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Blow Up of Solutions to Semilinear Wave Equations with Critical Exponent in High Dimensions**

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Abstract In this paper, the author considers the Cauchy problem for semilinear wave equations with critical exponent in $n \geq 4$ space dimensions. Under some positivity conditions on the initial data, it is proved that there can be no global solutions no matter how small the initial data are.

Keywords Semilinear wave equation, Critical exponent, Cauchy problem, Blow up
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1 Introduction

Consider the Cauchy problem for the semilinear wave equation

$$\Box u(t,x) = |u(t,x)|^p, \tag{1.1}$$

corresponding to initial conditions

$$u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad x \in R^n, \tag{1.2}$$

where $f, g \in C_0^\infty(R^n)$ and $\Box = \partial_t^2 - \triangle$ is the wave operator. In 1979, John showed that when $n = 3$ global solutions always exist if $p > 1 + \sqrt{2}$ and initial data are sufficiently small, and moreover, the global solutions do not exist if $p < 1 + \sqrt{2}$ and the initial data are not both identically zero. The number $1 + \sqrt{2}$ appears to have first arisen in Strauss' work on semilinear Schrödinger equations. Based on this, he made the insightful conjecture that when $n \geq 2$ global solutions of (1.1)–(1.2) should always exist if initial data are sufficiently small and p is greater than a critical power $p_0(n)$ which is the positive root of quadratics

$$(n-1)p^2 - (n+1)p - 2 = 0. \tag{1.3}$$

This conjecture was verified when $n = 2$ by Glassey [3], $n = 4$ by the author in [14] and finally for all $n \geq 4$ and $p \leq \frac{n+3}{n-1}$ by V. Georgiev, H. Lindblad and C. Sogge [1]. On the other hand, when $1 < p < p_0(n)$, there can be blow up for arbitarily small data. This was shown by Glassey [2] when $n = 2$ and by Sideris [6] for all $n \geq 4$. For the critical case $p = p_0(n)$, it was shown by Schaeffer [5] that there still can be blow up for small data if $n = 2$ or 3 (see also [8, 12, 13]). However, when $n \geq 4$ and $p = p_0(n)$, the problem has been left open for more than 20 years. The aim of this paper is to extend Sideris' blow up result to $p = p_0(n)$ for all $n \geq 4$.

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For the subcritical case, Sideris was able to find certain averaged quantities such that they satisfy differential inequalities which blow up. In the critical case, these differential inequalities do not blow up. The best thing that can be achieved by Sideris' method is as follows: Suppose we have a solution on the time interval $[0, T]$. Then we can get a sharp lower bound of certain averaged quantities which depends on T. The new idea is to get an upper bound of the averaged quantities which is independent of T. In this way, we can get an upper bound for T. This idea was inspired by the recent work of Q. S. Zhang [11] on the damped wave equation. The key step to accomplish this idea is to use positivity of some special functions which are homogeneous of degree $-q$ $(q > 0)$, radial symmetric and solve the homogenous linear wave equations. Such kind of special functions were used by the author in [13].

After the completion of this paper, we received a preprint of [10], similar result is proved by different method.

2 Main Result

In this section, we briefly review results proved by Sideris [6] and state our main theorem. We first recall the following local existence theorem of Sideris.

Proposition 2.1 *Let $f \in H^1(R^n)$, $g \in L^2(R^n)$, $n \geq 4$ with* $\operatorname{supp} f, g \subset \{|x| < k\}$ *and suppose $1 \leq p \leq \frac{n+3}{n-1}$. Then there exists a $T > 0$ and a unique solution $u(t) \in C([0,T]; L^{\frac{2(n+1)}{n-1}}(R^n))$ of the integral equation*

$$u(t) = u^0(t) + \int_0^t R(t-\tau) * |u(\tau)|^p d\tau \tag{2.1}$$

with $\operatorname{supp} u(t) \subset \{|x| < k + t\}$*, where* (2.1) *is the integral equation of* (1.1)–(1.2)*, that is, $u^0(t)$ is the solution of the homogeneous linear wave equation with initial data (f, g) and R is the fundamental solution of the wave operator.*

Our main result can be stated as follows:

Theorem 2.1 *Consider the Cauchy problem* (1.1)–(1.2). *Suppose that $p = p_0(n)$ and the initial data satisfy*

$$\int_{R^n} |x|^{\eta-1} f(x) dx > 0, \quad \int_{R^n} |x|^{\eta} g(x) dx > 0, \tag{2.2}$$

where η is 0 if n is odd and $\frac{1}{2}$ if n is even. Let $u(t)$ be a solution given by Proposition 2.1 *on the time interval $[0, T]$. Then there exists a positive constant C^* depending only on the initial data such that*

$$T \leq C^*. \tag{2.3}$$

This theorem is of the same form as that of Sideris'. The strange condition (2.2) comes from the following lemma of Sideris.

Lemma 2.1 *Let u^0 be the solution of the homogeneous linear wave equation with initial data (f, g). Suppose that* (2.2) *is satisfied. Then there exists a positive constant T_0 such that*

$$\int_{|x|>t} \int_{t-k}^{t} (t-\tau)^m u^0(\tau, x) d\tau dx \geq c_0 (t+k)^{\frac{n-1}{2}}, \quad \forall\, t \geq T_0, \tag{2.4}$$

where $m = \frac{n-5}{2}$ *if* n *is odd and* $m = \frac{n-4}{2}$ *if* n *is even, and* c_0 *is a positive constant independent of* t.

The number m in Lemma 2.1 appears naturally in the solution formula for the linear wave equation and we have the following lemma of Sideris.

Lemma 2.2 *Let* u *be the solution of* (2.1) *and let*

$$v(t) = u(t) - u^0(t). \tag{2.5}$$

Then

$$\int_0^t (t-\tau)^m v(\tau, x) d\tau \geq 0, \quad a.e.\ x. \tag{2.6}$$

We have

Corollary 2.1 *Under the assumptions of Lemma* 2.2,

$$\int_0^t (t-\tau)^{m+1} v(\tau, x) d\tau \geq 0, \quad a.e.\ x. \tag{2.7}$$

Proof In fact, we have

$$\int_0^t (t-\tau)^{m+1} v(\tau, x) d\tau = \int_0^t dt_1 \int_0^{t_1} (t_1-\tau)^m v(\tau, x) d\tau.$$

It follows from Lemmas 2.1 and 2.2 that

$$\int_{|x|>t} \int_{t-k}^t (t-\tau)^m |u(\tau, x)|^p d\tau dx \geq c(t+k)^{n-1-\frac{n-1}{2}p}, \quad \forall\, t \geq T_0. \tag{2.8}$$

In fact, it follows from Hölder's inequality that

$$\begin{aligned}
&\int_{|x|>t} \int_{t-k}^t (t-\tau)^m |u(\tau, x)|^p d\tau dx \\
&\geq c(t+k)^{-(n-1)(p-1)} \Big| \int_{|x|>t} \int_{t-k}^t (t-\tau)^m u(\tau, x) d\tau dx \Big|^p \\
&= c(t+k)^{-(n-1)(p-1)} \Big| \int_{|x|>t} \int_{t-k}^t (t-\tau)^m u^0(\tau, x) d\tau dx + \int_{|x|>t} \int_{t-k}^t (t-\tau)^m v(\tau, x) d\tau dx \Big|^p.
\end{aligned}$$

We have

$$\operatorname{supp} v(\tau) \subset \{|x| < \tau + k\}. \tag{2.9}$$

Thus

$$\int_{|x|>t} \int_{t-k}^t (t-\tau)^m v(\tau, x) d\tau dx = \int_{|x|>t} \int_0^t (t-\tau)^m v(\tau, x) d\tau dx \geq 0. \tag{2.10}$$

Therefore, (2.8) follows from Lemma 2.1.

3 Proof of Theorem 2.1

The key point is to seek a solution of the linear wave equation

$$\Box\phi = 0 \tag{3.1}$$

on the domain $|x| \le t$, $t \ge 0$ of the form

$$\phi = \phi_q = (t+|x|)^{-q} h\Big(\frac{2|x|}{t+|x|}\Big), \tag{3.2}$$

where $q > 0$. A simple calculation shows that $h = h_q$ satisfies the ordinary differential equations

$$z(1-z)h''(z) + \Big[n-1-\Big(q+\frac{n+1}{2}\Big)z\Big]h'(z) - \frac{n-1}{2}qh(z) = 0. \tag{3.3}$$

Therefore, we can take

$$h_q(z) = F\Big(q, \frac{n-1}{2}, n-1, z\Big), \tag{3.4}$$

where F is the hypergeometric function defined by

$$F(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} z^n \tag{3.5}$$

with $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1)$, $n \ge 1$. For $\alpha > \beta > 0$, one has the formula

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt, \quad |z| < 1. \tag{3.6}$$

Thus

$$h_q(z) = \frac{\Gamma(n-1)}{\Gamma^2(\frac{n-1}{2})} \int_0^1 t^{\frac{n-3}{2}}(1-t)^{\frac{n-3}{2}}(1-zt)^{-q}dt. \tag{3.7}$$

It follows that

$$h(z) > 0, \quad 0 \le z < 1. \tag{3.8}$$

Moreover, when

$$0 < q < \frac{n-1}{2}, \tag{3.9}$$

$h(z)$ is continous at $z = 1$. Thus

$$C_0^{-1} \le h(z) \le C_0, \quad 0 \le z \le 1 \tag{3.10}$$

for some positive constant C_0. When

$$q > \frac{n-1}{2}, \tag{3.11}$$

$h(z)$ behaves like $(1-z)^{\frac{n-1}{2}-q}$ when z is close to one. Thus

$$C_0^{-1}(1-z)^{\frac{n-1}{2}-q} \le h(z) \le C_0(1-z)^{\frac{n-1}{2}-q}, \quad 0 \le z \le 1, \tag{3.12}$$

for some positive constant C_0. One can also verify

$$\partial_t \phi_q(t,x) = -q\phi_{q+1}(t,x) \tag{3.13}$$

by (3.2) and (3.7).

To prove Theorem 2.1, we consider the functional

$$I(T) = \int_{2k}^{T} \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1) \int_{R^n} \Phi_q(t,x) \int_0^{2k} \sigma^{m+1} |u(t-\sigma,x)|^p d\sigma dx dt, \tag{3.14}$$

where

$$\frac{1}{p'} + \frac{1}{p} = 1, \quad q = n-1-\frac{2}{p-1}, \tag{3.15}$$

$\Phi_q(t,x) = \phi(t+k+1,x)$, and $\theta = \theta(\tau)$ is a cut-off function such that $\theta \in C^\infty(R)$, $0 \le \theta \le 1$, $\theta = 1$ if $\tau \le \frac{1}{2}$, $\theta = 0$ if $\tau \ge \frac{3}{4}$. We shall prove that when $p = p_0(n)$, $I(T)$ is bounded above by a positive constant C which is independent of T:

$$I(T) \le C. \tag{3.16}$$

Assume that (3.16) is valid, we now prove Theorem 2.1. Before doing that, we list the relations satisfied by p and q if $p = p_0(n)$:

$$q = \frac{n-1}{2} - \frac{1}{p}, \tag{3.17}$$

$$0 < q < \frac{n-1}{2}, \tag{3.18}$$

$$q+1 > \frac{n-1}{2}, \tag{3.19}$$

$$(n+1-q)\frac{1}{p'} = 2, \tag{3.20}$$

$$(n-1-q)\frac{1}{p'} = \frac{n+1}{2} - q - \frac{1}{p'}, \tag{3.21}$$

$$\Big(q - \frac{n-3}{2}\Big)p' = 1, \tag{3.22}$$

$$-(n-1) + \frac{n-1}{2}p + q = 1. \tag{3.23}$$

By (3.18) and (3.10), we have

$$\begin{aligned}
I(T) &\ge \int_{2k}^{T/2} dt \ln^{-1}(t+k+1)(t+k+1)^{-q} \int_0^{2k} \sigma^{m+1} \int_{R^n} |u(t-\sigma,x)|^p dx d\sigma \\
&\ge \int_{k}^{T/2-k} dt \ln^{-1}(t+2k+1)(t+2k+1)^{-q} \int_k^{2k} \sigma^{m+1} \int_{R^n} |u(t+k-\sigma,x)|^p dx d\sigma \\
&\ge k \int_{k}^{T/2-k} dt \ln^{-1}(t+2k+1)(t+2k+1)^{-q} \int_0^{k} (\sigma+k)^{m} \int_{R^n} |u(t-\sigma,x)|^p dx d\sigma \\
&\ge k \int_{k+T_0}^{T/2-k} dt \ln^{-1}(t+2k+1)(t+2k+1)^{-q} \int_0^{k} \sigma^{m} \int_{R^n} |u(t-\sigma,x)|^p dx d\sigma.
\end{aligned} \tag{3.24}$$

By (2.8)

$$\int_0^k \sigma^m \int_{R^n} |u(t-\sigma,x)|^p dx d\sigma \ge c(t+2k+1)^{n-1-\frac{n-1}{2}p}, \quad t \ge T_0. \tag{3.25}$$

It then follows from (3.23) that

$$\begin{aligned} I(T) &\geq ck \int_{k+T_0}^{T/2-k} dt \ln^{-1}(t+2k+1)(t+2k+1)^{-1} \\ &= ck\Big(\ln\ln\Big(\frac{T}{2}+k+1\Big) - \ln\ln(3k+T_0+1)\Big). \end{aligned} \tag{3.26}$$

This combining with (3.16) yields an upper bound

$$\ln\ln\Big(\frac{T}{2}+k+1\Big) \leq C_1, \tag{3.27}$$

where C_1 is a positive constant depending only on the initial data. This gives (2.3).

It remains to prove (3.16). In the following, we shall denote by C a positive constant independent of T. The meaning of C will change from step to step.

It is easy to see that

$$\int_0^{2k} \sigma^{m+1}|u(t-\sigma,x)|^p d\sigma = \Box w(t,x), \quad t \geq 2k, \tag{3.28}$$

where

$$w(t,x) = \int_0^{2k} \sigma^{m+1} u(t-\sigma,x)d\sigma. \tag{3.29}$$

Therefore, noting (3.13), we have

$$\begin{aligned} I(T) &= \int_{2k}^T \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1) \int_{R^n} \Phi_q(\partial_t^2 w - \triangle w)dxdt \\ &= \int_{2k}^T \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1) \int_{R^n} (\Phi_q\partial_t^2 w - \triangle\Phi_q w)dxdt \\ &= \int_{2k}^T \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1) \int_{R^n} (\Phi_q\partial_t^2 w - \partial_t^2\Phi_q w)dxdt \\ &= \int_{2k}^T \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1)\partial_t \int_{R^n} (\Phi_q\partial_t w - \partial_t\Phi_q w)dxdt \\ &= \int_{2k}^T \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1)\Big[\partial_t^2 \int_{R^n} \Phi_q w dx - 2\partial_t \int_{R^n} \partial_t\Phi_q w dx\Big]dt \\ &= \int_{2k}^T \theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1)\Big[\partial_t^2 \int_{R^n} \Phi_q w dx - 2\partial_t \int_{R^n} \Phi_{q+1} w dx\Big]dt. \end{aligned} \tag{3.30}$$

Integrating by parts, we get

$$I(T) = -\mathrm{I}_0 + \mathrm{I}_1 + \mathrm{I}_2, \tag{3.31}$$

where

$$\begin{aligned} \mathrm{I}_0 &= \ln^{-1}(3k+1) \int_{R^n} [\Phi_q(2k,x)w_t(2k,x) + q\Phi_{q+1}(2k,x)w(2k,x)]dx \\ &\quad + \ln^{-2}(3k+1)(3k+1)^{-1} \int_{R^n} \Phi(2k,x)w(2k,x)dx, \end{aligned} \tag{3.32}$$

$$\mathrm{I}_1 = -2q \int_{2k}^T \partial_t\Big[\theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1)\Big] \int_{R^n} \Phi_{q+1}(t,x)w(t,x)dxdt, \tag{3.33}$$

$$\mathrm{I}_2 = \int_{2k}^T \partial_t^2\Big[\theta^{2p'}\Big(\frac{t}{T}\Big) \ln^{-1}(t+k+1)\Big] \int_{R^n} \Phi_q(t,x)w(t,x)dxdt. \tag{3.34}$$

We first estimate I_0. We have

$$u = v + u^0. \tag{3.35}$$

Thus, by (2.7), we can get

$$\begin{aligned} w(2k,x) &= \int_0^{2k} \sigma^{m+1} v(2k-\sigma,x)d\sigma + \int_0^{2k} \sigma^{m+1} u^0(2k-\sigma,x)d\sigma \\ &\geq \int_0^{2k} \sigma^{m+1} u^0(2k-\sigma,x)d\sigma = f_1(x). \end{aligned} \tag{3.36}$$

Similarly, by (2.6), we have

$$\begin{aligned} w_t(2k,x) &= (m+1)\int_0^{2k} \sigma^{m} v(2k-\sigma,x)d\sigma + (m+1)\int_0^{2k} \sigma^{m} u^0(2k-\sigma,x)d\sigma \\ &\geq (m+1)\int_0^{2k} \sigma^{m} u^0(2k-\sigma,x)d\sigma = f_2(x). \end{aligned} \tag{3.37}$$

Thus

$$\begin{aligned} \mathrm{I}_0 &\geq \ln^{-1}(3k+1)\int_{R^n}[\Phi_q(2k,x)f_2(x) + q\Phi_{q+1}(2k,x)f_1(x)]dx \\ &\quad + \ln^{-2}(3k+1)(3k+1)^{-1}\int_{R^n}\Phi(2k,x)f_1(x)dx \\ &\geq -C. \end{aligned} \tag{3.38}$$

We now estimate I_1. By (3.19) and (3.12), we have

$$\Phi_{q+1}(t,x) \leq C_0(t+k+1)^{-\frac{n-1}{2}}(t+k+1-|x|)^{-(q-\frac{n-3}{2})}. \tag{3.39}$$

Similarly, by (3.18) and (3.10), we have

$$C_0^{-1}(t+k+1)^{-q} \leq \Phi_q(t,x) \leq C_0(t+k+1)^{-q}. \tag{3.40}$$

It then follows from Hölder's inequality that

$$\Big|\int_{R^n}\Phi_{q+1}(t,x)w(t,x)dx\Big| \leq \Big(\int_{|x|\leq t+k}\Phi_q\Big(\frac{\Phi_{q+1}}{\Phi_q}\Big)^{p'}dx\Big)^{\frac{1}{p'}}\Big(\int_{R^n}\Phi_q|w|^p dx\Big)^{\frac{1}{p}}. \tag{3.41}$$

We have

$$\Phi_q\Big(\frac{\Phi_{q+1}}{\Phi_q}\Big)^{p'} \leq C_0(t+k+1)^{-q+(q-\frac{n-1}{2})p'}(t+k+1)^{-p'(q-\frac{n-3}{2})}.$$

Noting (3.21) and (3.22), we have

$$\begin{aligned} \Big(\int_{|x|\leq t+k}\Phi_q\Big(\frac{\Phi_{q+1}}{\Phi_q}\Big)^{p'}dx\Big)^{\frac{1}{p'}} &\leq C(t+k+1)^{(n-1-q)\frac{1}{p'}+q-\frac{n-1}{2}}\Big(\int_0^{t+k}(t+k+1-r)^{-1}dr\Big)^{\frac{1}{p'}} \\ &\leq C(t+k+1)^{1-\frac{1}{p'}}(\ln(t+k+1))^{1-\frac{1}{p}}. \end{aligned}$$

Again, it follows from Hölder's ineqality that

$$\begin{aligned} \mathrm{I}_1 &\leq C\Big[T^{-1}\Big(\int_{T/2}^{T}(t+k+1)^{p'-1}dt\Big)^{\frac{1}{p'}} + \Big(\int_{2k}^{T}\ln^{-p'}(t+k+1)(t+k+1)^{-1}dt\Big)^{\frac{1}{p'}}\Big]I^{\frac{1}{p}}(T) \\ &\leq CI^{\frac{1}{p'}}(T). \end{aligned} \tag{3.42}$$

Noting (3.20), we have

$$\Big|\int_{R^n}\Phi_q w dx\Big| \le \Big(\int_{|x|\le t+k}\Phi_q dx\Big)^{\frac{1}{p'}}\Big(\int_{R^n}\Phi_q|w|^p dx\Big)^{\frac{1}{p}} \le C(t+k+1)^{\frac{n-q}{p'}}\Big(\int_{R^n}\Phi_q|w|^p dx\Big)^{\frac{1}{p}}$$
$$= C(t+k+1)^{2-\frac{1}{p'}}\Big(\int_{R^n}\Phi_q|w|^p dx\Big)^{\frac{1}{p}}.$$

Thus, in a similar way, we can get

$$\mathrm{I}_2 \le CI^{\frac{1}{p}}(T). \tag{3.43}$$

Therefore, it follows from (3.38), (3.42) and (3.43) that

$$I(T) \le C + CI^{\frac{1}{p}}(T). \tag{3.44}$$

Therefore, (3.16) follows by Young's ineqality.

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References

[1] Georgiev, V., Lindblad, H. and Sogge, C. D., Weighted Stricharz estimates and global existence for semilinear wave equations, *Amer. J. Math.*, **119**, 1997, 1291–1319.

[2] Glassey, R. T., Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.*, **177**, 1981, 323–340.

[3] Glassey, R. T., Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.*, **178**, 1981, 233–261.

[4] John, F., Blow up of solutions of nonlinear wave equations in three space dimensions, *Manuscripta Math.*, **28**, 1979, 235–268.

[5] Schaeffer, J., The equation $\Box u = |u|^p$ for the critical value of p, *Proc. Royal Soc. Edinburgh*, **101**, 1985, 31–34.

[6] Sideris, T. C., Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differential Equations*, **52**, 1984, 378–406.

[7] Strauss, W. A., Nonlinear scattering theory at low energy, *J. Funct. Anal.*, **41**, 1981, 110–133.

[8] Takamura, H., An elementary proof of the exponential blow-up for semilinear wave equations, *Math. Meth. Appl. Sci*, **17**, 1994, 239–249.

[9] Tataru, D., Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equations, *Trans. Amer. Math. Soc.*, **353**, 2001, 795–807.

[10] Yordanov, B. T. and Zhang, Q. S., Finite time blow up for critical wave equations in high dimensions, *Journal of Funct. Anal.*, **231**, 2006, 367–374.

[11] Zhang, Q. S., A blow-up result for a nonlinear wave equation with damping: The critical case, *C. R. Acad. Sci. Paris.*, **333**, 2001, 109–114.

[12] Zhou, Y., Blow up of classical solutions to $\Box u = |u|^{1+\alpha}$ in three space dimensions, *J. Part. Diff. Equations*, **5**, 1992, 21–32.

[13] Zhou, Y., Life span of classical solutions to $\Box u = |u|^p$ in two space dimensions, *Chin. Ann. Math.*, **14B**(2), 1993, 225–236.

[14] Zhou, Y., Cauchy problem for semilinear wave equations in four space dimensions with small initial data, *J. Part. Diff. Equations*, **8**, 1995, 135–144.