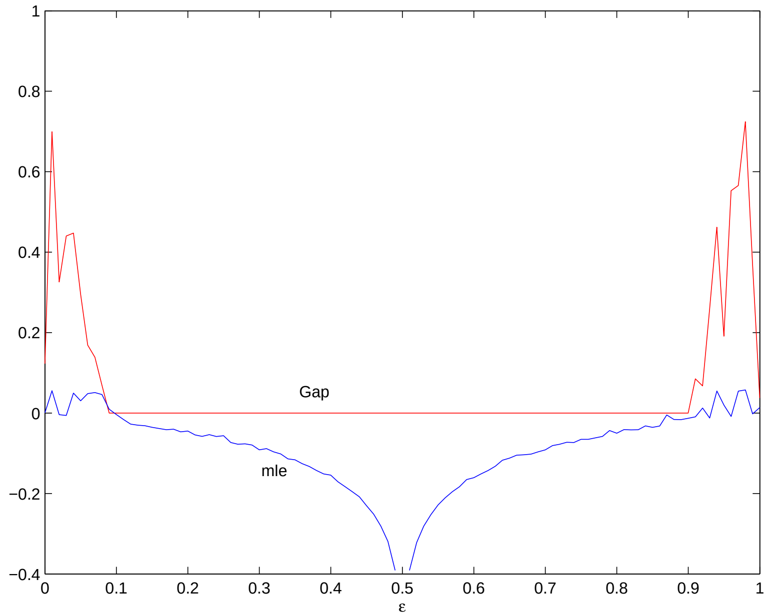

Figure 1 Gap varies through ε

Instead, negative Lyapunov exponent on $\widehat{x}(t)$ (blue line) in Figure 1 implies the coupled system is synchronized. Instead, positive Lyapunov exponent on $\widehat{x}(t)$ implies the coupled system can not be synchronized. Theorem 3.1 coincides with the simulation result.

4 Global Stability of Synchronization Manifold

In this section, we investigate global stability of the synchronization manifold.

Theorem 4.1 *Suppose that $B \in$ **A1**, $f \in F(\kappa)$, where $\kappa > 0$ is a constant. If there exist a positive number $b > \kappa$ and a positive definite matrix P such that*

$$(I_m - \Xi^\top + B^\top)P(I_m - \Xi + B) \leq \frac{1}{b^2}P, \tag{4.1}$$

where

$$\Xi = \begin{bmatrix} \xi_1 & \xi_2 & \cdots & \xi_m \\ \xi_1 & \xi_2 & \cdots & \xi_m \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1 & \xi_2 & \cdots & \xi_m \end{bmatrix},$$

then the synchronization manifold $\mathcal{S}$ is globally exponentially stable for the coupled system (1.1). *Moreover, the convergence rate is $O((\frac{\kappa}{b})^t)$.*

Proof Let $\delta x(t) = [\delta x^1(t), \cdots, \delta x^m(t)]$. Then

$$\delta x(t+1) = \{I_m - \Xi + B\}[F(x(t)) - F(\overline{x}(t))], \tag{4.2}$$

where

$$F(x(t)) = [f^\top(x^1(t)), \cdots, f^\top(x^m(t))]^\top, \quad F(\overline{x}(t)) = [f^\top(\overline{x}(t)), \cdots, f^\top(\overline{x}(t))]^\top.$$

Define $V(t) = \mathrm{tr}\{\delta x^\top(t) P \delta x(t)\}$. We have

$$\begin{aligned} V(t+1) &= \mathrm{tr}\{\delta x^\top(t+1) P \delta x(t+1)\} \\ &= \mathrm{tr}\{[F(x(t)) - F(\overline{x}(t))]^\top \{I_m - \Xi + B\}^\top P \{I_m - \Xi + B\} [F(x(t)) - F(\overline{x}(t))]\} \\ &\le \mathrm{tr}\Big\{\frac{1}{b^2}[F(x(t)) - F(\overline{x}(t))]^\top P [F(x(t)) - F(\overline{x}(t))]\Big\} \\ &\le \mathrm{tr}\Big\{\frac{\kappa^2}{b^2}\delta x^\top(t) P \delta x(t)\Big\} = \frac{\kappa^2}{b^2} V(t), \end{aligned}$$

which implies $\delta x^\top(t) = O((\frac{\kappa}{b})^t)$. Theorem 4.1 is proved.

If coupling matrix $A \in \mathbf{A2}$ or $A \in \mathbf{A3}$, we have the following corollaries, which are easier to use in practice.

Corollary 4.1 *Suppose $A \in \mathbf{A2}$, $f \in F(\kappa)$, where $\kappa > 0$ is a constant, $C = \mathrm{diag}\{c_1, c_2, \cdots, c_m\}$ is a positive diagonal matrix. Let $0 = \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_m$ be the eigenvalues of CA. If*

$$\max\{|1 + \lambda_2|, |1 + \lambda_m|\} < \frac{1}{\kappa}, \tag{4.3}$$

then system (1.3) *is globally stable exponentially.*

Proof It is calear that $\xi = [\xi_1, \cdots, \xi_m] = \frac{1}{m}[c_1^{-1}, \cdots, c_m^{-1}]^\top$ is the left eigenvector of CA corresponding to eigenvalue 0.

Let $\overline{A} = C^{\frac{1}{2}} A C^{\frac{1}{2}}$, which is symmetric. Its eigenvalue decomposition is $\overline{A} = Q \Lambda Q^{-1}$, where $Q = [q_1, \cdots, q_m]$, $\Lambda = \mathrm{diag}[0, \cdots, \lambda_m]$. We take $q_1 = \frac{1}{m}[c_1^{\frac{1}{2}}, \cdots, c_m^{\frac{1}{2}}]^\top$. Then the eigenvalue decomposition of CA is

$$CA = C^{\frac{1}{2}} \overline{A} C^{-\frac{1}{2}} = \{C^{\frac{1}{2}} Q\} \Lambda \{C^{\frac{1}{2}} Q\}^{-1},$$

where $C^{\frac{1}{2}} Q = [v_1, \cdots, v_m]$. Because $\xi = [\xi_1, \cdots, \xi_m]$ is the left eigenvector of CA corresponding to eigenvalue 0 satisfying $\xi^\top v_1 = 1$, $\xi^\top v_i = 0$ for all $i = 2, \cdots, m$, we have

$$\{C^{\frac{1}{2}} Q\}^{-1} \Xi \{C^{\frac{1}{2}} Q\} = E_m^1,$$

where E_m^1 is the matrix having all entries $E_m^1(i,j) = 0$ except $E_m^1(1,1) = 1$. Therefore,

$$\begin{aligned} &(I_m - \Xi + CA)^\top C^{-1} (I_m - \Xi + CA) \\ &= Q C^{\frac{1}{2}} (I_m - E_m^1 + \Lambda)^\top Q^\top C^{\frac{1}{2}} C^{-1} C^{\frac{1}{2}} Q (I_m - E_m^1 + \Lambda) \{C^{\frac{1}{2}} Q\}^{-1} \\ &= C^{-\frac{1}{2}} Q (I_m - E_m^1 + \Lambda)^\top (I_m - E_m^1 + \Lambda) Q^\top C^{-\frac{1}{2}} \\ &\le \max_{i=2,\cdots,m} (1 + \lambda_i)^2 C^{-\frac{1}{2}} Q Q^\top C^{-\frac{1}{2}} \le \max_{i=2,\cdots,m} (1 + \lambda_i)^2 C^{-1}. \end{aligned}$$

Corollary 4.1 is a direct consequence of Theorem 4.1.

Corollary 4.2 *Suppose A satisfies Condition* **A3**, $f \in F(\kappa)$, *where $\kappa > 0$ is a constant. Let $C = \mathrm{diag}\{\frac{1}{k_1}, \cdots, \frac{1}{k_m}\}$ and $0 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ be the eigenvalues of CA. If*

$$\max\{|1+\varepsilon\lambda_2|, |1+\varepsilon\lambda_m|\} < \frac{1}{\kappa}, \tag{4.4}$$

then system (1.2) *is globally stable exponentially.*

Remark 4.1 In discussion of global stability of synchronization manifold, two factors play key role. One is the dynamical property of identical system of each node $s(t+1) = f(s(t))$, which is described by the Lipschitz constant κ of $f(\cdot)$. The other is the coupling configuration, which is described by a linear matrix inequality (LMI) (4.1).

5 Conclusions

In this paper, we present a new approach to analyze synchronization of LCMLs. A projection of the trajectory of the coupled system $\widehat{x}(t)$ on the synchronization manifold is introduced. Decomposition of $R^{n,m}$ into synchronization manifold $\mathcal{S}$ and transverse subspace $\mathcal{L}$ is proposed. Based on the decomposition, stability analysis of the synchronization manifold reduces to proving that the component in $\mathcal{L}$ converges to zero.

By this new approach, we discuss both local and global synchronization for the LCMLs. Some simple criteria to guarantee local and global stability of the synchronization manifold are given.

Appendix

Proof of Lemma 2.1 Since A satisfies Condition **A1**, $[1, 1, \cdots, 1]^\top$ is the right eigenvector of A corresponding to eigenvalue 0 with multiplicity 1.

Because $\mathrm{Rank}(A) = m-1$, there exists a nonsingular $(m-1)\times(m-1)$ minor of A. Without loss of generality, we suppose

$$A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix},$$

where $\overline{A}_{22} \in R^{m-1,m-1}$ is nonsingular. Now, we construct a nonsingular matrix

$$\widetilde{A} = \begin{bmatrix} 1 & A_{12} \\ 0 & \overline{A}_{22} \end{bmatrix},$$

and let ξ be the unique solution of the linear equation $\xi^\top \widetilde{A} = (1, 0, 0, \cdots, 0)$.

Let

$$\xi = [\xi_1, \cdots, \xi_n]^\top = [\xi_1, \xi'^\top]^\top,$$

where $\xi' \in R^{m-1}$. We will prove that all $\xi_i \geq 0$. First, it is easy to see that $\xi_1 = 1$ and $\xi' = -(\overline{A}_{22}^\top)^{-1} A_{12}^\top$. Because $A \in$ **A1**, $-\overline{A}_{22}^\top$ is a nonsingular M matrix (see [20, pp. 21–22]), which implies that all elements of $-(\overline{A}_{22}^\top)^{-1}$ are nonnegative. Moreover, all elements of A_{12} are also nonnegative. Thus, all $\xi_i \geq 0$. With Condition **A1**, the first column of A is a linear combination of the remaining columns. Therefore, $\xi^\top A = 0$, which means that ξ is the left eigenvector corresponding to the eigenvalue 0.

If the eigenvector ξ is of the form $\xi^\top = [\xi_+^\top, 0]$, where $\xi_+ = [\xi_1, \xi_2, \cdots, \xi_p]^\top \in R^p$ with all $\xi_i > 0$ for $i = 1, 2, \cdots, p$, writing

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in R^{p,p}$, then by $\xi^\top A = 0$ we have

$$\xi_+^\top A_{11} = 0, \quad \xi_+^\top A_{12} = 0.$$

If $A_{12} \neq 0$, then A_{11} is a nonsingular M matrix. Hence, $\xi_+ = 0$, which contradicts the assumption that all entries of ξ_+ are greater than 0. Therefore, $A_{12} = 0$.

On the other hand, if A is reducible, then without loss of generality A can be written as

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in R^{p,p}$ and $1 \le p \le m-1$. Let

$$\xi^\top = [{\xi^1}^\top, {\xi^2}^\top].$$

From $\xi^\top A = 0$, we have

$${\xi^1}^\top A_{11} + {\xi^2}^\top A_{21} = 0, \quad {\xi^2}^\top A_{22} = 0.$$

Because A_{22} is non-singular, we conclude that

$${\xi^2}^\top = 0 \quad \text{and} \quad {\xi^1}^\top A_{11} = 0.$$

Lemma 2.1 is proved.

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