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Bott Generator and Its Application to an Almost Complex Structure on S^6 ***

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Abstract The Bott generator of the homotopy group $\pi_{2k-1}\mathrm{U}(\infty)$ is used to construct an almost complex structure on S^6, which is integrable except a small neighborhood.

Keywords Bott generator, Almost complex structure, Integrability
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1 The Bott Generator of $\pi_{2k-1}\mathrm{U}(\infty)$

Let $\mathrm{U}(k)$ be the unitary group. The homotopy sequence of the fibration $\mathrm{U}(k) \to \mathrm{U}(k+1) \xrightarrow{\pi} S^{2k+1}$ implies $\pi_{2k-1}\mathrm{U}(k) \cong \pi_{2k-1}\mathrm{U}(k+1)$, consequently

$$\pi_{2k-1}\mathrm{U}(k) \cong \pi_{2k-1}\mathrm{U}(\infty) \cong \widetilde{K}(S^{2k}). \tag{1.1}$$

The celebrated Bott periodicity theorem (see [3]) asserts $\widetilde{K}(S^{2k}) \cong \mathbb{Z}$. A natural question is to search for an explicit generator of the infinite group $\pi_{2k-1}U(k)$. Note that the unitary group $\mathrm{U}(k)$ is a deformation retract of the general linear group $\mathrm{GL}(k;\mathbb{C})$, and that $\mathrm{U}(k)$ is diffeomorphic to $S^1 \times \mathrm{SU}(k)$. It turns out that

$$\pi_{2k-1}\mathrm{SU}(k) \cong \pi_{2k-1}\mathrm{U}(k) \cong \pi_{2k-1}\mathrm{GL}(k;\mathbb{C}) \cong \mathbb{Z} \tag{1.2}$$

for $k > 1$.

Following Atiyah [1], we consider continuous maps

$$f : S^{2k-1} \to \mathrm{GL}(k;\mathbb{C}). \tag{1.3}$$

The first row of the matrix f defines a map

$$f_1 : S^{2k-1} \to \mathbb{C}^k - \{0\},$$

so that $\frac{f_1}{|f_1|}$ is a map from S^{2k-1} to itself, whose mapping degree makes sense. We then put

$$\deg f = \frac{(-1)^{k-1}}{(k-1)!}\deg \frac{f_1}{|f_1|}. \tag{1.4}$$

The theorem of Bott, described by Atiyah [1], is then as follows

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Theorem 1.1 *The map* $f : S^{2k-1} \to \mathrm{GL}(k;\mathbb{C})$ *can be deformed into another map* $g : S^{n-1} \to \mathrm{GL}(N;\mathbb{C})$ *if and only if* $\deg f = \deg g$*; moreover there exists a map with any given degree.*

In fact if $2N > n$, one can show that $f : S^{2k-1} \to \mathrm{GL}(k;\mathbb{C})$ is always deformed into a map g so that

$$g(x) = \mathrm{diag}(I, h(x)), \tag{1.5}$$

where $h(x) \in \mathrm{GL}([\frac{n}{2}];\mathbb{C})$. If m is odd, the Bott periodicity theorem says that every map $f : S^{m-1} \to \mathrm{GL}(N;\mathbb{C})$ is deformed into a constant map for $2N \geq m$.

To understand the reason for the unexpected factor $(k-1)!$ in the definition of the degree, we look at the homotopy sequence of the fibration $\mathrm{U}(k-1) \to \mathrm{U}(k) \xrightarrow{\pi} S^{2k-1}$:

$$\pi_{2k-1}\mathrm{U}(k-1) \to \pi_{2k-1}\mathrm{U}(k) \xrightarrow{\pi_*} \pi_{2k-1}S^{2k-1} \to \pi_{2k-2}\mathrm{U}(k-1) \to 0. \tag{1.6}$$

Since $\pi_{2k-2}\mathrm{U}(k-1) \cong \mathbb{Z}_{(k-1)!}$ and $\pi_{2k-1}\mathrm{U}(k-1)$ is isomorphic to 0 or $\mathbb{Z}_2$ (see [4]), it follows that the map $\pi_* : \pi_{2k-1}\mathrm{U}(k) \to \pi_{2k-1}S^{2k-1} \cong \mathbb{Z}$ is given by sending 1 to $(k-1)!$.

We now return to the generator of $\pi_{2k-1}\mathrm{U}(\infty)$. Suppose then that

$$f : S^{n-1} \to \mathrm{GL}(N;\mathbb{C}), \quad g : S^{m-1} \to \mathrm{GL}(M;\mathbb{C})$$

are two given maps. One defines their tensor product $f * g : S^{m+n-1} \to \mathrm{GL}(2MN;\mathbb{C})$ by

$$(x, y) \mapsto \begin{pmatrix} f(x) \otimes \mathrm{Id}_M & -\mathrm{Id}_N \otimes g^*(y) \\ \mathrm{Id}_N \otimes g(y) & f^*(x) \otimes \mathrm{Id}_M \end{pmatrix},$$

where f and g are extended homogeneously to all of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, $f^*(x)$ is the transposed conjugate of the matrix $f(x)$ and Id_M denotes the identity matrix of degree M. It is easy to check that for $(x, y) \neq (0, 0)$, $f * g(x, y)$ is nonsingular, thereby $f * g$ defines a continuous map $S^{m+n-1} \to \mathrm{GL}(2MN;\mathbb{C})$.

If m and n are both even, one has the multiplicative formula

$$\deg(f * g) = \deg f \cdot \deg g. \tag{1.7}$$

It then follows that $\deg(e_k) = 1$, where $e_1 : S^1 \to \mathrm{GL}(1, \mathbb{C})$ is the standard map $e_1(z) = z$ and the map

$$e_k = e_1 * \cdots * e_1 \ (k\text{-times}) : S^{2k-1} \to \mathrm{GL}(2^{k-1};\mathbb{C})$$

is its k-fold power. Therefore, product theory furnishes the Bott generating map, meaning a generating element of the infinite cyclic group of homotopy classes of maps from S^{2k-1} to $\mathrm{GL}(2^{k-1}, \mathbb{C})$.

In order to get a generating element of $\pi_{2k-1}\mathrm{U}(k)$, it suffices to deform $e_k : S^{2k-1} \to \mathrm{GL}(2^{k-1};\mathbb{C})$ into a map

$$G_k : S^{2k-1} \to \mathrm{GL}(k;\mathbb{C}) \hookrightarrow \mathrm{GL}(2^{k-1};\mathbb{C}),$$

where the notation $\hookrightarrow$ is the inclusion.

We conclude this section with an interesting relation with the gauge theory given by the general index theory for families in [2]. Let Y be a fiber bundle over S^2 with fiber S^{2k-2}, and let V be a complex vector bundle over Y of rank k with a unitary connection, so that the gauge group is $\mathrm{U}(k)$ or $\mathrm{SU}(k)$. Let $\mathcal{G}$ be the group of gauge transformations. According to [2], one has

$$\pi_1(\mathcal{G}) \cong \pi_{2k-1}\mathrm{SU}(k) \cong \mathbb{Z}.$$

Therefore, the Bott generator of $\pi_{2k-1}\mathrm{SU}(k)$ will bring us an interpretation of a generator of the fundamental group of the group of gauge transformations.

2 The Bott Generator of $\pi_5\mathrm{SU}(3)$

The main result of this section is as follows

Proposition 2.1 *$G(z) = z\,z^t + h(\overline{z})$ is the Bott generator of the homotopy group $\pi_5\mathrm{SU}(3)$, where $z = (z_1, z_2, z_3)^t \in S^5 \subset \mathbb{C}$, and*

$$h(\overline{z}) = \begin{pmatrix} 0 & -\overline{z}_3 & \overline{z}_2 \\ \overline{z}_3 & 0 & -\overline{z}_1 \\ -\overline{z}_2 & \overline{z}_1 & 0 \end{pmatrix}.$$

Proof First a straightforward computation gives that

$$\det G(z) = |z|^4 = 1$$

for $z \in S^5$. Next, from the following clear equalities

$$\begin{cases} h(z)z = 0, \\ z\,\overline{z}^t - h(\overline{z})\,h(z) = |z|^2\, I, \end{cases} \tag{2.1}$$

it follows that $G(z)\,\overline{G(z)}^t = I$ and then $G(z)$ belongs to $\mathrm{SU}(3)$ for each $z \in S^5$.

It remains to show that $G(z) : S^5 \to \mathrm{SU}(3)$ is homotopic to the Bott fenerator $e_3 : S^5 \to \mathrm{GL}(4;\mathbb{C})$, where we use the inclusion $\mathrm{SU}(3) \hookrightarrow \mathrm{GL}(3;\mathbb{C}) \hookrightarrow \mathrm{GL}(4;\mathbb{C})$. Recall that $e_3 = e_1 * e_2$ is given by

$$e_3(z) = \begin{pmatrix} z_1 & 0 & -\overline{z}_2 & -\overline{z}_3 \\ 0 & z_1 & z_3 & -z_2 \\ z_2 & -\overline{z}_3 & \overline{z}_1 & 0 \\ z_3 & \overline{z}_2 & 0 & \overline{z}_1 \end{pmatrix}. \tag{2.2}$$

Let us introduce five maps $T_j = T_j(z) : S^5 \to \mathrm{GL}(4;\mathbb{C})$ defined by

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & \overline{z}_1 & \overline{z}_2 & \overline{z}_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -z_1 & 1 & 0 & 0 \\ -z_2 & 0 & 1 & 0 \\ -z_3 & 0 & 0 & 1 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 1 & -z_1 & -z_3 & z_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $\det T_2(z) = \det T_3(z) = \det T_4(z) = 1$ for all $z \in \mathbb{R}^6$, it follows that T_j $(j = 2, 3, 4) : S^5 \to \mathrm{GL}(4 : \mathbb{C})$ is homotopic to the constant map I. Moreover, since $\mathrm{GL}(4;\mathbb{C})$ is connected, T_1 and

T_5 are both homotopic to the constant map I, too. Therefore we arrive at

$$e_3 \simeq T_3T_2T_1e_3T_4T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1^2 & z_1z_2-\overline{z}_3 & z_1z_3+\overline{z}_2 \\ 0 & z_2z_1+\overline{z}_3 & z_2^2 & z_2z_3-\overline{z}_1 \\ 0 & z_3z_1-\overline{z}_2 & z_3z_2+\overline{z}_1 & z_3^2 \end{pmatrix},$$

which completes the proof.

3 An Interesting Almost Complex Structure on S^6

We start with some preparations. Following [5], let us embed $\mathrm{gl}(m;\mathbb{C})$ into $\mathrm{gl}(2m;\mathbb{R})$ in a natural way: $A+\sqrt{-1}B \mapsto \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$, where A, B belong to $\mathrm{gl}(m;\mathbb{R})$. Clearly any P in $\mathrm{GL}(2m;\mathbb{R})$ has a unique decomposition

$$P = P_1 + P_2S, \tag{3.1}$$

where P_j $(j=1,2)$ belongs to $\mathrm{gl}(m;\mathbb{C})$, $S = \mathrm{diag}(I_m, -I_m)$. The formula that $QS = S\overline{Q}$ for $Q \in \mathrm{gl}(m;\mathbb{C})$ will be useful later, where $\overline{Q}$ denotes the complex conjugate of Q.

In order to define an almost complex structure on S^6, we need a coordinate system $S^6 = (S^6 - A_+) \cup (S^6 - A_-)$ with $f_+ : \mathbb{R}^6 \to S^6 - \{A_+\}$ and $f_- : \mathbb{R}^6 \to S^6 - \{A_-\}$, defined by

$$\begin{aligned} f_+(u) &= (1+|u|^2)^{-1}(2u^t, |u|^2-1)^t, \\ f_-(v) &= (1+|v|^2)^{-1}(2\overline{v}^t, 1-|v|^2)^t, \end{aligned} \tag{3.2}$$

where $u = (u_1, \cdots, u_6)^t$, $A_+ = (0, \cdots, 0, 1)^t$, $A_- = (0, \cdots, 0, -1)^t$, $\overline{v} = (v_1, v_2, v_3, -v_4, -v_5, -v_6)^t$. It follows from (3.2) that the coordinate change

$$f_-^{-1} \circ f_+ : \mathbb{R}^6 - \{0\} \to \mathbb{R}^6 - \{0\}$$

is given by $u \mapsto v = \frac{\overline{u}}{|u|^2}$. Clearly we have the Jacobian $(\frac{\partial v_i}{\partial u_j}) = SA$, where $A = |u|^{-4}(|u|^2 I - 2u\,u^t)$. It is not difficult to verify that $|u|^2 SA$ belongs to $\mathrm{SO}(6)$ for each $u \neq 0 \in \mathbb{R}^6$.

We now define our almost complex structure J on S^6 in terms of the coordinate system $v_1, \cdots, v_6$ simply by

$$J\Big(\frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_6}\Big) = \Big(\frac{\partial}{\partial v_1}, \cdots, \frac{\partial}{\partial v_6}\Big) J_0 \quad \text{for } |v| \le 1, \tag{3.3}$$

where $J_0 = \left(\begin{smallmatrix} 0 & -I_m \\ I_m & 0 \end{smallmatrix}\right)$. By the coordinate change $v = \frac{\overline{u}}{|u|^2}$, (3.3) is equivalent to

$$J\Big(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_6}\Big) = \Big(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_6}\Big) Q^{-1} J_0 Q \tag{3.4}$$

for $1 \le |u| \le +\infty$, where $Q = SA$.

We have got an almost complex structure J on $\{u \mid u \in \mathbb{R}^6 \cup \{\infty\}, 1 \le |u| \le +\infty\} \subset S^6$, and the next thing to do is to extend smoothly the definition of the matrix (of functions) $Q^{-1}J_0Q$ to the field $\{u \mid u \in \mathbb{R}^6, 0 \le |u| < 1\}$. The idea originates from the homotopy theory. As it is well known, one has two one to one correspondences:

$$\{X^{-1}J_0X \mid X \in \mathrm{SO}(6)\} \longleftrightarrow \{Y_{6\times 6} \mid Y^2 = -I, Y^t = -Y\} \longleftrightarrow \mathrm{SO}(6)/\mathrm{U}(3) \cong \mathbb{C}P^3.$$

The fibration $\mathrm{U}(3) \hookrightarrow \mathrm{SO}(6) \to \mathrm{SO}(6)/\mathrm{U}(3) = \mathbb{C}P^3$ has its homotopy sequence:

$$0 \cong \pi_6\mathbb{C}P^3 \to \pi_5\mathrm{U}(3) \to \pi_5\mathrm{SO}(6) \to \pi_5\mathbb{C}P^3 \cong 0.$$

Thus we get an isomorphism $\pi_5\mathrm{U}(3) \cong \pi_5\mathrm{SO}(6)$. Recall that $G(z) = |z|^{-2} z\, z^t + |z|^{-1} h(\overline{z})$ is the Bott generator of $\pi_5\mathrm{U}(3)$, and $|u|^2 AS$ is the "positive" generator of $\pi_5\mathrm{SO}(6)$ (see [6]). These arguments implies that $G(|u|^2AS)^{-1}$ is homotopic to the constant map I, and then

$$I \cong G(|u|^2AS)^{-1} = G(|u|^2AS)^t = |u|^2GQ. \tag{3.5}$$

For the sake of convenience, we rewrite A as the form in (3.1):

$$A = |z|^{-4}\{|z|^2 I - (z\overline{z}^t + zz^t S)\}, \tag{3.6}$$

where $z = (z_1, z_2, z_3)^t = (u_1 + \sqrt{-1}u_4, u_2 + \sqrt{-1}u_5, u_3 + \sqrt{-1}u_6)^t \in \mathbb{C}^3$. Therefore we have from (3.5) that

$$\begin{aligned} I \simeq |z|^2GQ &= (|z|^{-2} z\, z^t + |z|^{-1}h(\overline{z}))(|z|^{-2}(|z|^2 I - \overline{z}z^t)S - |z|^{-2}\overline{z}\,\overline{z}^t) \\ &= |z|^{-2}(|z|h(\overline{z})S - z\overline{z}^t). \end{aligned} \tag{3.7}$$

We are now in a position to construct our extension as follows.

Lemma 3.1 *Let f and g be real smooth functions of $|z|^2, z \in \mathbb{C}^3$, with boundary conditions:*
(1) $f \equiv 1$ for $|z| \le \frac{\varepsilon}{2}$, $f \equiv 0$ for $|z| \ge \varepsilon$, and $1 > f > 0$ for $\varepsilon > |z| > \frac{\varepsilon}{2}$;
(2) $g \equiv 0$ for $|z| \le \frac{\varepsilon}{2}$, $g \equiv 1$ for $|z| \ge \varepsilon$, and $1 > g > 0$ for $\varepsilon > |z| > \frac{\varepsilon}{2}$.
Let $P(z) = (f\,\sqrt{-1}\,I - g\, z\overline{z}^t) + |z| g\, h(\overline{z})S$. For any $z \ne 0 \in \mathbb{C}^3$, $P(z)$ is, up to an non-zero factor, an element of SO(6).

Proof We observe that $P^t = (-f\sqrt{-1}\,I - gz\,\overline{z}^t) - |z|gh(\overline{z})S$, and find that

$$\begin{aligned} P\,P^t &= ((f\sqrt{-1}\,I - gz\,\overline{z}^t) + |z|gh(\overline{z})S)((-f\sqrt{-1}\,I - gz\,\overline{z}^t) - |z|gh(\overline{z})S) \\ &= f^2 I + g^2|z|^2(z\,\overline{z}^t - h(\overline{z})h(z)) = (f^2 + g^2|z|^4)I, \end{aligned}$$

which is nonsingular owing to the definitions of f and g. It completes the proof.

Summarizing the arguments above, we have established

Theorem 3.1 *There is an almost complex structure J on $S^6 = \mathbb{R}^6 \cup \{\infty\} = \{u \mid 0 \le |u| \le +\infty\}$ defined by*

$$J\Big(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_6}\Big) = \Big(\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_6}\Big)P^{-1}J_0P,$$

where P is given in Lemma 3.1. *J is integrable for $|u| \le \frac{\varepsilon}{2}$ or $\varepsilon \le |u| \le +\infty$.*

The proof of this theorem is omitted.

4 Integrability of the Almost Complex Structure *J*

Proposition 4.1 *The almost complex structure J defined in Lemma* 3.1 *is not integrable for $\frac{\varepsilon}{2} < |u| < \varepsilon$.*

Proof Observe that if $f \neq 0$ (in fact it holds for $\frac{\varepsilon}{2} < |z| < \varepsilon$), then the complex matrix $f\sqrt{-1}\,I - gz\overline{z}^t$ has its inverse as

$$(f\sqrt{-1}\,I - gz\overline{z}^t)^{-1} = \frac{-\sqrt{-1}}{f}\,I - \frac{g}{f(f+g|z|^2\sqrt{-1})}z\overline{z}^t.$$

Thus the almost complex structure defined by $P(z) = (f\,\sqrt{-1}\,I - gz\overline{z}^t) + g\,|z|h(\overline{z})\,S$ is equivalent to that of

$$(f\,\sqrt{-1}\,I - g\,z\overline{z}^t)^{-1}P = I - g\,f^{-1}\,|z|\,\sqrt{-1}\,h(\overline{z})\,S.$$

However, the non-integrability of $I - g\,f^{-1}\,|z|\,\sqrt{-1}\,h(\overline{z})\,S$ is an immediate consequence of the following criterion (see [5]).

Lemma 4.1 *$I_m + VS$ is integrable if and only if $T^i_{jk} = T^i_{kj}$ for all $1 \leq i, j, k \leq m$, where $(v_{ij})_k = \frac{\partial}{\partial z_k}(v_{ij})$, $(v_{ij})_{\overline{k}} = \frac{\partial}{\partial \overline{z}_k}(v_{ij})$, and $T^i_{jk} = (v_{ij})_{\overline{k}} - \sum\limits_s (v_{ij})_s\, v_{sk}$.*

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