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Basic Results of Gabor Frame on Local Fields***

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Abstract Basic facts for Gabor frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ on local field are investigated. Accurately, that the canonical dual of frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in \mathbf{P}}$ also has the Gabor structure is showed; that the product ab decides whether it is possible for $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in \mathbf{P}}$ to be a frame for $L^2(K)$ is discussed; some necessary conditions and two sufficient conditions of Gabor frame for $L^2(K)$ are established. An example is finally given.

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1 Introduction

While working on some deep questions in non-harmonic Fourier series, Duffin and Schaeffer [1–2] introduced the concept of a frame for Hilbert spaces. Outside of this area, this idea has been lost until Daubechies, Grossman and Meyer [3] brought attention to it in 1986. They showed that Duffin and Schaeffer's defintion was an abstraction of a concept given by Gabor [4] in 1946 for doing signal analysis. Today the frames introduced by Gabor are called Gabor frames and play an important role in signal analysis (see [5–9]). Over the last fifth years, there has been a tremendous influx of outstanding researchers into Gabor frames in $L^2(\mathbf{R}^d)$.

Although there are many results for Gabor frame on $\mathbf{R}^d$, the counterparts on local field are not yet reported. So this paper is concerned with Gabor frame on local field. Recently, we established the orthonormal wavelet construction from multiresolution analysis on local field in [10], and discussed wavelet bases on local field in [11]. As one of a series of works on local field, the objective of this paper is to investigate the most fundamental facts for Gabor frame on local field. Accurately, that canonical dual of frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ also has the Gabor structure is showed; that product ab decides whether it is possible for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ to be a frame for $L^2(K)$ is discussed; some necessary conditions and two sufficient conditions of Gabor frame for $L^2(K)$ are established. An example is also presented.

The layout of this paper is as follows. Section 1 briefly introduces some notations of local fields to be used throughout the paper. In order to research Gabor frame from our setting, some basic facts for the frame in abstract Hilbert space are listed, and the most basic works of Gabor frame on local fields are contributed in Section 2, and indeed these basic works are

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some parts of major conclusions in this paper. Some necessary conditions of Gabor frame on local fields are established in Section 3. Section 4 is devoted to discussion of sufficient condition for Gabor frame on local fields, and in result, two sufficient conditions are given. We end the paper with an example.

2 Some Notations of Local Fields

In this section, we list some notations of local fields to be used throughout the paper. For more details please refer to [10–12].

A local field means an algebraic field and a topological space with the topological properties of locally compact, non-discrete, complete and totally disconnected, denoted by K. The additive and multiplicative groups of K are denoted by K^+ and K^*, respectively. dx is the normalized Haar measure on K^+. $|\alpha|$ is the absolute value or valuation of α in K, which is also a non-archimedian norm on K. But $|E|$ is the Haar measure of $E \subset K$. $\mathfrak{p}$ is a fixed prime element of K. We have the fact $|\mathfrak{p}| = q^{-1}$ with that $q = p^c$, p is a prime number and c is a positive integer.

Every $\mathfrak{P}^k = \{x \in K : |x| \le q^{-k}\}$ is a compact subgroup of K^+. $\mathfrak{O} = \mathfrak{P}^0$ is the ring of integers in K. So, $|\mathfrak{O}| = 1$ and $|\mathfrak{P}^k| = q^{-k}$.

χ is a fixed character on K^+ that is trivial on $\mathfrak{O}$ but is non-trivial on $\mathfrak{P}^{-1}$. $\chi_y(x) := \chi(yx)$ for $x, y \in K$.

The "natural" order on the sequence $\{u(n) \in K\}_{n=0}^{\infty}$ is endowed as follows.

We recall $\mathfrak{P}$ is the prime ideal in $\mathfrak{O}$, $\mathfrak{O}/\mathfrak{P} \cong \mathrm{GF}(q) = \Gamma$, $q = p^c$, p a prime, c a positive integer and $\rho : \mathfrak{O} \longrightarrow \Gamma$ the cannonical homomorphism of $\mathfrak{O}$ on to Γ. Note that $\Gamma = \mathrm{GF}(q)$ is a c-dimensional vector space over $\mathrm{GF}(p) \subset \Gamma$. We choose a set $\{1 = e_0, e_1 \cdots, e_{c-1}\} \subset \mathfrak{O}^* = \mathfrak{O}\backslash\mathfrak{P}$ such that $\{\rho(e_k)\}_{k=0}^{c-1}$ is a basis of $\mathrm{GF}(q)$ over $\mathrm{GF}(p)$.

Definition 2.1 *For n, $0 \le n < q$, $n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}$, $0 \le a_k < p$ and $k = 0, \cdots, c-1$, we define*

$$u(n) = (a_0 + a_1\varepsilon_1 + \cdots + a_{c-1}\varepsilon_{c-1})\mathfrak{p}^{-1}, \quad 0 \le n < q. \tag{2.1}$$

For $n = b_0 + b_1 q + \cdots + b_s q^s$, $0 \le b_k < q$, $n \ge 0$, we set

$$u(n) = u(b_0) + \mathfrak{p}^{-1}u(b_1) + \cdots + \mathfrak{p}^{-s}u(b_s).$$

Hereafter we will denote $\chi_{u(n)}$ by χ_n $(n \ge 0)$. We also often use the following number set throughout this paper: $\mathbf{P} = \{0, 1, 2, \cdots\}$.

Definition 2.2 *We say a function f defined on K with period a if $f(x + u(l)a) = f(x)$ for all $x \in K$ and $l \in \mathbf{P}$.*

3 Some Basic Facts

We first list some facts for the frame in abstract Hilbert space (cf. [7]).

Definition 3.1 *Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence in Hilbert space H.*

(i) *$\{f_k\}_{k=1}^{+\infty}$ is called a frame for H if there exist constants $C, \widetilde{C} > 0$ such that*

$$C\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le \widetilde{C}\|f\|^2, \quad f \in H.$$

The numbers C, $\widetilde{C}$ are called frame bounds. They are not unique. The optimal upper frame bound is the infimum over all upper frame bounds, and the optimal lower frame bound is the supremum over all lower frame bounds.

(ii) *A frame is tight if we can choose $C = \widetilde{C}$ as frame bounds.*

(iii) $\{f_k\}_{k=1}^{+\infty}$ *is called a Bessel sequence for H if only the right inequality holds in the above formula. $\widetilde{C}$ is called a Bessel bound for $\{f_k\}_{k=1}^{+\infty}$.*

(iv) $\{f_k\}_{k=1}^{+\infty}$ *is called a Riesz basis for H if $\{f_k\}_{k=1}^{+\infty} = \{Ue_k\}_{k=1}^{+\infty}$, where $\{e_k\}_{k=1}^{+\infty}$ is an orthonormal basis for H and $U:\ H \to H$ is a linearly bounded bijective operator.*

Suppose that $\{f_k\}_{k=1}^{+\infty}$ is a frame in Hilbert space H, then the operator $S:\ H \to H,\quad Sf = \sum\limits_{k=1}^{+\infty}\langle f, f_k\rangle f_k$ is called the frame operator associated with the frame $\{f_k\}_{k=1}^{+\infty}$. It is well known that S is linearly bounded, invertible, self-adjoint and positive, and $\{S^{-1}f_k\}_{k=1}^{+\infty}$ is also a frame with bounds $\widetilde{C}^{-1}, C^{-1}$ in H, which is called the canonical dual of $\{f_k\}_{k=1}^{+\infty}$, and $\forall f \in H$, $f = \sum\limits_{k=1}^{+\infty}\langle f, S^{-1}f_k\rangle f_k = \sum\limits_{k=1}^{+\infty}\langle f, f_k\rangle S^{-1}f_k$, where all series converge unconditionally. Moreover, $\{S^{-\frac{1}{2}}f_k\}_{k=1}^{+\infty}$ is a tight frame with bound 1, and $\forall\, f \in H$, $f = \sum\limits_{k=1}^{+\infty}\langle f, S^{-\frac{1}{2}}f_k\rangle S^{-\frac{1}{2}}f_k$. The following facts will be used.

Proposition 3.1 *Assume that $\{f_k\}_{k=1}^{+\infty}$ is a Bessel sequence with bound C for a Hilbert space H. Then (1) $\|f_k\|^2 \le C$ $(k = 1, 2, \cdots)$; (2) if $\|f_k\|^2 = C$ for some k, then $\langle f_k,\ f_j\rangle = 0$ for $j \ne k$.*

Proposition 3.2 *Assume that $\{f_k\}_{k=1}^{+\infty}$ is a sequence for a Hilbert space H. Then $\{f_k\}_{k=1}^{+\infty}$ is a Riesz basis for H if and only if $\{f_k\}_{k=1}^{+\infty}$ is complete in H, and there exist constants $C,\ \widetilde{C} > 0$ such that for every finite scalar sequence $\{c_k\}$, $C\sum\limits_k |c_k|^2 \le \left\|\sum\limits_k c_k f_k\right\|^2 \le \widetilde{C}\sum\limits_k |c_k|^2$. The numbers $C, \widetilde{C}$ are called Riesz basis bounds. Moreover, a Riesz basis must be a frame, and frame bounds are just Riesz basis bounds; a Riesz basis with bound $C = \widetilde{C} = 1$ is an orthonormal basis for H.*

Proposition 3.3 *Suppose that H is a Hilbert space. Let $U : H \to H$ be a bounded operator, and assume that $\langle Ux, x\rangle = 0$ for all $x \in H$. If H is a complex Hilbert space, then $U = 0$; if H is a real Hilbert space and U is self-adjoint, then $U = 0$.*

Now, we turn to Gabor frame on local fields.

Definition 3.2 *A Gabor frame on local fields is a frame for $L^2(K)$ of the form $\{\chi_m(bx)g(x-u(n)a)\}_{m,n\in\,\mathbf{P}}$, where a, b are fixed elements in K, and g is a fixed function in $L^2(K)$. The function g is called the window function or the generator.*

For a Gabor frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbf{P}}$, we concern with whether the frame operator S commutes with the operator $E_{u(m)b}T_{u(n)a}$, where $T_{u(n)a}f(x) = f(x - u(n)a)$, $E_{u(m)b}f(x) = \chi(u(m)bx)f(x)$.

Lemma 3.1 *Let $g \in L^2(K)$ and $a, b \in K\backslash\{0\}$ be given, and assume that $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbf{P}}$ is a frame with frame operator S. Then $\forall\, m, n \in \mathbf{P}$, $SE_{u(m)b}T_{u(n)a} = E_{u(m)b}T_{u(n)a}S$.*

Proof Let $f \in L^2(K)$. It is apparent that

$$E_w T_v f(x) = \chi(wx) f(x-v) \quad \text{and} \quad T_v E_w f(x) = \overline{\chi(vw)} E_w T_v f(x). \tag{3.1}$$

By (3.1), we have

$$\begin{aligned} & S E_{u(m)b} T_{u(n)a} f \\ = & \sum_{m',\, n' \in \mathbf{P}} \langle E_{u(m)b} T_{u(n)a} f,\ E_{u(m')b} T_{u(n')a} g \rangle E_{u(m')b} T_{u(n')a} g \\ = & \sum_{m',\, n' \in \mathbf{P}} \langle f,\ \chi\{[u(m') - u(m)] b u(n) a\} E_{[u(m') - u(m)]b} T_{[u(n') - u(n)]a} g \rangle E_{u(m')b} T_{u(n')a} g. \end{aligned}$$

Performing the change of variables $u(m') \to u(m') + u(m)$, $u(n') \to u(n') + u(m)$ and using (3.1), we have

$$\begin{aligned} S E_{u(m)b} T_{u(n)a} f &= \sum_{m',\, n' \in \mathbf{P}} \overline{\chi\{u(m') b u(n) a\}}\, \langle f,\ E_{u(m')b} T_{u(n')a} g \rangle E_{[u(m') + u(m)]b} T_{[u(n') + u(n)]a} g \\ &= \sum_{m',\, n' \in \mathbf{P}} \langle f,\ E_{u(m')b} T_{u(n')a} g \rangle E_{u(m)b} T_{u(n)a} E_{u(m')b} T_{u(n')a} g \\ &= E_{u(m)b} T_{u(n)a} S f. \end{aligned}$$

This completes the proof.

As a consequence of Lemma 3.1, S^{-1} commutes with the operator $E_{u(m)b} T_{u(n)a}$. Consequently, $S^{-\frac{1}{2}}$ also commutes with $E_{u(m)b} T_{u(n)a}$. Thus, Lemma 3.1 has the following consequence.

Theorem 3.1 *If $\{E_{u(m)b} T_{u(n)a} g\}_{m,\, n \in \mathbf{P}}$ is a Gabor frame, then the canonical dual has Gabor structure and is given by $\{E_{u(m)b} T_{u(n)a} S^{-1} g\}_{m,\, n \in \mathbf{P}}$, where $g \in L^2(K)$ and $a, b \in K \backslash \{0\}$. The canonical tight frame associated with $\{E_{u(m)b} T_{u(n)a} g\}_{m,\, n \in \mathbf{P}}$ is $\{E_{u(m)b} T_{u(n)a} S^{-\frac{1}{2}} g\}_{m,\, n \in \mathbf{P}}$.*

In order to prove results in Sections 3 and 4, we need an identity which is stated in Lemma 3.4 latter. First, we have

Lemma 3.2 *Let $f,\ g \in L^2(K)$, $a, b \in K \setminus \{0\}$ and $k \in \mathbf{P}$ be given. Then the series*

$$\sum_{n \in \mathbf{P}} f(x - u(n)a) \cdot \overline{g(x - u(n)a - b^{-1}u(k))}, \quad x \in K \tag{3.2}$$

converges absolutely for a.e. $x \in K$, and it defines a function with period a, whose restriction to the set $G_a := \{x \in K : |x| \le |a|\}$, belongs to $L(G_a)$. In fact,

$$\sum_{n \in \mathbf{P}} |f(x - u(n)a) \cdot \overline{g(x - u(n)a - b^{-1}u(k))}\,| \in L(G_a).$$

Proof Since $f,\ T_{b^{-1}u(k)} g \in L^2(K)$, we have $f \cdot \overline{T_{b^{-1}u(k)} g} \in L(K)$. Thus

$$\int_{G_a} \sum_{n \in \mathbf{P}} |f(x - u(n)a) \cdot \overline{g(x - u(n)a - b^{-1}u(k))}\,| dx = \int_K |f(x) \cdot \overline{g(x - b^{-1}u(k))}\,| dx < \infty.$$

Consequently, $\sum\limits_{n\in \mathbf{P}} |f(x-u(n)a)\cdot \overline{g(x-u(n)a-b^{-1}u(k))}\,| < +\infty$, a.e. $x \in G_a$. Obviously, if the series in (3.2) converges for a.e. $x \in G_a$, then by Definition 2.2, it defines a function with period a. Hence the proof is complete.

Lemma 3.3 *Let $f, g \in L^2(K)$, $a, b \in K \setminus \{0\}$ and $n \in \mathbf{P}$ be given. We consider the function $F_n \in L(G_{b^{-1}})$ defined by $F_n(x) = \sum\limits_{k\in\mathbf{P}} f(x-b^{-1}u(k))\cdot \overline{g(x-u(n)a-b^{-1}u(k))}$. Then, for any $m \in \mathbf{P}$, $\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle = \int_{G_{b^{-1}}} F_n(x)\overline{\chi_m(bx)}\,dx$. In particular, the m-th Fourier coefficient of $F_n(x)$ with respect to the orthonormal basis $\{|b|^{\frac{1}{2}}\chi_m(bx)\}_{m\in\mathbf{P}}$ for $L^2(G_{b^{-1}})$ is $C_m = |b|^{\frac{1}{2}}\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle$.*

Proof We have already seen in Lemma 3.2 that the series defining F_n converges absolutely for a.e. $x \in K$. Now

$$\begin{aligned}
\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle &= \int_K f(x)\cdot \overline{g(x-u(n)a)}\cdot \overline{\chi_m(bx)}dx \\
&= \sum_{k\in \mathbf{P}} \int_{G_{b^{-1}}} f(x-b^{-1}u(k))\cdot \overline{g(x-u(n)a-b^{-1}u(k))}\cdot \overline{\chi_m(bx)}dx \\
&= \int_{G_{b^{-1}}} F_n(x)\cdot \overline{\chi_m(bx)}dx.
\end{aligned}$$

The proof is complete.

Given $a, b \in K \setminus \{0\}$ and $g \in L^2(K)$, we will often use the following functions defined by

$$G(x) = \sum_{n\in \mathbf{P}} |g(x-u(n)a)|^2, \quad x \in K, \tag{3.3}$$

$$H_k(x) = \sum_{n\in \mathbf{P}} g(x-u(n)a)\cdot \overline{g(x-u(n)a-b^{-1}u(k))}, \quad x \in K,\ k = 0, 1, \cdots . \tag{3.4}$$

It is obvious that $G(x)$ and $H_k(x)$ are bounded functions with period a, and $G(x) = H_0(x)$.

Lemma 3.4 *Let $a, b \in K \setminus \{0\}$ and $g \in L^2(K)$ be given. Suppose that f is a bounded measurable function with compact support. Then*

$$\begin{aligned}
&\sum_{m,\, n\in \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \\
=&\frac{1}{|b|}\int_K |f(x)|^2 G(x)dx + \frac{1}{|b|}\sum_{k\in \mathbf{P}\setminus\{0\}} \int_K \overline{f(x)}\cdot f(x-b^{-1}u(k))H_k(x)dx.
\end{aligned}\tag{3.5}$$

Proof Let $n \in \mathbf{P}$, and consider the b^{-1}-periodic function

$$F_n(x) = \sum_{k\in \mathbf{P}} f(x-b^{-1}u(k))\cdot \overline{g(x-u(n)a-b^{-1}u(k))}.$$

We have already given a general argument for F_n being well defined point-wise a.e., but our present assumptions give more. In fact, the compact support of f implies that $f(x-b^{-1}u(k))$ can be non-zero only for finitely many k-values. The number of k-values for which $f(x-b^{-1}u(k)) \neq 0$ is uniformly bounded, i.e., there is a constant C such that at most C k-values appear, independently of the chosen x. It follows that F_n is bounded, so $F_n \in L(G_{b^{-1}}) \bigcap L^2(G_{b^{-1}})$. In fact, even

$$\sum_{k\in \mathbf{P}} |f(x-b^{-1}u(k)) \cdot \overline{g(x-u(n)a-b^{-1}u(k))}\,| \in L(G_{b^{-1}}) \cap L^2(G_{b^{-1}}).$$

By Lemma 3.3, for all $m, n \in \mathbf{P}$,

$$\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle = \int_{G_{b^{-1}}} F_n(x)\overline{\chi_m(bx)}\,dx. \tag{3.6}$$

Since $\{|b|^{\frac{1}{2}}\chi_m(bx)\}_{m\in \mathbf{P}}$ is an orthonormal basis for $L^2(G_{b^{-1}})$, Parseval's theorem gives

$$\sum_{m\in \mathbf{P}} \Big| \int_{G_{b^{-1}}} F_n(x)\overline{\chi_m(bx)}\,dx \Big|^2 = \frac{1}{|b|} \int_{G_{b^{-1}}} |F_n(x)|^2 dx. \tag{3.7}$$

The assumption on f being a bounded measurable function with compact support will justify all interchanges of integration and summation in final calculation by the observation that

$$\sum_{k\in \mathbf{P}} \int_K |\overline{f(x)} \cdot f(x-b^{-1}u(k))| \cdot \sum_{k\in \mathbf{P}} |g(x-u(n)a) \cdot \overline{g(x-u(n)a-b^{-1}u(k))}\,|dx < \infty.$$

Now in virtue of (3.6) and (3.7), we have

$$\sum_{m,\,n\in \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 = \sum_{m,\,n\in \mathbf{P}} \Big| \int_{G_{b^{-1}}} F_n(x)\overline{\chi_m(bx)}dx \Big|^2 = \frac{1}{|b|} \sum_{n\in \mathbf{P}} \int_{G_{b^{-1}}} |F_n(x)|^2 dx.$$

Writing

$$|F_n(x)|^2 = F_n(x)\overline{F_n(x)} = \sum_{k\in \mathbf{P}} \overline{f(x-b^{-1}u(k))} \cdot g(x-u(n)a-b^{-1}u(k))F_n(x),$$

we continue with

$$\begin{aligned}
&\sum_{m,\,n\in \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \\
=&\frac{1}{|b|} \sum_{n\in \mathbf{P}} \int_{G_{b^{-1}}} \sum_{k\in \mathbf{P}} \overline{f(x-b^{-1}u(k))} \cdot g(x-u(n)a-b^{-1}u(k)) \cdot F_n(x)dx \\
=&\frac{1}{|b|} \sum_{n\in \mathbf{P}} \int_K \overline{f(x)} \cdot g(x-u(n)a) \cdot F_n(x)dx \\
=&\frac{1}{|b|} \int_K |f(x)|^2 G(x)dx + \frac{1}{|b|} \sum_{k\in \mathbf{P}\backslash\{0\}} \int_K \overline{f(x)} \cdot f(x-b^{-1}u(k))H_k(x)dx.
\end{aligned}$$

This is as desired.

4 Necessary Conditions for Gabor Frame for $L^2(K)$

We now move to the question about how to obtain Gabor frames $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ for $L^2(K)$. One of the most fundamental results says that the product ab decides whether it is possible for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ to be a frame for $L^2(K)$.

Theorem 4.1 *Let $g \in L^2(K)$ and $a, b \in K \setminus \{0\}$ be given.*

(i) If $|ab| > 1$, then $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is not a frame for $L^2(K)$.

(ii) If $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a frame, then $|ab| = 1$ if and only if $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a Riesz basis.

Proof Assume that $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a Gabor frame for $L^2(K)$. We begin with an observation concerning the canonical tight frame $\{E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\}_{m,\,n\in \mathbf{P}}$ associated with $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$. Firstly, we apply Lemma 3.4 on the frame $\{E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\}_{m,\,n\in \mathbf{P}}$. For an arbitrary bounded and measurable function f with support in a subgroup $G_{b^{-1}}$, we obtain that

$$\int_K |f(x)|^2 dx = \sum_{m,\,n\in \mathbf{P}} |\langle f, E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\rangle|^2 = \frac{1}{|b|}\int_K |f(x)|^2 \sum_{n\in \mathbf{P}} |S^{-\frac{1}{2}}g(x-u(n)a)|^2 dx.$$

Thus, this gives that $\sum\limits_{n\in \mathbf{P}} |S^{-\frac{1}{2}}g(x-u(n)a)|^2 = |b|$, a.e., $x \in K$, consequently, $\|S^{-\frac{1}{2}}g\|^2 = \int_{G_a} \sum\limits_{n\in \mathbf{P}} |S^{-\frac{1}{2}}g(x-u(n)a)|^2 dx = |ab|$.

(i) We have to prove that $|ab| \le 1$ for the arbitrary given frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$. Since $\{E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\}_{m,\,n\in \mathbf{P}}$ is a tight frame with bound 1, the fact Proposition 3.1(1) implies that $\|S^{-\frac{1}{2}}g\| \le 1$. It follows from $\|S^{-\frac{1}{2}}g\|^2 = |ab|$ that $|ab| \le 1$ as desired.

(ii) We have to prove that a frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a Riesz basis if and only if $|ab| = 1$. Firstly, assume that $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a Riesz basis. Then by Proposition 3.2, $\{E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\}_{m,\,n\in \mathbf{P}}$ is also a Riesz basis having bound $C = \widetilde{C} = 1$. Consequently, this implies that $\|S^{-\frac{1}{2}}g\| = 1$. Since we have already given a completely general proof for the equality $\|S^{-\frac{1}{2}}g\|^2 = |ab|$, we have $|ab| = 1$ as desired.

For the other implication, we now assume that $|ab| = 1$. Then $\|S^{-\frac{1}{2}}g\|^2 = |ab| = 1$, and therefore $\|E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\|^2 = 1$ for all $m, n \in \mathbf{P}$. Using Proposition 3.1(2), we conclude that $\{E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\}_{m,\,n\in \mathbf{P}}$ is an orthonormal basis for $L^2(K)$. Notice that $S^{\frac{1}{2}}$ is linearly bounded and invertible. Hence by Definition 3.1(iv), $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}} = \{S^{\frac{1}{2}}E_{u(m)b}T_{u(n)a}S^{-\frac{1}{2}}g\}_{m,\,n\in \mathbf{P}}$ is a Riesz basis.

Remark 4.1 Theorem 4.1 shows that it is only possible for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ to be a frame if $|ab| \le 1$, and the frame is over-complete if $|ab| < 1$.

Remark 4.2 One can actually prove a stronger result that in Theorem 4.1(i): when $|ab| > 1$, the family $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ can not even be complete in $L^2(K)$.

The assumption $|ab| \le 1$ is not enough for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ to be a frame, even if $g \neq 0$. For example, let $\Phi_{\mathfrak{P}^2}$ be the characteristic function on $\mathfrak{P}^2$, $a = \mathfrak{p}$ and $b = e_0$, the unit of multiplication of K. Then the sequence of functions $\{E_{u(m)b}T_{u(n)a}\Phi_{\mathfrak{P}^2}\}_{m,\,n\in \mathbf{P}}$ is not complete in $L^2(K)$ and can not form a frame. The following Theorem 4.2 gives a necessary condition for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ to be a frame for $L^2(K)$.

Theorem 4.2 *Let $g \in L^2(K)$ and $a, b \in K\setminus\{0\}$ be given. Assume that $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a frame with bounds A and B. Then*

$$|b|A \le G(x) \le |b|B, \quad a.e., \tag{4.1}$$

where $G(x)$ is the same as in (3.3). More precisely, if the upper bound in (4.1) is violated, then $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is not a Bessel sequence; if the lower bound is violated, then $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ does not satisfy the lower frame bound condition.

Proof The proof is by contradiction. Assume that the upper condition in (4.1) is not true. Then there exists a measurable set $\Delta \subseteq K$ with positive measure such that $G(x) > |b|B$ on Δ. We can assume that Δ is contained in a ball Γ with diameter of $|b|^{-1}$. Set $\Delta_0 = \{x \in \Delta : G(x) \geq 1 + |b|B\}$ and $\Delta_k = \{x \in \Delta : \frac{1}{k+1} + |b|B \leq G(x) < \frac{1}{k} + |b|B\},\ k = 1, 2, \cdots$. Then we obtain a partition of Δ into disjoint measurable sets. At least one of them, say, Δ_l, has positive measure. Now consider the function $f = \Phi_{\Delta_l}$, the characteristic function on Δ_l, and note that $\|f\|^2 = |\Delta_l|$. For $n \in \mathbf{P},\ f \cdot T_{u(n)a}g$ has support in Δ_l. Since Δ_l is contained in a ball Γ with diameter of $|b|^{-1}$ and $\{|b|^{\frac{1}{2}}\chi_m(bx)\}_{m\in\mathbf{P}}$ constitutes an orthonormal basis for $L^2(\Gamma)$ for every ball Γ of diameter $|b|^{-1}$, we have

$$\sum_{m\in \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 = \sum_{m\in \mathbf{P}} |\langle f\,\overline{T_{u(n)a}g},\ E_{u(m)b}\rangle|^2 = \frac{1}{|b|}\int_K |f(x)|^2|g(x-u(n)a)|^2dx.$$

Thus

$$\sum_{m,\,n\in\ \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 = \frac{1}{|b|}\int_{\Delta_l} |f(x)|^2G(x)dx \geq \Big(B + \frac{1}{|b|(l+1)}\Big)\|f\|^2.$$

Consequently, B can not be an upper frame bound for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$. A similar proof shows that if the lower condition in (4.1) is violated, then A can not be a lower frame bound for $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$. The proof is complete.

Corollary 4.1 *A function g generating the frame $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is necessarily bounded.*

5 Sufficient Conditions for Gabor Frame for $L^2(K)$

Two sufficient conditions for Gabor frame for $L^2(K)$ will be established in this section. The first sufficient condition is as follows.

Theorem 5.1 *Let $g \in L^2(K)$ and $a, b \in K \setminus \{0\}$ be given. Suppose that there are constants $A, B > 0$ such that*

$$A \leq G(x) \leq B, \quad a.e.\ x \in K, \tag{5.1}$$

$$\sum_{k\in\ \mathbf{P}\setminus\{0\}} \|H_k\|_\infty < A. \tag{5.2}$$

Then $\{E_{u(m)b}T_{u(n)a}g\}_{m,\,n\in \mathbf{P}}$ is a Gabor frame for $L^2(K)$.

Proof We are going to process (3.5) in Lemma 3.4. Note that

$$\begin{aligned}
&\frac{1}{|b|}\sum_{k\in\ \mathbf{P}\setminus\{0\}} \int_K |f(x)| \cdot |f(x-b^{-1}u(k))| \cdot |H_k(x)|dx \\
&\leq \frac{1}{|b|}\sum_{k\in\ \mathbf{P}\setminus\{0\}} \Big\{\int_K |f(x)|^2|H_k(x)|dx\Big\}^{\frac{1}{2}} \times \Big\{\int_K |f(x-b^{-1}u(k))|^2|H_k(x)|dx\Big\}^{\frac{1}{2}} \\
&\leq \frac{1}{|b|}\Big\{\int_K |f(x)|^2\Big[\sum_{k\in\ \mathbf{P}\setminus\{0\}} |H_k(x)|\Big]dx\Big\}^{\frac{1}{2}} \times \Big\{\int_K |f(x-b^{-1}u(k))|^2\Big[\sum_{k\in\ \mathbf{P}\setminus\{0\}} |H_k(x)|\Big]dx\Big\}^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{1}{|b|}\Big\{\int_K |f(x)|^2\Big[\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big]dx\Big\}^{\frac{1}{2}} \times \Big\{\int_K |f(x)|^2\Big[\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x+b^{-1}u(k))|\Big]dx\Big\}^{\frac{1}{2}}$$

$$\leq \frac{1}{|b|}\Big\{\int_K |f(x)|^2\Big[\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big]dx\Big\}^{\frac{1}{2}} \times \Big\{\int_K |f(x)|^2\Big[\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big]dx\Big\}^{\frac{1}{2}}$$

$$\leq \frac{1}{|b|}\sum_{k\in \mathbf{P}\setminus\{0\}}\int_K |f(x)|^2|H_k(x)|dx \leq \frac{1}{|b|}\sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty \|f\|^2. \tag{5.3}$$

Hence, it follows from (3.5) and (5.3) that

$$\sum_{m,\, n\in\, \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \leq \frac{1}{|b|}\|f\|^2\Big\{\|G\|_\infty + \sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty\Big\}$$

and

$$\sum_{m,\, n\in\, \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \geq \frac{1}{|b|}\|f\|^2\Big\{\|G\|_\infty - \sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty\Big\}.$$

Consequently, by (5.1) and (5.2) we have

$$\begin{aligned}\frac{1}{|b|}\Big\{A - \sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty\Big\}\|f\|^2 &\leq \sum_{m,\, n\in\, \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \\ &\leq \frac{1}{|b|}\Big\{B + \sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty\Big\}\|f\|^2.\end{aligned}$$

Thus, $\{E_{u(m)b}T_{u(n)a}g\}_{m,\, n\in\, \mathbf{P}}$ is a frame for $L^2(K)$, because the set of f considered is dense in $L^2(K)$. The proof is complete.

Next, we state the second sufficient condition which is more general than the first one.

Theorem 5.2 *Let $g \in L^2(K)$ and $a, b \in K \setminus \{0\}$. Suppose that*

$$B := \frac{1}{|b|}\sup_{x\in G_a}\sum_{k\in\, \mathbf{P}} |H_k(x)| < +\infty. \tag{5.4}$$

Then $\{E_{u(m)b}T_{u(n)a}g\}_{m,\, n\in \mathbf{P}}$ is a Bessel sequence with upper frame bound B. If in addition,

$$A := \frac{1}{|b|}\inf_{x\in G_a}\Big\{G(x) - \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big\} > 0, \tag{5.5}$$

then $\{E_{u(m)b}T_{u(n)a}g\}_{m,\, n\in \mathbf{P}}$ is a frame for $L^2(K)$ with bounds $A,\ B$.

Proof Consider a function $f \in L^2(K)$ which is continuous and has compact support. By Lemma 3.4, we want to estimate the second term of right side in (3.5). For $k \in \mathbf{P}$, note that

$$H_k(x) = \sum_{n\in\, \mathbf{P}} T_{u(n)a}g(x)\cdot \overline{T_{u(n)a+b^{-1}u(k)}g(x)}. \tag{5.6}$$

Hence, we have

$$\begin{aligned}\sum_{k\in \mathbf{P}\setminus\{0\}} |T_{-b^{-1}u(k)}H_k(x)| &= \sum_{k\in \mathbf{P}\setminus\{0\}} \Big|T_{-b^{-1}u(k)}\sum_{n\in\, \mathbf{P}} T_{u(n)a}g(x)\cdot \overline{T_{u(n)a+b^{-1}u(k)}g(x)}\Big| \\ &= \sum_{k\in \mathbf{P}\setminus\{0\}} \Big|\sum_{n\in\, \mathbf{P}} T_{u(n)a-b^{-1}u(k)}g(x)\cdot \overline{T_{u(n)a}g(x)}\Big|.\end{aligned}$$

Replacing $u(k)$ with $-u(k)$ (which is allowed), we obtain

$$\begin{aligned}\sum_{k\in \mathbf{P}\setminus\{0\}} |T_{-b^{-1}u(k)}H_k(x)| &= \sum_{k\in \mathbf{P}\setminus\{0\}} \Big|\sum_{n\in \mathbf{P}} T_{u(n)a+b^{-1}u(k)}g(x)\,\overline{T_{u(n)a}g(x)}\Big| \\ &= \sum_{k\in \mathbf{P}\setminus\{0\}} \Big|\sum_{n\in \mathbf{P}} \overline{T_{u(n)a+b^{-1}u(k)}g(x)}\cdot T_{u(n)a}g(x)\Big| \\ &= \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|.\end{aligned}$$

So, by Cauchy-Schwartz's inequality,

$$\begin{aligned}&\Big|\sum_{k\in \mathbf{P}\setminus\{0\}} \int_K \overline{f(x)}\cdot f(x-b^{-1}u(k))H_k(x)dx\Big| \\ \le& \sum_{k\in \mathbf{P}\setminus\{0\}} \int_K |f(x)|\cdot|T_{b^{-1}u(k)}f(x)|\cdot|H_k(x)|dx \\ \le& \sum_{k\in \mathbf{P}\setminus\{0\}} \Big\{\int_K |f(x)|^2|H_k(x)|dx\Big\}^{\frac12}\Big\{\int_K |T_{b^{-1}u(k)}f(x)|^2|H_k(x)|dx\Big\}^{\frac12} \\ \le& \Big\{\sum_{k\in \mathbf{P}\setminus\{0\}} \int_K |f(x)|^2|H_k(x)|dx\Big\}^{\frac12}\Big\{\sum_{k\in \mathbf{P}\setminus\{0\}} \int_K |T_{b^{-1}u(k)}f(x)|^2|H_k(x)|dx\Big\}^{\frac12} \\ =& \Big\{\int_K |f(x)|^2 \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|dx\Big\}^{\frac12}\Big\{\int_K |f(x)|^2 \sum_{k\in \mathbf{P}\setminus\{0\}} |T_{-b^{-1}u(k)}H_k(x)|dx\Big\}^{\frac12} \\ =& \int_K |f(x)|^2 \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|dx. \end{aligned}\tag{5.7}$$

By (3.5), (5.7) and the condition (5.4), we have

$$\sum_{m,\,n\in \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \le \frac{1}{|b|}\int_K |f(x)|^2\cdot\Big\{G(x)+\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big\}dx \le B\|f\|^2.$$

Since this estimate holds on a dense subset of $L^2(K)$, it holds on $L^2(K)$. This proves the first part. If in addition (5.5) is satisfied, we again consider a continuous function f with compact support, and obtain that

$$\sum_{m,\,n\in \mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 \ge \frac{1}{|b|}\int_K |f(x)|^2\cdot\Big\{G(x)-\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big\}dx \ge A\|f\|^2.$$

Thus, the proof is complete.

Remark 5.1 Let us compare Theorems 5.1 and 5.2. Using the definitions of $G(x)$ and $H_k(x)$, we see that the conditions in Theorem 5.1 imply that

$$\sup \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)| \le \sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty < \inf G(x).$$

So, the conditions in Theorem 5.2 hold.

Remark 5.2 The advantage of Theorem 5.2 is that we compare the functions $G(x)$ and $\sup \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|$ point-wise rather than requiring that the supremum of $\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|$ is smaller than the infimum of $G(x)$. For a given function $g \in L^2(K)$ and a fixed element a in K, this will usually imply that we can prove that $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in \mathbf{P}}$ is a frame for a larger range of the parameter b. The example in next section demonstrates this in practice.

Corollary 5.1 *Let $a, b \in K \setminus \{0\}$ be given. Suppose that $g \in L^2(K)$ is of support in a ball with diameter of $|b|^{-1}$, and the function $G(x)$ satisfies (5.1) for $A, B > 0$. Then $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbf{P}}$ is a frame for $L^2(K)$ with bounds A, B. The frame operator and its inverse are given by*

$$Sf = \frac{G(\cdot)}{|b|}f \quad \text{and} \quad S^{-1}f = \frac{|b|}{G(\cdot)}f, \quad f \in L^2(K),$$

respectively.

Proof $\{E_{u(m)b}T_{u(n)a}g\}_{m,n\in\mathbf{P}}$ is a frame by Theorem 5.2, because

$$H_k(x) = \sum_{n\in\mathbf{P}} g(x - u(n)a) \cdot \overline{g\Big(x - u(n)a - \frac{u(k)}{b}\Big)} = 0 \quad \text{for all } k \in \mathbf{P} \setminus \{0\}.$$

Given a continuous function f with compact support, Lemma 3.4 implies that

$$\langle Sf, f\rangle = \sum_{m,n\in\mathbf{P}} |\langle f,\ E_{u(m)b}T_{u(n)a}g\rangle|^2 = \frac{1}{|b|}\int_K |f(x)|^2 G(x)dx. \tag{5.8}$$

By continuity of S, (5.8) holds for all $f \in L^2(K)$. It follows from Proposition 3.3 that S acts by multiplication with the function $\frac{G(\cdot)}{|b|}$. The proof is complete.

6 An Example

Let $a = b = e_0$, the unit of multiplication of K. Define

$$g(x) = \begin{cases} 1 + q|x|, & x \in \mathfrak{D}, \\ \dfrac{1}{q(q-1)}(1 + q|x'|), & x \in \mathfrak{P}^{-1} \setminus \mathfrak{D}, \\ 0, & \text{otherwise}, \end{cases}$$

where we use the expression of $x \in \mathfrak{P}^{-1} \setminus \mathfrak{D}$ that $x = x' - u(l)$ with $x' \in \mathfrak{D}$ and $l \in \{1, 2, \cdots, q-1\}$.

Consider for $n, k \in \mathbf{P}$, the function $x \longrightarrow g(x - u(n))g(x - u(n)a - u(k))$ for $x \in \mathfrak{D}$. Due to the compact support of g, it must be zero if $n \notin \{0, 1, \cdots, q-1\}$ or $k \notin \{0, 1, \cdots, q-1\}$. So we work out that

$$G(x) = \sum_{n\in\mathbf{P}} |g(x - u(n))|^2 = \sum_{n=0}^{q-1} |g(x - u(n))|^2 = \Big(1 + \frac{1}{q^2(q-1)}\Big)(1 + q|x|)^2, \quad x \in \mathfrak{D},$$

and for $k \in \{1, 2, \cdots, q-1\}$,

$$H_k(x) = \sum_{n=0}^{q-1} g(x - u(n))g(x - u(n) - u(k)) = \Big(\frac{2}{q(q-1)} + \frac{q-2}{q^2(q-1)^2}\Big)(1 + q|x|)^2, \quad x \in \mathfrak{D}.$$

Thus

$$\sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)| = \sum_{k=1}^{q-1} H_k(x) = \Big(\frac{2}{q} + \frac{q-2}{q^2(q-1)}\Big)(1+q|x|)^2, \quad x \in \mathfrak{D}.$$

Therefore,

$$\inf_{x\in\mathfrak{D}} \Big[G(x) - \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big] = \frac{q-2}{q} + \frac{3-q}{q^2(q-1)} > 0,$$

$$\sup_{x\in\mathfrak{D}} \Big[G(x) + \sum_{k\in \mathbf{P}\setminus\{0\}} |H_k(x)|\Big] = \Big(1 + \frac{2}{q} + \frac{1}{q^2}\Big)(1+q)^2 < \infty.$$

Theorem 5.2 now shows that $\{E_{u(m)}T_{u(n)}g\}_{m,\,n\in \mathbf{P}}$ is a frame for $L^2(K)$ with bounds

$$A = \frac{q-2}{q} + \frac{3-q}{q^2(q-1)} \quad \text{and} \quad B = \Big(1 + \frac{2}{q} + \frac{1}{q^2}\Big)(1+q)^2.$$

But

$$\inf_{x\in\mathfrak{D}} G(x) = 1 + \frac{1}{q^2(q-1)} < 2,$$

$$\sum_{k\in \mathbf{P}\setminus\{0\}} \|H_k\|_\infty = \Big(\frac{2}{q} + \frac{q-2}{q^2(q-1)}\Big)(1+q)^2 > 2(1+q).$$

So the condition of Theorem 5.1 is not satisfied.

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