The Degree of Symmetry of Certain Compact Smooth Manifolds II **

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Abstract We give the sharp estimates for the degree of symmetry and the semi-simple degree of symmetry of certain compact fiber bundles with non-trivial four dimensional fibers in the sense of cobordism, by virtue of the rigidity theorem of harmonic maps due to Schoen and Yau (Topology, 18, 1979, 361–380). As a corollary of this estimate, we compute the degree of symmetry and the semi-simple degree of symmetry of $\mathbb{C}P^2 \times V$, where V is a closed smooth manifold admitting a real analytic Riemannian metric of non-positive curvature. In addition, by the Albanese map, we obtain the sharp estimate of the degree of symmetry of a compact smooth manifold with some restrictions on its one dimensional cohomology.

Keywords Degree of symmetry, Fiber bundle, Cobordism, Non-positive Curvature,
Harmonic map, First cohomology
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1 Introduction

Let M^n be a closed, connected and smooth n-manifold and $N(M^n)$ the degree of symmetry of M^n, that is, the maximum of the dimensions of the isometry groups of all possible Riemannian metrics on M^n. Of course, $N(M)$ is the maximum of the dimensions of the compact Lie groups which can act effectively and smoothly on M. The semi-simple degree of symmetry $N_s(M)$ is defined similarly, where we consider only actions of semi-simple compact Lie groups on M. The following is well known:

$$N(M^n) \le \frac{1}{2}n(n+1). \tag{1.1}$$

In addition, if the equality holds, then M^n is diffeomorphic to the standard sphere S^n or the real projective space $\mathbb{R}P^n$. In [10] H. T. Ku, L. N. Mann, J. L. Sicks and J. C. Su obtained similar results on a product manifold $M^n = M_1^{n_1} \times M_2^{n_2}$ ($n \ge 19$), where M_i is a compact connected smooth manifold of dimension n_i: they showed that

$$N(M) \le \frac{1}{2}n_1(n_1+1) + \frac{1}{2}n_2(n_2+1), \tag{1.2}$$

and that if the equality holds, then M^n is a product of two spheres, two real projective spaces or a sphere and a real projective space. A preliminary lemma for the proof of Ku-Mann-Sicks-Su's results claims that if M^n ($n \geq 19$) is a compact connected smooth n-manifold which is not diffeomorphic to the complex projective space $\mathbb{C}P^m$ ($n = 2m$), then

$$N(M^n) \leq \frac{1}{2}k(k+1) + \frac{1}{2}(n-k)(n-k+1) \tag{1.3}$$

holds for each $k \in \mathbf{N}$ such that the k-th Betti number b_k of M is nonzero.

Let V be a closed, connected and smooth manifold which admits a real analytic Riemannian metric of non-positive curvature. It was noted in Remark 1.2 of [17] that by the results in [3] and [11] the following holds:

Fact 1.1 $N(V)$ *equals the rank of Center* $\pi_1(V)$ *and any connected compact Lie groups which can act effectively and smoothly on* V *must be a torus group.*

Let E be a closed smooth fiber bundle over V with connected fiber F. In Theorem 1.1 of [17] the author generalized partially Ku-Mann-Sicks-Su's result (1.2), showing the corresponding sharp estimates of $N(E)$ and $N_s(E)$ by assuming that the fiber F satisfies various topological properties. In particular, part of the statements of Theorem 1.1 and Corollary 1.1 in [17] says:

Fact 1.2 *Suppose that E is oriented and that F is an orientable $4m$-manifold ($m \geq 5$) of nonzero signature. Then the following statements hold:*

$$N(E) \leq N(V) + 4m(m+1), \quad N_s(E) \leq 4m(m+1). \tag{1.4}$$

In particular, if V is orientable, then

$$N(\mathbb{C}P^{2m} \times V) = N(V) + 4m(m+1), \quad N_s(\mathbb{C}P^{2m} \times V) = 4m(m+1). \tag{1.5}$$

The case of $1 \leq m \leq 4$ could not be covered in [17] because the author used Ku-Mann-Sick-Su's result (1.3), in which the dimension of the manifold is assumed to be ≥ 19. In this paper we shall show that Fact 1.2 also holds for $m = 1$. That is,

Theorem 1.1 *Let V be a closed, connected and smooth manifold which can admit a real analytic Riemannian metric of non-positive curvature and E be a closed smooth fiber bundle over V such that the fiber F of E is connected. Suppose that E is oriented and that the fiber F has dimension 4 and has nonzero signature. Then the following hold:*

$$N(E) \leq N(V) + 8, \quad N_s(E) \leq 8. \tag{1.6}$$

In particular, if V is oriented, then

$$N(\mathbb{C}P^2 \times V) = N(\mathbb{C}P^2) + N(V) = N(V) + 8, \quad N_s(\mathbb{C}P^2 \times V) = N_s(\mathbb{C}P^2) = 8. \tag{1.7}$$

In fact, the assumption in (1.7) that V is oriented can be removed due to the following

Theorem 1.2 *Let V be a closed, connected and smooth manifold which can admit a real analytic Riemannian metric of non-positive curvature and E be a closed smooth fiber bundle over V such that the fiber F of E is connected. Suppose that the fiber F has dimension 4 and is not cobordant mod 2 to either 0 or $\mathbb{R}P^4$. Then both (1.6) and (1.7) hold.*

Remark 1.1 The assumption in Theorem 1.1 that F^4 is orientable and has nonzero signature is independent of the assumption in Theorem 1.2 that F^4 is not cobordant mod 2 to either 0 or $\mathbb{R}P^4$. For example, the oriented 4-manifold $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ has signature 2 and is cobordant mod 2 to 0; $\mathbb{R}P^2 \times \mathbb{R}P^2$ is a non-orientable 4-manifold which is not cobordant to either 0 or $\mathbb{R}P^4$. However, since the group Ω_4 of the oriented cobordism class of 4-manifolds is isomorphic to $\mathbb{Z}$ and the isomorphism is given by the signature, an oriented compact 4-manifold having zero signature is cobordant to zero in Ω_4.

By Remark 1.4 in [17] the connectedness of F is necessary for the validity of Theorems 1.1 and 1.2.

D. Burghelea and R. Schultz [1] showed that $N_s(M) = 0$ if there exist $\alpha_1, \cdots, \alpha_n$ in $H^1(M; \mathbb{R})$ with $\alpha_1 \cup \cdots \alpha_n \neq 0$. In Theorem 1.2 of [17] Burghelea-Schultz's result was generalized to the following

Fact 1.3 *Let M be an n-dimensional closed connected smooth manifold. If there exist $\alpha_1, \cdots, \alpha_k$ in $H^1(M; \mathbb{R})$ with $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$, then the following hold:*

$$N(M) \leq \frac{1}{2}(n - k + 1)(n - k) + k,$$

$$N_s(M) \begin{cases} \leq \frac{1}{2}(n - k + 1)(n - k), & \text{if } n - k > 1, \\ = 0, & \text{otherwise.} \end{cases}$$

Further assuming $b_1(M) > k$, we obtain the following

Theorem 1.3 *Let M be an n-dimensional closed connected smooth manifold and $k \geq 3$ be an integer. If the first Betti number $b_1(M)$ of M is greater than k and there exist $\alpha_1, \cdots, \alpha_k$ in $H^1(M; \mathbb{R})$ with $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$ in $H^k(M; \mathbb{R})$, then*

$$N(M) \leq \begin{cases} \frac{1}{2}(n - k + 1)(n - k) + k - 2, & \text{if } n \geq k + 3, \\ n, & \text{if } n = k + 2 \text{ or } k + 1, \\ n - 2, & \text{if } n = k. \end{cases} \tag{1.8}$$

Remark 1.2 It is implied by the assumption of Theorem 1.3 that M has dimension $\geq k$. We make the assumption of $k \geq 3$ in Theorem 1.3 because of some known facts in [17]. Precisely speaking, the statements (ii) and (iii) of Theorem 1.2 in [17] give the best possible estimates for Case $k = 0$ or 1 and Case $k = 2$, respectively. Moreover, to get those estimates there, the author only needs to assume that $b_1(M) > k$ because the non-vanishing property of the cup product $\alpha_1 \cup \cdots \cup \alpha_k$ is superfluous in the sense that it does not give any more restrictions to the degree of symmetry of M.

Remark 1.3 By the definition of degree of symmetry, it is easy to see that for a product manifold $M_1 \times M_2$, where M_i is a compact connected smooth manifold, the following holds:

$$N(M_1 \times M_2) \geq N(M_1) + N(M_2). \tag{1.9}$$

Let Σ_g be an oriented closed surface of genus g and $M^n = S^{n-k} \times T^{k-2} \times \Sigma_g$ ($n \geq k+3$, $g \geq 2$). Then M^n satisfies the assumption of Theorem 1.3. Since by Fact 1.1 $N(T^{k-2} \times \Sigma_g) = k - 2$, by (1.9) and Theorem 1.3 we obtain the equality

$$N(S^{n-k} \times T^{k-2} \times \Sigma_g) = \frac{1}{2}(n - k + 1)(n - k) + k - 2,$$

combining which with the equalities

$$N(T^n) = n, \quad N(T^{n-2} \times \Sigma_g) = n - 2,$$

we can see that the estimate (1.8) is best possible.

This paper is organized as follows. In Section 2, we prepare for the following sections. In particular, we cite some results in [17] and prove a key lemma (cf. Lemma 2.2) for Theorems 1.1 and 1.2. In Section 3, we prove Theorem 1.1 (1.2) with the help of this key lemma and the oriented (unoriented) cobordism theory. In Section 4, we prove Theorem 1.3 by virtue of the unique continuation property of harmonic maps.

2 Preliminaries

For a closed Riemannian manifold M let $I^0(M)$ be the identity component of the isometry group of M. The following proposition will provide the frame for the proof of Theorems 1.1, 1.2.

Proposition 2.1 (Cf. [16, Theorem 4]) *Suppose that M, N are closed real analytic Riemannian manifolds and the sectional curvature of N is non-positive. Suppose that $h : M \to N$ is a surjective harmonic map and its induced map $h_* : \pi_1(M) \to \pi_1(N)$ is surjective. Then the space of surjective harmonic maps homotopic to h is represented by $\{\beta \circ h \mid \beta \in I^0(N)\}$, where $I^0(N)$ is a torus group of dimension equaling both the rank of $\mathrm{Center}\,\pi_1(N)$ and the degree of symmetry of N.*

We cite a topological result from [17].

Proposition 2.2 (Cf. [17, Proposition 3.1]) *Let $p_0 : E \to B$ be a fiber bundle over a compact connected smooth manifold B such that the fiber of E is also connected. Then any continuous map homotopic to $p_0 : E \to B$ is surjective.*

We cite a lemma in [17], which is also necessary for Theorems 1.1, 1.2.

Lemma 2.1 (Cf. [17, Lemma 2.1]) *Let M be a connected Riemannian manifold and f a smooth map from M to a smooth manifold N. Suppose that $y \in N$ is a regular value of f and F is a connected component of the submanifold $f^{-1}(y)$ of M. If an isometry α of M satisfies that $h \circ \alpha = h$ and that $\alpha(x) = x$ for any $x \in F$, then α is the identity map of M.*

Lemma 2.2 (Key Lemma of Theorems 1.1 and 1.2) *Let Y be a closed connected smooth 4-manifold not diffeomorphic to either S^4 or $\mathbb{R}P^4$. Then $N(Y) \leq 8$. The equality $N(Y) = 8$ holds if and only if Y is diffeomorphic to $\mathbb{C}P^2$. Moreover, $N_s(\mathbb{C}P^2) = 8$.*

Proof By (1.1) $N(Y) \leq 9$. Then $N(Y) \leq 8$ follows from Theorem A′ in [8] which claims that there exists no 4-dimensional Riemannian manifold having a 9-dimensional isometry group. If Y is a Riemannian manifold whose isometry group has dimension 8, then by Theorem 5 in [8] Y is a Kählerian space with positive constant holomorphic sectional curvatures. Since the holomorphic sectional curvature of a Kähler manifold determines completely its Riemannian curvature tensor (cf. [18, Lemma 7.19]), Y has positive sectional curvature and then by the theorem of Synge (cf. [2, Theorem 5.9]) Y is simply connected. By the theorem of Cartan-Ambrose-Hicks (cf. [2, Theorem 1.36]), Y is isometric to the Kähler manifold $\mathbb{C}P^2$ with the Fubini-Study metric. Since the compact Lie group SU(3) acting isometrically on $\mathbb{C}P^2$ is semi-simple, $N_s(\mathbb{C}P^2) = N(\mathbb{C}P^2) = 8$.

We do some preparations for the proof of Theorem 1.3 in the following.

For a compact oriented Riemannian manifold M with nonzero first Betti number $b_1(M)$, let $\mathcal{H}$ be the real vector space of all harmonic 1-forms on M and ν the natural projecion from the universal covering $\widetilde{M}$ of M. For $x_0 \in \widetilde{M}$, set $p_0 = \nu(x_0)$. We define a smooth map $\widetilde{a} : \widetilde{M} \to \mathcal{H}^*$ from $\widetilde{M}$ to the dual space $\mathcal{H}^*$ of $\mathcal{H}$ by a line integral

$$\widetilde{a}(x)(\omega) = \int_{x_0}^{x} \nu^* \omega, \quad x \in \widetilde{M}, \ \omega \in \mathcal{H}.$$

For $\sigma \in \pi_1(M)$,

$$\widetilde{a}(\sigma x) = \widetilde{a}(x) + \psi(\sigma)$$

holds, where $\psi(\sigma)(\omega) = \int_{x_0}^{\sigma x_0} \nu^* \omega$, so that ψ is a homomorphism from $\pi_1(M)$ into $\mathcal{H}^*$ as an additive group. It is a fact that $\Delta = \psi(\pi_1(M))$ is a lattice in the vector space $\mathcal{H}^*$, and clearly this vector space has a natural Euclidean metric from the global inner product of 1-forms on M. With the quotient metric, we call the torus $A(M) = \mathcal{H}^*/\Delta$ the Albanese torus of the Riemannian manifold M. By the above relation between $\widetilde{a}$ and ψ, we obtain a map $a : M \to A(M)$ satisfying $\widetilde{a}(x) \in a \circ \nu(x)$ for any $x \in \widetilde{M}$. We call the map a the Albanese map. From the very construction of a, we see that the map it induces on fundamental groups

$$a_* : \pi_1(M) \to \pi_1(A(M))$$

is surjective and that a^* maps the space of harmonic 1-forms on $A(M)$ isomorphically onto $\mathcal{H}$. By [14, Corollary 1], the Albanese map is harmonic. Set

$$r_a := \max\{\operatorname{rank} da(p) \mid p \in M\}.$$

Lemma 2.3 (Cf. [17, Lemma 4.3]) *Let M be an n-dimensional oriented compact Riemannian manifold with nonzero first Betti number b_1. Let $a : M \to A(M)$ be the Albanese map. Suppose there exist $\alpha_1, \cdots, \alpha_k$ in $H^1(M; \mathbb{R})$ with $\alpha_1 \cup \cdots \cup \alpha_k \neq 0$ in $H^k(M; \mathbb{R})$. Then $r_a \geq k$ holds.*

Lemma 2.4 (Cf. [17, Lemmas 4.1 and 4.2]) *Let M be a non-orientable compact manifold and $\pi : M' \to M$ be its orientable double covering. Then the following statements hold:*
(i) $N(M) \leq N(M')$,
(ii) $b_1(M) \leq b_1(M')$,
(iii) *If M has the property that there exist k one dimensional real cohomology classes $\alpha_1, \cdots, \alpha_k$ of M such that $\alpha_1 \cup \cdots \cup \alpha_k$ is nonzero in $H^k(M; \mathbb{R})$, then so does M'.*

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 For the proof of (1.6), by Corollary A.1 in Appendix we have only to show that for any real analytic Riemannian metric on E and any compact semi-simple subgroup G of $I^0(E)$, the following inequalities hold:

$$\dim I^0(E) \leq N(V) + 8, \quad \dim G \leq 8. \tag{3.1}$$

Since the fiber F is connected, the fiber bundle projection $p : E \to V$ induces a surjective map $p_* : \pi_1(E) \to \pi_1(V)$. Using a well-known result by Eells-Sampson [5], we see that there exist harmonic maps homotopic to $p : E \to V$. By Proposition 2.2, each of them is surjective and

then satisfies the assumptions of Proposition 2.1. Taking a harmonic map $h : E \to V$ homotopic to $p : E \to V$, by Proposition 2.1, for any $\alpha \in I^0(E)$ we can find a unique $\rho(\alpha) \in I^0(V)$ with $h \circ \alpha = \rho(\alpha) \circ h$. We see that $\rho : I^0(E) \to I^0(V)$ is a Lie group homomorphism. Since G is semi-simple and $I^0(N)$ is a torus group, the restriction of ρ to G must be trivial. That is, G is contained in $\mathrm{Ker}\,\rho$. Therefore, the proof of (3.1) is completed if we can show that $\mathrm{Ker}\,\rho$, which acts isometrically on E, has dimension ≤ 8.

Choosing a smooth homotopy $P : E \times [0, 1] \to V$ between p and h, we can also choose a regular value y of P by Sard's theorem since P is surjective. Then we have the following

Claim 1 $P^{-1}(y)$ is an oriented submanifold in $E \times [0, 1]$ with boundary $p^{-1}(y) + h^{-1}(y)$. That is, there exists an oriented cobordism in E between F and $h^{-1}(y)$.

Proof of Claim 1 Since y is the regular value of P and $P^{-1}(y)$ is non-empty, it is easy to see that $P^{-1}(y)$ is a submanifold of $E \times [0, 1]$ with boundary

$$\partial P^{-1}(y) \cap E \times 0 + \partial P^{-1}(y) \cap E \times 1 = p^{-1}(y) + h^{-1}(y) \cong F + h^{-1}(y).$$

Note that the normal bundle of $P^{-1}(y)$ in E is trivial. Since E is orientable, so is $P^{-1}(y)$. We have proved the claim.

By Hirzebruch's signature theorem (cf. [7, Theorem 8.2.2]) and Claim 1, up to the sign difference, the signature of $\widetilde{F}$ equals that of F so that there exists a connected component F^* of $\widetilde{F}$ having nonzero signature. Hence F^* is not diffeomorphic to either S^4 or $\mathbb{R}P^4$. By Lemma 2.1 $\mathrm{Ker}\,\rho$ acts effectively on F^*. Moreover, Lemma 2.2 tells us that $\mathrm{Ker}\,\rho$ has dimension ≤ 8. We complete the proof of (3.1) and (1.6). Since $\mathbb{C}P^2$ has signature 1, (1.7) follows from (1.6) and (1.9). The proof of Claim 1 is complete.

Proof of Theorem 1.2 Repeating the part of the proof of Theorem 1.1 before Claim 1, we can see that $P^{-1}(y)$ is a submanifold of $E \times [0, 1]$ with boundary $p^{-1}(y) + h^{-1}(y)$. That is, $h^{-1}(y)$ is cobordant mod 2 to F. Since F is not cobordant mod 2 to either 0 or $\mathbb{R}P^4$, there exists a connected component F^* of $h^{-1}(y)$ such that F^* is not diffeomorphic to either S^4 or $\mathbb{R}P^4$. Since $\mathrm{Ker}\,\rho$ acts effectively on F^*, by Lemma 2.2 $\mathrm{Ker}\,\rho$ has dimension ≤ 8. Therefore the inequalities in (1.6) hold. To show (1.7), we only need to show $\mathbb{C}P^2$ is not cobordant mod 2 to zero or $\mathbb{R}P^4$, which follows from that $w_2^2[\mathbb{R}P^4] = 0$ and $w_2^2[\mathbb{C}P^2] \neq 0$.

4 Proof of Theorem 1.3

Proof of Theorem 1.3 By Lemma 2.4, we may assume M is an oriented Riemannian manifold. Let $a : M \to A(M)$ be the Albanese map and b_1 the first Betti number of M. By Corollary A.1 we have only to consider the analytic Riemannian metric on M. For any $\gamma \in I^0(M)$, $a \circ \gamma$ is also a harmonic map from M to the Albanese torus $A(M)$ and homotopic to a. By [14, Lemma 3] there is a unique translation $\rho(\gamma)$ of the torus $A(M)$ such that

$$a \circ \gamma = \rho(\gamma) \circ a.$$

Then we have a Lie group homomorphism $\rho : I^0(M) \to T^{b_1}$, where the torus group T^{b_1} is the translation group of the Albanese torus $A(M)$.

Recall $r_a = \max\{\mathrm{rank}\,da(p) \mid p \in M\}$. We claim that

$$\dim \mathrm{Ker}\,\rho \leq \frac{1}{2}(n - r_a + 1)(n - r_a), \quad \dim \mathrm{Im}\,\rho \leq r_a. \tag{4.1}$$

Proof of (4.1) Since the proof of Lemma 2.3 in [17] has some ambiguity, we give a clear and rigorous proof here. We first prove $\dim \operatorname{Im} \rho \leq r_a$. Let r be the dimension of $\operatorname{Im} \rho$. As a connected subgroup of T^{b_1}, $\operatorname{Im} \rho$ is an r-dimensional torus group acting freely on $a(M)$ by the definition of ρ. Choose a point x in M. Then $y := a(x)$ is in $a(M)$ and the orbit $(\operatorname{Im} \rho)(y)$ of y with respect to the $\operatorname{Im} \rho$ action is an r-dimensional subtorus contained in $a(M)$. More precisely, by the definition of ρ, we have

$$(\operatorname{Im} \rho)(y) = a(I^0(M)(x)),$$

where $I^0(M)(x)$ is the orbit of x with respect to the $I^0(M)$ action on M. Let a_1 be the restriction of the Albanese map a to the submanifold $I^0(M)(x)$ of M. Then $r_a \geq r_{a_1} = \dim \operatorname{Im} \rho = r$.

Then we show that $K := \operatorname{Ker} \rho$ has dimension $\leq \frac{1}{2}(n - r_a + 1)(n - r_a)$ by investigating the K action on M. Since M is connected, the orbit space M/K with the induced topology is also connected. By [9, Theorem 4.27], the union set of all principal orbits of the K action is an open dense subset of M. Taking an open set $U \subset M$, at any point of which the map da has rank equal to r_a, we can choose a point p in U such that the orbit $K(p)$ is principal. By the definition of ρ, $K(p)$ is contained in the inverse image of $a(p)$ under a. Since $\operatorname{rank} da(p) = r_a$, the submanifold $K(p)$ has dimension not exceeding $n - r_a$. Since K acts effectively on the principal orbit $K(p)$, the dimension of K can not exceed $\frac{1}{2}(n - r_a + 1)(n - r_a)$ by (1.1). By now we have completed the proof of (4.1).

We see from Lemma 2.3 that $r_a \geq k$.

Case 1 If $r_a \geq k + 1$, then from (4.1)

$$\dim I(M) = \dim \operatorname{Ker} \rho + \dim \operatorname{Im} \rho \leq \frac{1}{2}(n - k - 1)(n - k) + k + 1.$$

Case 2 Suppose $r_a = k$ in what follows. We claim that $\dim \operatorname{Im} \rho$ will be less than $k - 1$, which together with (4.1) imply the following estimate

$$\dim I^0(M) \leq \frac{1}{2}(n - k + 1)(n - k) + k - 2.$$

Otherwise, suppose $\dim \operatorname{Im} \rho \geq k - 1$. Recall that the Lie group $\operatorname{Im} \rho$ acting on $A(M)$ in fact acts on the image $a(M)$ of a. Hence we can assume that there exists a subgroup T^{k-1} of the translation group T^{b_1} which acts freely and isometrically on $a(M)$. Since both M and $A(M)$ are real analytic, a theorem of Morrey [13] shows that the harmonic mapping a is in fact real analytic. By the well-known theorems in real analytic geometry (cf. [12]) we know that both M and $A(M)$ can be triangulated so that $a(M)$ is a compact connected simplicial subcomplex of dimension k in $A(M)$. We write the orbit space of the free and isometric T^{k-1} actions on $A(M)$ and $a(M)$ by $A(M)/T^{k-1}$ and $a(M)/T^{k-1}$ respectively, in which the former is in fact also a flat torus of dimension $b_1 - k + 1$. Since the natural projection map $\pi : A(M) \to A(M)/T^{k-1}$ is totally geodesic, we see that by a result in [4] the composition map $\pi \circ a : M \to A(M)/T^{k-1}$ is a harmonic map, whose image is $a(M)/T^{k-1}$, the orbit space of the free T^{k-1} action on the simplicial subcomplex $a(M)$ of dimension k in $A(M)$. Hence $a(M)/T^{k-1}$, the image of $\pi \circ a$ in $A(M)/T^{k-1}$, has dimension 1 so that the differential of harmonic map $\pi \circ a$ has rank ≤ 1 at any point of M. By the unique continuation property of the harmonic maps (cf. [15, Theorem 3]), we see that $\pi \circ a$ maps M onto a closed geodesic of $A(M)/T^{k-1}$, which means that $a(M)$ is a principal T^{k-1}-bundle over S^1. Since S^1 is connected, there exists a section on this bundle such

that $a(M)$ is a trivial T^{k-1}-bundle, i.e. a k-dimensional torus. This contradicts the surjectivity of the homomorphism $a_* : \pi_1(M) \to \pi_1(A(M)) \cong \mathbb{Z}^{b_1}$ ($b_1 > k$). Hence, we have proved the claim. In particular, if the dimension n of the manifold M equals k, then r_a must be k and the estimate dim $I^0(M) \leq k - 2$ follows from the claim.

Combining Cases 1 and 2, we can see that the dimension of $I^0(M)$ is dominated by the maximum of those two integers

$$\frac{1}{2}(n - k - 1)(n - k) + k + 1, \quad \frac{1}{2}(n - k)(n - k + 1) + k - 2,$$

provided dim $M = n \geq k + 1$. By some simple computations, we complete the proof.

Appendix   Real Analytic Group Action

In this appendix, we will prove the following theorem.

Theorem A.1 *Let G be a compact Lie group acting smoothly and effectively on a compact smooth manifold M. Then there exists a real analytic manifold M' on which G acts real analytically so that there exists a G-isomorphism between (M, G) and (M', G). That is, there is a diffeomorphism $f : M \to M'$ which satisfies for any $x \in M$ and $g \in G$,*

$$f(gx) = g(f(x)).$$

Although the theorem should not be new, the author provides a proof here since he does not know any reference of the theorem. We put the proof of Theorem A.1 afterward. Firstly, from it we have the following

Corollary A.1 *The degree of symmetry of M equals the maximum of the dimensions of the isometry groups of all the real analytic Riemannian metrics on M.*

Proof   Let G be a compact Lie group acting smoothly and effectively on a compact smooth manifold M. By Theorem A.1 there exists another real analytical G action on the unique real analytic structure of M compatible to the existed smooth structure of M. Moreover, the new G action is equivariant to the old one on the smooth structure of M. Thus the new one is also effective. Taking a real analytic Riemannian metric on M, by the invariant integration on G we can construct a new real analytic one on which G acts isometrically. The corollary is proved.

Let $f : M \to N$ be a smooth map between smooth manifolds M and N. It is transverse to a submanifold $A \subset N$ if and only if whenever $f(x) = y \in A$, then the tangent space to N at y is spanned by the tangent space to A at y and the image of the tangent space to M at x. That is,

$$T_y A + df(T_x M) = T_y N.$$

Lemma A.1 (Cf. [6, Theorem 1.3.3])   *Let $f : M \to N$ be a smooth map and $A \subset N$ a submanifold of codimension ℓ. If f is transverse to A, then $f^{-1}(A)$ is a submanifold of M of codimension ℓ.*

Lemma A.2 (Cf. [9, Theorem 4.12])   *Let G be a compact Lie group and M a compact manifold on which G acts smoothly. Then there exists a representation space (V, μ) of G and a smooth G-embedding $\iota : M \to V$. That is, for any $x \in M$ and $g \in G$,*

$$\iota(gx) = \mu(g)(\iota(x)).$$

Moreover, if the G-action on M is effective, then the representation (V, μ) of G is faithful.

Let G be a compact group acting smoothly on two manifolds M and N. A smooth map $f : M \to N$ is a G-map if and only if for any $x \in M$ and $g \in G$ the following holds:

$$g(f(x)) = f(gx).$$

Lemma A.3 (An Equivariant Version of Theorem 2.5.2 in [6]) *Let G be a compact Lie group acting isometrically on Euclidean spaces $\mathbb{R}^q$ and $\mathbb{R}^s$. Let $M \subset \mathbb{R}^q$ be a G-invariant compact submanifold of codimension > 0 and E a G-invariant tubular neighborhood of M in $\mathbb{R}^q$. Let $f : E \to W$ be a smooth G-map into a G-invariant open set W of $\mathbb{R}^s$. Let $v : \mathbb{R}^q \to \mathbb{R}$ be a smooth G-invariant function with support in E, equal to 1 on a G-invariant compact neighborhood K of M. Set $h(x) = v(x)f(x) = v(x)(f_1(x), \cdots, f_s(x))$. Let $\delta : \mathbb{R}^q \to \mathbb{R}$, $\delta(x) = \exp(-|x|^2)$. Let $C = 1/\int_{\mathbb{R}^q} \delta$. Let $\epsilon > 0$. Then for $k > 0$ sufficiently large,*

$$\psi(x) = (\psi_1(x), \cdots, \psi_s(x)) := (h_1(x), \cdots, h_s(x)) * (Ck^q\delta(kx))$$

is an analytic G-map and satisfies $\|\psi - f\|_{C^1, K} < \epsilon$.

Proof The proof of the analytic property of ψ and the estimate $\|\psi - f\|_{C^1, K} < \epsilon$ for $k > 0$ large enough is straightforward. We have only to show that $h * \delta$ is G-invariant. Since $v(gx) = v(x)$, $h = vf : E \to \mathbb{R}^s$ is G-map. For any $g \in G$, since g acts on $\mathbb{R}^q$ isometrically and δ is a radial function on $\mathbb{R}^q$,

$$
\begin{aligned}
(h * \delta)(gx) &= \int_{\mathbb{R}^q} h(y)\delta(gx - y)\, dy = \int_{\mathbb{R}^q} h(gz)\delta(g(x - z))\, dz \\
&= \int_{\mathbb{R}^q} h(gz)\delta(x - z)\, dz = g\left(\int_{\mathbb{R}^q} h(z)\delta(x - z)\, dz \right) \\
&= g(h * \delta(x)).
\end{aligned}
$$

Proof of Theorem A.1 By Lemma A.2, there exists a faithful representation space V of G and a G-embedding $\iota : M \to V$. By the invariant integration on G, we can induce a G-invariant inner product $(\,,\,)$ on V, equipped with which V becomes a Euclidean space $\mathbb{R}^q$ and G becomes a subgroup of $\mathrm{O}(q)$. We shall not distinguish M and $\iota(M)$ in what follows. Let k be the codimension of M in $\mathbb{R}^q$. Since M is a G-invariant submanifold, by [9, Theorem 4.8] there exists a G-invariant normal tubular neighborhood E of M in $\mathbb{R}^q$ which can be identified with a G-invariant neighborhood of the zero section of the normal bundle of M in $\mathbb{R}^q$. Let $p : E \to M$ be the restriction of the bundle projection to E, which is a G-map.

Let $G_{q,k}$ be the Grassmann manifold of k-dimensional linear subspaces of $\mathbb{R}^q$ and $E_{q,k} \to G_{q,k}$ be the Grassmann bundle. Then the fiber of $E_{q,k}$ over the k-plane $P \subset \mathbb{R}^q$ is the set of pairs (P, x) where $x \in P$. Then the G-action on $\mathbb{R}^q$ induces the natural real analytic actions on $G_{q,k}$ and $E_{q,k}$ respectively such that the bundle projection $E_{q,k} \to G_{q,k}$ is a G-map. Let $h : M \to G_{q,k}$ be the map sending $x \in M$ to the k-plane normal to M at x and $f : E \to E_{q,k}$ be the natural map covering h; thus

$$f(y) = (h \circ p(y), y) \in E_{q,k} \subset G_{q,k} \times \mathbb{R}^q.$$

Since G acts isometrically on M, as a linear map on $\mathbb{R}^q$, $dg = g$ maps the k-plane normal to M at x to the one normal to M at gx. Therefore, both h and f are G-maps. Moreover, f is transverse to the zero section $G_{q,k} \subset E_{q,k}$ and

$$f^{-1}(G_{q,k}) = M.$$

Now we embed $E_{q,k}$ analytically into $\mathbb{R}^s$ with $s = q^2 + q$. For this it suffices to embed $G_{q,k}$ into $\mathbb{R}^{q^2}$. This is done by mapping a k-plane $P \in G_{q,k}$ to the linear map $\mathbb{R}^q \to \mathbb{R}^q$ given by the orthogonal projection on P. There exists a natural isometric G-action on $\mathbb{R}^s$ such that the embedding $E_{q,k} \subset \mathbb{R}^s$ is a G-map. Then we can find a G-invariant normal tubular neighborhood W of $E_{q,k}$. Let $\Pi : W \to E_{q,k}$ be the real analytic G-invariant projection.

Let $f' : E \to W$ be the extension of the map $f : E \to E_{q,k}$ to W. It follows from Lemma A.3 that f' can be approximated near M by an analytic G-map $\psi : E \to W$. Then $\phi = \Pi \circ \psi$ is an analytic G-invariant approximation of $f : E \to E_{q,k}$. Put $M' = \phi^{-1}(G_{q,k})$. If ϕ is sufficiently C^1 close to f, then ϕ is also transverse to $G_{q,k}$, so by Lemma A.1 M' is a real analytic submanifold of codimension k in $E \subset \mathbb{R}^q$. Moreover, the restriction of G-map $p : E \to M$ to M' is a G-isomorphism from M to M'.

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