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The Hausdorff Dimension of Sections***

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Abstract The notion of finite-type open set condition is defined to calculate the Hausdorff dimensions of the sections of some self-similar sets, such as the dimension of intersection of the Koch curve and the line $x = a$ with $a \in \mathbb{Q}$.

Keywords Hausdorff dimension, Self-similar set, Section, Open set condition
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1 Introduction

Let $\{S_i\}_{i=1}^m$ be an iterated function system (IFS) of contractive similitudes on $\mathbb{R}^n$ with the same contraction ratio $\rho \in (0, 1)$ defined by

$$S_i(x) = \rho R_i x + b_i, \quad 1 \le i \le m, \tag{1.1}$$

where $b_i \in \mathbb{R}^n$ and R_i is an $n \times n$ orthogonal matrix for each i. Then there exists a unique non-empty compact set $E \subset \mathbb{R}^n$ such that $E = \bigcup\limits_{i=1}^m S_i(E)$ (see [4]). This set E is called the attractor of the IFS. Assume that (1.1) satisfies the open set condition (OSC), i.e., there exists a non-empty open set V such that

$$\bigcup_i S_i(V) \subset V, \quad S_i(V) \cap S_j(V) = \emptyset, \quad \forall\, i \ne j. \tag{1.2}$$

Suppose L is an $(n-1)$-plane, let

$$\Gamma_L = \{S_{i_1\cdots i_k}^{-1}(L) : S_{i_1\cdots i_k}^{-1}(L) \cap E \ne \emptyset, i_1 \cdots i_k \in \{1, \cdots, m\}^k, k \ge 1\}.$$

We say that the section $E \cap L$ is of finite type, if $\#\Gamma_L < \infty$.

Let $\Delta = \{L : L \cap E \ne \emptyset, L \cap V \ne \emptyset\}$. Notice that $E \cap L = \bigcup\limits_{i=1}^m [S_i(E) \cap L]$. Let $\Lambda = \Big\{L : E \cap L = \bigcup\limits_{i:\, S_i^{-1}(L) \in \Delta} [S_i(E) \cap L]\Big\}$. Here $S_i^{-1}(L) \in \Delta$ if and only if $S_i(V) \cap L \ne \emptyset$ and $S_i(E) \cap L \ne \emptyset$. Write $\Omega^0(L) = \{L\}$. By induction for every $k \ge 0$, let

$$\Omega^{k+1}(L) = \{S_i^{-1}(L') : L' \in \Omega^k(L) \text{ and } S_i^{-1}(L') \in \Delta\}.$$

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Definition 1.1 *Assume that the IFS* (1.1) *with attractor E satisfies the OSC* (1.2)*. Suppose L is an $(n-1)$-plane. We say that the section $E \cap L$ holds the finite-type open set condition (FOSC), if*

$$\bigcup_{k=0}^{\infty} \Omega^k(L) \subset \Lambda \quad \textit{and} \quad \# \bigcup_{k=0}^{\infty} \Omega^k(L) < \infty. \tag{1.3}$$

Assume that $E \cap L$ satisfies the FOSC. We can define the transition matrix $A(L)$ on $\Sigma_L = \bigcup_{k=0}^{\infty} \Omega^k(L)$ such that the entry of $A(L)$ in the row w.r.t. L_1 and the column w.r.t. L_2 is

$$\#\{i : \ 1 \le i \le m, \ S_i^{-1}(L_1) = L_2\}. \tag{1.4}$$

The following is the main result about the FOSC.

Theorem 1.1 *Suppose E is the attractor of the IFS* (1.1) *satisfying the OSC, and L is an $(n-1)$-plane. If $E \cap L$ satisfies the FOSC, then* $\dim_H(E \cap L) = \frac{\log \lambda}{-\log \rho}$, *where λ is the spectral radius of $A(L)$.*

Notice that in the above theorem, we do not need the irreducible condition on the transition matrix $A(L)$.

We organize the paper as follows. In Section 2, we prove the above theorem by using a graph-directed construction satisfying the OSC, although $A(L)$ may not be irreducible. In Section 3, we obtain Proposition 3.1 for sections of finite type, and verify the FOSC for some special sections, for example, the sections of Sierpinski carpet and the intersection of the Koch curve with the line $x = c$ ($c \in \mathbb{Q} \cap (0,1)$).

2 Proof of Theorem 1.1

We will recall some important concepts and results related to the graph-directed sets (see [1–7]). Assume that there exist N complete metric spaces $\{(X_i, d_{X_i})\}_{1 \le i \le N}$ isometric to $\mathbb{R}^n$, where $N \in \mathbb{N}$. Suppose that $\{1, \cdots, N\}$ is the vertex set of a directed graph G. For any $1 \le i, j \le N$, let $\Gamma_{i,j} = \{e' : \ e' \in G$ is a directed edge from i to $j\}$. For any edge $e \in \Gamma_{i,j}$, there is a corresponding similitude $T_e : X_j \to X_i$ with the similarity ratio $\rho_e \in (0,1)$, that is, $d_{X_i}(T_e(x), T_e(y)) = \rho_e d_{X_j}(x, y)$, $\forall\, x, y \in X_j$. The compact sets $\{E_i\}_{i=1}^N$ are called the graph-directed sets, if for each i,

$$E_i = \bigcup_j \bigcup_{e \in \Gamma_{i,j}} T_e(E_j).$$

Fixed $s \ge 0$, we will obtain an $N \times N$ matrix $B(s) = (b_{ij})_{1 \le i,j \le N}$ with the entry $b_{ij} = \sum_{e \in \Gamma_{ij}} (\rho_e)^s$, where ρ_e is the contraction ratio of T_e .

We say that a family $\{T_e\}_{e \in G}$ of similitudes satisfies the OSC, if there exists a family $\{U_i\}_{i=1}^N$ of non-empty open sets such that for each i,

(1) $\bigcup_{e \in \Gamma_{i,j}} T_e(U_j) \subset U_i$;

(2) $\bigcup_{e \in \Gamma_{i,j}} T_e(U_j)$ is a disjointed union.

We say that the family $\{T_e\}_{e\in G}$ satisfies the strong open set condition (see [2, 8]), if the above conditions (1), (2) and the following condition hold:

(3) $U_i \cap E_i \neq \emptyset$ for every i.

A directed graph G is said to be strongly connected provided that for any vertices x, y of G there is a directed path from x to y. We say that the graph-directed construction is irreducible, if the corresponding graph is strongly connected.

In fact, for the graph-directed sets in the Euclidean space, the strong open set condition is equivalent to the open set condition (see [6]). Then according to [2, Theorem 3.14], we have the following lemma.

Lemma 2.1 *Suppose the graph-directed construction is irreducible and the family* $\{T_e\}_{e\in G}$ *satisfies the open set condition. Let* $\dim E_i = t$*. Then* t *is the unique value such that the spectral radius of* $B(t)$ *equals to* 1*, where* $B(t)$ *is defined as above.*

Remark 2.1 In the preceding lemma, if all the similarity ratios are the same, i.e., $\rho_e = \rho$, then there exists a nonnegative irreducible integer matrix $\mathcal{C}$ such that $B(s) = \rho^s \mathcal{C}$. Furthermore for each i, we get that $\dim E_i = \frac{-\log \rho(\mathcal{C})}{\log \rho}$, where $\rho(\mathcal{C})$ is the spectral radius of the nonnegative matrix $\mathcal{C}$.

Graph-directed Construction Let Σ_L be the vertex set. For any vertex $L \in \Sigma_L$, we assign a natural space L isometric to $\mathbb{R}^{n-1}$. For any $L_1, L_2 \in \Sigma_L$ and i with $S_i^{-1}(L_1) = L_2$, we define an edge from vertex L_1 to vertex L_2, whose corresponding similitude from space L_2 to space L_1 is $S_i|_{L_2} : L_2 \to L_1$.

Proof of Theorem 1.1

Step 1 $\{E \cap L'\}_{L'\in\Sigma_L}$ are graph-directed sets satisfying the OSC.

Suppose V is a non-empty open set of $\mathbb{R}^n$ satisfying

$$\bigcup_i S_i(V) \subset V, \quad S_i(V) \cap S_j(V) = \emptyset, \quad \forall\, i \neq j.$$

By the definition of the FOSC, we obtain a family $\{L'\}_{L'\in\Sigma_L}$ of metric spaces isometric to $\mathbb{R}^{n-1}$.

It follows from the definition of the FOSC that

$$E \cap L' = \bigcup_{S_i^{-1}(L')\in\Sigma_L} S_i[S_i^{-1}(L') \cap E]. \tag{2.1}$$

Given $L' \in \Sigma_L$, by the definition of the FOSC, we get a non-empty open subset $V_{L'} = V \cap L'$ of L'. Because $\bigcup\limits_{i=1}^{m} [S_i(V) \cap L] \subset V \cap L$, the disjoint union

$$\bigcup_{S_i^{-1}(L')\in\Sigma_L} S_i[S_i^{-1}(L') \cap V] = \bigcup_{S_i^{-1}(L')\in\Sigma_L} [S_i(V) \cap L'] \subset V \cap L'.$$

Therefore, the open set condition holds for $\{E \cap L'\}_{L'\in\Sigma_L}$.

We write the transition matrix $A(L) = (a_{L_1,L_2})_{L_1,L_2\in\Sigma_L}$. And $a^{(n)}_{L',L''}$ the entry of $[A(L)]^n$ in the row w.r.t. L' and the column w.r.t. L''.

Step 2 $\dim_H(E\cap L) \le \frac{\log\lambda}{-\log\rho}$.

Notice that

$$E\cap L = \bigcup_{S^{-1}_{i_1\cdots i_k}(L)\in\Sigma_L} S_{i_1\cdots i_k}[S^{-1}_{i_1\cdots i_k}(L)\cap E], \tag{2.2}$$

where the diameter of $S_{i_1\cdots i_k}[S^{-1}_{i_1\cdots i_k}(L)\cap E]$ is less than $\rho^k|E|$.

For any $\delta>0$ and integer k large enough with $\rho^k<\delta$, we have

$$\begin{aligned}\mathcal{H}^s_\delta(E\cap L) &\le \sum_{L_{i_1}\cdots L_{i_k}} a_{L,L_{i_1}}a_{L_{i_1},L_{i_2}}\cdots a_{L_{i_{k-1}},L_{i_k}}(\rho^k|E|)^s = \sum_{L'\in\Sigma_L} a^{(k)}_{L,L'}(\rho^k|E|)^s\\ &\le \sum_{L',L''\in\Sigma_L} a^{(k)}_{L',L''}(\rho^k|E|)^s = \|A(L)^k\|(\rho^k|E|)^s,\end{aligned}$$

where the norm $\|B\| = \|(b_{ij})_{ij}\| = \sum_{i,j}|b_{ij}|$.

Let λ be the spectral radius of $A(L)$. Then

$$\lambda = \lim_{k\to\infty}\|A(L)^k\|^{1/k}.$$

Letting $\delta\to 0$, we have $k\to\infty$, and thus

$$\dim_H(E\cap L) \le \lim_{k\to\infty}\frac{\log\|A(L)^k\|}{-k\log\rho} = \frac{\log\lambda}{-\log\rho}.$$

Step 3 $\dim_H(E\cap L) \ge \frac{\log\lambda}{-\log\rho}$.

Under some permutation of Σ_L, we write $A(L)$ in the following shape

$$A(L) = \begin{pmatrix} A_{11} & & \\ \vdots & \ddots & \\ A_{l1} & \cdots & A_{ll}\end{pmatrix},$$

where A_{ii} is an irreducible square matrix for each $1\le i\le l$.

Then the maximal spectral radius of A_{ii} $(1\le i\le l)$ equals to λ, the spectral radius of $A(L)$. Without loss of generality, we assume that the spectral radius of A_{jj} is λ for some j and let Σ_j be the irreducible branch with respect to A_{jj}.

Therefore for $L'\in\Sigma_j$, we have

$$\bigcup_{S_i^{-1}(L')\in\Sigma_j} S_i[E\cap S_i^{-1}(L')] \subset E\cap L'. \tag{2.3}$$

Hence $E\cap L'$ includes $a_{L',L''}$ copies of $E\cap L''$ with contraction ratio ρ.

Suppose $\{B_{L'}\}_{L'\in\Sigma_j}$ are graph-directed sets according to Σ_j and A_{jj} with

$$\bigcup_{S_i^{-1}(L')\in\Sigma_j} S_i[B_{S_i^{-1}(L')}] = B_{L'}, \tag{2.4}$$

where $B_{L'}$ exactly includes $a_{L',L''}$ copies of $B_{L''}$ with similarity ratio ρ whenever $L', L'' \in \Sigma_j$. In the same way, the open set condition holds for the graph-directed sets. By the irreducibility of A_{jj} and Remark 2.1, we have

$$\dim B_{L'} = \frac{-\log \lambda}{\log \rho}. \tag{2.5}$$

By (2.3) and (2.4), we have $B_{L'} \subset E \cap L'$. Then

$$\dim_H E \cap L' \geq \dim_H B_{L'}. \tag{2.6}$$

We need only to prove

$$\dim_H E \cap L \geq \dim_H E \cap L'. \tag{2.7}$$

In fact, since

$$E \cap L = \bigcup_{S^{-1}_{i_1 \cdots i_k}(L) \in \Sigma_L} S_{i_1 \cdots i_k}[S^{-1}_{i_1 \cdots i_k}(L) \cap E] \tag{2.8}$$

and $L' \in \bigcup_k \Omega^k(L)$, there is one copy of $L' \cap E$ contained in $E \cap L$, and then inequality (2.7) holds. Using (2.5)–(2.7), we have

$$\dim(E \cap L) \geq \frac{-\log \lambda}{\log \rho}.$$

3 Applications

In this section we will give some examples satisfying finite-type open set condition and calculate the Hausdorff dimension.

Suppose $\{S_i\}_{i=1}^m$ are similitudes defined by (1.1) with its attractor E. For $c \in \mathbb{R}$, $K = (k_1, \cdots, k_n) \in \mathbb{R}^n$ with $K \neq 0$, the $(n-1)$-plane $L_{K,c}$ is defined by

$$L_{K,c} = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid k_1x_1 + \cdots + k_nx_n = c\ \}.$$

For the attractor E, set $E_{K,c} = E \cap L_{K,c}$, the intersection of the self-similar set E with the hyperplane $L_{K,c}$. Given a sequence $i_1 \cdots i_l \in \{1, \cdots, m\}^l$, let

$$L_{K_l,c_l} = S^{-1}_{i_1 \cdots i_l}(L_{K,c}).$$

Since $S^{-1}_{i_{l+1}}(L_{K_l,c_l}) = L_{K_{l+1},c_{l+1}}$, there are the following recurrence relations:

$$K_{l+1} = K_l R_{i_{l+1}} \quad \text{and} \quad c_{l+1} = \rho^{-1}c_l - \rho^{-1}K_l b_{i_{l+1}}.$$

Remark 3.1 If the set $\{R_1, \cdots, R_m\}$ is contained in a finite subgroup of $O(n)$, then $\{K_l\}_{l=0}^\infty$ is a finite set. At this time we need only to check whether $\{c_l\}_l$ is discrete or not. In fact, if the section $E \cap L_{K,c}$ is non-empty, then the corresponding c shall be bounded.

Proposition 3.1 *Suppose R_i is identical transformation for each i, $\rho^{-1} \in \mathbb{N}$, and $K = (k_1, k_2, \cdots, k_n) \in \mathbb{Q}^n$, $c \in \mathbb{Q}$, then $E \cap L_{K,c}$ is of finite type.*

Proof Since R_i is identical transformation and $K = (k_1, k_2, \cdots, k_n) \in \mathbb{Q}^n$ is invariant, we need only to consider the discreteness of the set of all the parameter c with $L_{K,c} \cap E$ non-empty.

Notice that $c \in \mathbb{Q}$, and $b_i \in \mathbb{Q}^n$ for each $1 \le i \le m$. We assume that

$$K = \frac{K'}{N}, \quad c = \frac{M}{N} \quad \text{and} \quad b_i = \frac{M_i}{N},$$

where $N \in \mathbb{N}$, $M \in \mathbb{Z}$, $K' \in \mathbb{Z}^n$ and $M_i \in \mathbb{Z}^n$ for each $1 \le i \le m$.

Let $\Theta = \{\frac{a}{N^2} : a \in \mathbb{Z}\}$. Given a sequence $i_1 \cdots i_l$, we will show that $c_l \in \Theta$ for any l, where $L_{K_l, c_l} = S^{-1}_{i_1 \cdots i_l}(L_{K,c})$.

In fact, as $c_0 = c = \frac{M}{N} \in \Theta$ and ρ^{-1} is an integer, we have

$$c_{l+1} = \rho^{-1}\Big[\frac{a_l}{N^2} - \frac{K' M_{i_{l+1}}}{N^2}\Big] = \frac{a_{l+1}}{N^2} \in \Theta.$$

The distance of different elements in Θ is at least $\frac{1}{N^2}$.

That means $E \cap L_{K,c}$ is of finite type.

Example 3.1 (Sierpinski carpet)

Suppose $\{S_i\}_{i=1}^4$ are similitudes defined by:

$$S_1\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad S_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix},$$
$$S_3\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}, \quad S_4\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}.$$

Then the corresponding attractor E is called the Sierpinski carpet. The IFS satisfies the open set condition with the corresponding open set $U = (0,1) \times (0,1)$.

Given a planar line $L : y = \frac{2}{5}x$, we will consider the section $E \cap L$.

By the definition, we provide three different types:

$$L_1 = L = \Big\{y = \frac{2}{5}x\Big\}, \quad L_2 = \Big\{y = \frac{2}{5}x + \frac{2}{5}\Big\}, \quad L_3 = \Big\{y = \frac{2}{5}x + \frac{4}{5}\Big\}.$$

It is easy to check that the finite-type open set condition holds for the section $E \cap L$. The corresponding transition matrix is $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ with its spectral radius $\lambda = 1.4655\cdots$. It follows from Theorem 1.1 that

$$\dim_H(E \cap L) = \frac{\log \lambda}{\log 3} = 0.34793\cdots.$$

Example 3.2 (Koch curve)

In this example, we will prove that the Koch curve intersected by line $L : x = c$ with $c \in \mathbb{Q}$ is of finite type.

The Koch curve F can be generated by the following similitudes:

$$S_1\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix}, \quad S_4(x) = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix},$$
$$S_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix},$$
$$S_3\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix},$$

where $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ are contained in a finite subgroup G of $O(2)$. Under the action of G, from the initial vector $K = (1,0)^T$, we obtain six points as follows

$$\Big\{(1,0)^T,\ \Big(\frac{1}{2}, \frac{\sqrt{3}}{2}\Big)^T,\ \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2}\Big)^T,\ (-1,0)^T,\ \Big(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big)^T,\ \Big(\frac{1}{2}, -\frac{\sqrt{3}}{2}\Big)^T\Big\}.$$

These points are symmetric with respect to x-axis, i.e., under the action $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the set of these six points is invariant.

Using the recurrence relation $K_{l+1} = K_l R_{i_{l+1}}$, we conclude that K_{l+1} belongs to the six points. Thus

$$K_l b_{i_{l+1}} \in \Big\{0,\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{1}{6}\Big\}.$$

Let

$$\Theta^* = \Big\{\frac{a}{6p}\ \Big|\ a \in \mathbb{Z}\Big\}.$$

By induction, supposing $c_l \in \Theta^*$, we have

$$c_{l+1} = 3c_l - 3K_l b_{i_{l+1}} \in \Theta^*.$$

In the same way as Proposition 3.1, we verify that this section is of finite type.

Notice that Koch curve F satisfies the open set condition, and the corresponding open set V is the interior of ΔABC whose vertices are $A = (0,0)^T, B = (1,0)^T, C = (\frac{1}{2}, \frac{\sqrt{3}}{6})^T$. In order to verify the finite-type open set condition, we consider the small triangle $S_i(\Delta ABC)$, $i = 1,2,3,4$; Fix a planar line $L : x = c$. If any line $L' \in \Gamma_L$ induced by L does not occur the following two exceptional cases:

(1) $L' \cap S_i(\Delta ABC)$ is a singleton for some i (here $\dim_H(L' \cap F) = 0$),

(2) $L' \cap S_i(\Delta ABC) = S_i(AB), S_i(AC)$ or $S_i(BC)$ (here $\dim_H(L' \cap F) = \frac{\log 2}{\log 3}$),

then we can conclude that

$$L' \cap F \neq \emptyset \quad \text{and} \quad L' \cap V \neq \emptyset \quad \text{for any } L' \in \Gamma_L.$$

And thus the section $F \cap L$ satisfies the finite-type open set condition and the dimension of the section can be obtained by Theorem 1.1.

Given a line $L : x = \frac{11}{20}$, then Γ_L is composed of the following lines:

$$L_1 : x = \frac{11}{20}, \quad L_2 : y = \frac{1}{\sqrt{3}}\Big(x - \frac{7}{10}\Big), \quad L_3 : y = \frac{1}{\sqrt{3}}\Big(x - \frac{1}{10}\Big), \quad L_4 : y = \frac{1}{\sqrt{3}}\Big(x - \frac{3}{10}\Big),$$
$$L_5 : y = -\frac{1}{\sqrt{3}}\Big(x - \frac{7}{10}\Big), \quad L_6 : x = \frac{17}{20}, \quad L_7 : y = \frac{1}{\sqrt{3}}\Big(x - \frac{9}{10}\Big),$$
$$L_8 : y = -\frac{1}{\sqrt{3}}\Big(x - \frac{1}{10}\Big), \quad L_9 : y = -\frac{1}{\sqrt{3}}\Big(x - \frac{3}{10}\Big), \quad L_{10} : y = -\frac{1}{\sqrt{3}}\Big(x - \frac{9}{10}\Big).$$

Here the exceptional cases do not appear, and thus the section $F \cap L$ satisfies the finite-type open set condition. The corresponding transition matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the spectral radius of A is $1.6265766\cdots$. Hence

$$\dim_H(E \cap L) = \frac{\log 1.6265766\cdots}{\log 3} = 0.442811\cdots.$$

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