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A Remark on Steinness***

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Abstract In this paper, the authors prove that if M^n is a complete noncompact Kähler manifold with a pole p, and its holomorphic bisectional curvature is asymptotically nonnegative to p, then it is a Stein manifold.

Keywords Holomorphic bisectional curvature, Plurisubharmonic, Smooth, Exhausting
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1 Introduction

Recently, Ni and Tam proved (see [1]):

Theorem 1.1 *Let M^n be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that M has a pole. Then M is Stein.*

Now we use elementary method to improve above theorem, that is,

Theorem 1.2 *Let M^n be a complete noncompact Kähler manifold with a pole p, its bisectional curvature is asymptotically nonnegative to p, then it is Stein.*

Remark 1.1 In fact, Theorem 1.2 is held under the bisectional curvature $\geq -\frac{k}{r^2(p,x)}$.

2 Preliminary

Before we prove the theorem, we shall give some definitions and a lemma.

Definition 2.1 *We say that the bisectional curvature is asymptotically nonnegative if*

$$\mathrm{bi}\, K_M(x) \geq -\lambda(r(x)),$$

where $\lambda(\cdot)$ is a nonnegative and nonincreasing function on $[0,+\infty)$ and

$$\int_0^\infty r\lambda(r)dr < +\infty, \quad r(x) = \mathrm{dist}(p,x)$$

and p is a fixed point in M.

Definition 2.2 *If there exists $p \in M$, such that the exponential mapping $\exp_p : M_p \longrightarrow M$ is a diffeomorphism, then we say M has a pole p.*

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Now, we will prove the next lemma.

Lemma 2.1 *Suppose $f(t)$ is a C^2 function on $(0,T)$, $\frac{d}{dt}(-f(t)) - \frac{f^2(t)}{2} \geq -\frac{k}{t^2}$, and $\lim\limits_{t\to 0^+} f(t) = +\infty$. Then*

$$f(t) > 0, \tag{2.1}$$

where $k \geq 0$, T can tend to $+\infty$.

Proof We use comparison method to discuss the following inequality.

Assume

$$b_1(t) = \frac{-1+\sqrt{1+2k}}{2k}t, \quad b_2(t) = \frac{-1-\sqrt{1+2k}}{2k}t. \tag{2.2}$$

Then

$$\frac{db_i}{dt} + \frac{k}{t^2}b_i^2 - \frac{1}{2} = 0, \quad i = 1,2 \tag{2.3}$$

and

$$b_1(0) = b_2(0) = 0, \tag{2.4}$$

$$\dot{b}_1(t) = \frac{1}{2} - \frac{k}{t^2}b_1^2 = \frac{-1+\sqrt{1+2k}}{2k}, \tag{2.5}$$

$$\dot{b}_2(t) = \frac{1}{2} - \frac{k}{t^2}b_2^2 = \frac{-1-\sqrt{1+2k}}{2k}. \tag{2.6}$$

Let $A = \frac{1}{f(t)}$. Therefore

$$\frac{dA}{dt} + \frac{k}{t^2}A^2 - \frac{1}{2} \geq 0 \tag{2.7}$$

and $A(0) = 0$.

Then

$$\dot{A}(0) \geq \frac{1}{2} - k\dot{A}^2(0), \tag{2.8}$$

i.e., $\dot{A}(0) \geq \dot{b}_1(0)$ or $\dot{A}(0) \leq \dot{b}_2(0)$.

We will prove $\dot{A}(0) \leq \dot{b}_2(0)$ is not valid.

Conversely, if $\dot{A}(0) \leq \dot{b}_2(0) < 0$, we have

$$A(t) = \dot{A}(0)t + O(t^2),$$

and then there exists an ϵ satisfying $0 < \epsilon \leq T$ such that $A(t) < 0$, $\forall\, t \in (0,\epsilon)$, i.e.,

$$f(t) < 0, \quad \forall\, t \in (0,\epsilon). \tag{2.9}$$

This is a contradiction to $\lim\limits_{t\to 0^+} f(t) = +\infty$ which is known.

When $\dot{A}(0) \geq \dot{b}_1(0)$, we will prove $A(t) \geq b_1(t) > 0$, for $t \in [0,T)$.

Let $E = \{t \in (0,T) \mid \dot{A}(t) < \dot{b}_1(t)\}$. If $E = \emptyset$, then $\dot{A}(t) \geq \dot{b}_1(t)$, $\forall\, t \in [0,T)$, and $A(0) = b_1(0) = 0$. Then $A(t) \geq b_1(t) > 0$, $\forall\, t \in [0,T)$.

If E is a nonempty open set of $[0,T)$, then $\forall\, a \in E$, we have $A(a) \geq b_1(a)$. Otherwise, we have

$$\dot{b}_1(a) > \dot{A}(a) \geq \frac{1}{2} - k\frac{A^2}{t^2}\Big|_{t=a} \geq \frac{1}{2} - k\frac{b_1^2}{t^2}\Big|_{t=a}.$$

Obviously, it is impossible.

For any $c \in [0,T)\backslash E$, setting $t_0 = \mathrm{Sup}(E \cap [0,c])$, we have

$$A(c) - b_1(c) = \int_{t_0}^{c} \big(\dot{A}(s) - \dot{b}_1(s)\big) ds + A(t_0) - b_1(t_0).$$

By continuity, $A(t_0) - b_1(t_0) \geq 0$ and $(t_0, c] \subset [0,T)\backslash E$, so that $\dot{A}(s) - \dot{b}_1(s) \geq 0$. Then $A(c) - b_1(c) \geq 0$, i.e., $A(c) \geq b_1(c) > 0$.

Thus $f(t) > 0$, $\forall\, t \in (0,T)$, i.e.,

$$f(t) > 0, \quad \forall\, t \in [0,T). \tag{2.10}$$

3 The Proof of Theorem 1.2

Now we will prove Theorem 1.2.

Proof of Theorem 1.2 A result of Grauert [2] says that a complex manifold which admits a smooth strictly plurisubharmonic exhaustion function is Stein. Next we will prove the square of the distance function satisfies these three conditions.

Firstly, the square of the distance function $r^2(x) = r^2(p,x)$ between p and x is smooth because p is a pole and obviously it is exhausting.

Secondly, we will prove that $r^2(p,x)$ is strictly plurisubharmonic. $\forall\, x \in M$, there exists a normal geodesic γ_x with the minimal length which issues from p to x. For every nonzero vector $X \in T_x^{(1,0)}(M)$, we have $X = X_1 - \sqrt{-1}\, JX_1$ such that X_1 is the unit vector of $T_x(M)$. Now we choose a normal basis $\{e_1(x), e_2(x), \cdots, e_n(x), e_{n+1}(x), \cdots, e_{2n}(x)\} = \{e_1(x), e_2(x), \cdots, e_n(x), Je_1(x), \cdots, Je_n(x)\}$ on $T_x(M)$, such that $X_1 = e_1(x),\ JX_1 = e_{n+1}(x)$. And $\{e_i(t)\}_{1\leq i\leq 2n}$ is the normal basis on $\gamma_x(t)$ that was obtained by the parallel translation of $\{e_i(x)\}_{1\leq i\leq 2n}$ along $\gamma_x(t)$. Since M is Kähler, the parallel translation preserves the complex structure, so that $J[e_i(t)] = e_{n+i}(t),\ 1 \leq i \leq n$, is valid on $\gamma_x(t)$. We denote by $\{e_i(0)\}_{1\leq i\leq 2n}$ the parallel vector basis field at $p \in M$. Let $U(\gamma_x)$ be an open neighborhood of γ_x. For any $y \in U(\gamma_x)$, there is a minimal length normal geodesic from p to y. We translate $\{e_i(0)\}_{1\leq i\leq 2n}$ parallel along γ_y. Then we obtain a normal basis $\{e_i(y)\}_{1\leq i\leq 2n}$ on $T_y(M)$, $\forall\, y \in U(\gamma_x)$, such that $Je_i(y) = e_{n+i}(y),\ 1 \leq i \leq n$. Hence $\{e_i(y)\}_{1\leq i\leq 2n},\ y \in U(\gamma_x)$, is a normal frame on $U(\gamma_x)$.

Let

$$\begin{aligned} &\nabla_{e_i(t)}\dot{\gamma}(t) = u_{ij}(t)e_j(t), \\ &\nabla_{\dot{\gamma}(t)}e_i(t) = \lambda_{ij}(t)e_j(t) = 0. \end{aligned} \tag{3.1}$$

Then we have

$$\begin{aligned} u_{ij}(t) &= \langle \nabla_{e_i}\dot{\gamma}(t), e_j\rangle = e_i\langle \dot{\gamma}(t), e_j\rangle - \langle \dot{\gamma}(t), \nabla_{e_i}e_j\rangle \\ &= (e_ie_jr) - (\nabla_{e_i}e_jr) = Hr(e_i,e_j) = u_{ji}(t), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \langle R(e_1,\dot{\gamma}(t))\dot{\gamma}(t), e_1\rangle &= -\langle \nabla_{\dot{\gamma}(t)}\nabla_{e_1}\dot{\gamma}(t), e_1\rangle - \langle \nabla_{\nabla_{e_1}\dot{\gamma}(t)}\dot{\gamma}(t), e_1\rangle + \langle \nabla_{\nabla_{\dot{\gamma}(t)}e_1}\dot{\gamma}(t), e_1\rangle \\ &= \frac{d}{dt}(-u_{11}) - \sum_j u_{1j}u_{j1} \\ &\leq \frac{d}{dt}(-u_{11}) - u_{11}u_{11}. \end{aligned} \tag{3.3}$$

Similarly, we have

$$\langle R(e_{n+1},\dot{\gamma}(t))\dot{\gamma}(t),e_{n+1}\rangle \le \frac{d}{dt}(-u_{n+1,n+1}) - u_{n+1,n+1}u_{n+1,n+1}, \tag{3.4}$$

$$\begin{aligned} -\frac{k}{r^2} &\le \langle R(e_1,\dot{\gamma}(t))\dot{\gamma}(t),e_1\rangle + \langle R(e_{n+1},\dot{\gamma}(t))\dot{\gamma}(t),e_{n+1}\rangle \\ &\le \frac{d}{dt}(-u_{11}-u_{n+1,n+1}) - \frac{(u_{11}+u_{n+1,n+1})^2}{2}. \end{aligned} \tag{3.5}$$

Assume $f(t)=u_{11}+u_{n+1,n+1}$, and $\lim\limits_{t\to 0^+} f(t)=+\infty$. By Lemma 2.3 and (3.5), we have

$$D^2r(X_1,X_1)+D^2r(JX_1,JX_1) = Hr(e_1,e_1)+Hr(e_{n+1},e_{n+1}) > 0. \tag{3.6}$$

And

$$\begin{aligned} &D^2r^2(X_1,X_1)+D^2r^2(JX_1,JX_1) \\ &= 2\Big\langle X_1,\frac{\partial}{\partial r}\Big\rangle^2 + 2\Big\langle JX_1,\frac{\partial}{\partial r}\Big\rangle^2 + 2r[D^2r(X_1,X_1)+D^2r(JX_1,JX_1)] \\ &> 0. \end{aligned} \tag{3.7}$$

Because

$$\sqrt{-1}\,\partial\overline{\partial}r^2(X,\ \overline{X}) = D^2r^2(X_1,\ X_1)+D^2r^2(JX_1,\ JX_1), \tag{3.8}$$

where $X=X_1-\sqrt{-1}JX_1$, r^2 is strictly plurisubharmonic.

Thus M is a Stein manifold.

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