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Anti-integrability for the Logistic Maps

Yi-Chiuan CHEN*

Abstract The embedding of the Bernoulli shift into the logistic map $x \mapsto \mu x(1-x)$ for $\mu > 4$ is reinterpreted by the theory of anti-integrability: it is inherited from the anti-integrable limit $\mu \to \infty$.

Keywords Logistic maps, Hyperbolicity, Symbolic dynamics, Anti-integrable limit
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1 Introduction

There is no doubt that one of the most extensively studied dynamical systems is the family of logistic maps on the interval $[0,1]$:

$$x_{i+1} = f_\mu(x_i) = \mu x_i(1-x_i), \quad \mu \geq 0, \tag{1.1}$$

which, as the parameter μ increases, exhibits chaos via the route of period-doubling bifurcation (see for example [9, 12, 16]). When μ exceeds 4, it is well-known, e.g. [7, p. 46, 13, p. 124, 18, p. 38], that the map restricted to the invariant set $\Lambda_\mu := \bigcap\limits_{n=0}^{\infty} f_\mu^{-n}([0,1])$ is topologically conjugate to the Bernoulli shift with two symbols, namely the following diagram commutes

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\sigma} & \Sigma_2 \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ \Lambda_\mu & \xrightarrow{f_\mu} & \Lambda_\mu \end{array}$$

The meaning of the diagram is as follows. Let Σ_2 denote the space of sequences of 0's and 1's, $\Sigma_2 = \{\mathbf{a} \mid \mathbf{a} = \{a_i\}_{i=0}^{\infty},\ a_i = 0 \text{ or } 1\}$, σ be the Bernoulli shift acting in such a way that $\sigma(\mathbf{a}) = \{a_1, a_2, a_3, \ldots\} = \{\sigma(\mathbf{a})_i\}_{i=0}^{\infty}$ with $\sigma(\mathbf{a})_i = a_{i+1}$, and h, called the conjugacy, be the homeomorphism from the compact set Σ_2 (with the product topology) to Λ_μ. Partition the interval $[0,1]$ into two parts $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$; then an orbit $\mathbf{x} = \{x_i\}_{i=0}^{\infty}$ of $x_0 \in \Lambda_\mu$ under the map f_μ gives a sequence $\mathbf{a}$ of symbols by assigning for each $i \geq 0$ that $a_i = 0$ if $x_i < \frac{1}{2}$ or $a_i = 1$ if $x_i > \frac{1}{2}$. The conjugacy indicates there is a one to one correspondence between the orbit with initial point x_0 and the sequence $\mathbf{a}$ of symbols. We call $\mathbf{a} = h^{-1}(x_0)$ the itinerary of the orbit determined by x_0. The conjugacy also indicates Λ_μ is a Cantor set. (A simple proof of Λ_μ being a Cantor set, without using h, was given by Zeller and Thaler [19].)

In this paper, we present a new and simpler approach to the embedding of the Bernoulli shift inspired by the concept of anti-integrable limit of Aubry. Our approach is analytical and has an

Manuscript received April 5, 2005. Published online March 5, 2007.
*Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan, China.
E-mail: YCChen@math.sinica.edu.tw

advantage that it manifests a vivid picture on how the map is conjugate to the shift dynamics when μ is greater than 4. This concept was first utilised in 1990 by Aubry and Abramovici [1] to investigate chaotic orbits in the standard map. It has been extended, for example, to prove the existence of cantori for symplectic maps of arbitrary dimension (see [15]), to construct a non-zero measure set of drifting orbits in a symplectic map (see [8]), to show the existence of multi-bump trajectories in time-dependent Lagrangian systems (see [3, 5]), and to show the embedding of symbolic dynamics into scattering billiards (see [4]) and non-autonomous maps (see [6]). Here, we are going to show the family of logistic maps is also anti-integrable, and all its dynamics for μ larger than 4 can be reinterpreted as they are inherited from the anti-integrable limit $\mu \to \infty$. The philosophy behind is as follows: when μ becomes larger and larger, the invariant set Λ_μ becomes smaller and smaller, and eventually collapses to two points $x = 0$ and 1 when $\mu \to \infty$. One can think that the dynamics of the system also "collapses" at the singular limit $\mu \to \infty$, and "reduces" (see [1]) to the Bernoulli shift.

In the next section, the anti-integrable limit for the logistic maps and how the dynamics of the family collapses are formulated rigorously. Then we give a proof for the embedding of the Bernoulli shift.

2 Anti-integrability

Our results are true for all μ greater than 4, but in order to process without encountering any obstacle and to have self-contained proofs, we only consider the case $\mu > 2 + \sqrt{5}$. The reason is because we require a crucial fact that the non-trivial tangent orbits of the map are unbounded when $\mu > 4$ and this fact is easily obtained in the considered case. We point out at the end of this paper that the required fact amounts to the repelling hyperbolicity of the set Λ_μ for f_μ with $\mu > 4$.

A dynamical systems is, in Aubry's sense (see [1]), at the anti-integrable limit if it becomes non-deterministic and virtually a Bernoulli shift. The picture of the limit is particularly clear in our case.

Theorem 2.1 *The logistic map f_μ is anti-integrable at the limit $\mu \to \infty$.*

To see this, let $\epsilon = \frac{1}{\mu}$; then $\{x_i\}$, $i \geq 0$, is an orbit of (1.1) if and only if

$$x_i(1 - x_i) = \epsilon x_{i+1}. \tag{2.1}$$

When $\epsilon = 0$, the above difference equation becomes an algebraic one, and we have $x_i = 0$ or 1 for each i. Whether x_i is equal to 0 or 1 does not determine the value of x_{i+1}, therefore the system at $\epsilon = 0$ (or $\mu = \infty$) is non-deterministic. For any $\{x_i\} \in \Sigma_2$ we have $x_{i+1} = \sigma(\{x_i\})_i$, therefore the system is virtually a Bernoulli shift with two symbols. So, we have the theorem.

Any anti-integrable orbit of the logistic map is then precisely a point lying in the space Σ_2. In the sequel, we show that all the anti-integrable orbits persist as long as μ is greater than $2 + \sqrt{5}$.

We rewrite the map (1.1) as another map $F(\,\cdot\,, \epsilon)$ in the space $l_\infty := \{\mathbf{x} \mid \mathbf{x} = \{x_0, x_1, x_2, \ldots\},$

$x_i \in \mathbb{R},$ bounded$\}$ of bounded sequences with the sup norm:

$$F : l_\infty \times \mathbb{R} \to l_\infty,$$
$$(\mathbf{x}, \epsilon) \mapsto F(\mathbf{x}, \epsilon) = \{F_0(\mathbf{x}, \epsilon), F_1(\mathbf{x}, \epsilon), F_2(\mathbf{x}, \epsilon), \cdots\}$$

with $F_i(\mathbf{x}, \epsilon) = -\epsilon x_{i+1} + x_i(1 - x_i)$. It is readily to see that $\mathbf{x}$ is an orbit of (1.1) if and only if it solves $F(\mathbf{x}, \epsilon) = \mathbf{0}$. Theorem 2.1 can be rephrased as the following.

Proposition 2.1 *$F(\mathbf{x}^\dagger, 0) = \mathbf{0}$ if and only if $x_i^\dagger = 0$ or 1 for every $i \geq 0$.*

We proceed to show, by employing the implicit function theorem, that any $\mathbf{x}^\dagger$ in the proposition can be continued to ϵ less than $\frac{1}{2+\sqrt{5}}$.

Theorem 2.2 *Providing $\epsilon < \frac{1}{2+\sqrt{5}}$, then for any anti-integrable orbit $\mathbf{x}^\dagger$, there corresponds a unique C^1-family of points $\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger) = \{x_i^*(\epsilon; \mathbf{x}^\dagger)\}$ in l_∞ parametrized by ϵ such that $\mathbf{x}^*(0; \mathbf{x}^\dagger) = \mathbf{x}^\dagger$ and $F(\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger), \epsilon) = \mathbf{0}$. Conversely, if $\mathbf{x}_\epsilon$ is such a point that $F(\mathbf{x}_\epsilon, \epsilon) = \mathbf{0}$, then there exists a unique $\mathbf{x}^\dagger$ such that $F(\mathbf{x}^\dagger, 0) = \mathbf{0}$ and $\mathbf{x}^*(\epsilon; \mathbf{x}^\dagger) = \mathbf{x}_\epsilon$.*

The proof is as follows. F certainly is a C^1-map with its derivative at $\mathbf{x}$ a linear map

$$D_\mathbf{x} F(\mathbf{x}, \epsilon) : l_\infty \hookleftarrow, \quad \xi \mapsto \Big\{ \sum_{j=0}^{\infty} D_{x_j} F_i(\mathbf{x}, \epsilon) \xi_j \Big\}_{i=0}^{\infty},$$

or realized in matrix form as

$$D_\mathbf{x} F(\mathbf{x}, \epsilon)\xi = \begin{pmatrix} 1-2x_0 & -\epsilon & 0 & 0 & \cdots \\ 0 & 1-2x_1 & -\epsilon & 0 & \cdots \\ 0 & 0 & 1-2x_2 & -\epsilon & \cdots \\ 0 & 0 & 0 & 1-2x_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \vdots \end{pmatrix}.$$

Thus it is easy to see $D_\mathbf{x} F(\mathbf{x}^\dagger, 0)$ is invertible because it is a diagonal matrix with entries ± 1. So, the theorem is true for ϵ sufficiently small.

When ϵ is not zero, $D_\mathbf{x} F(\mathbf{x}, \epsilon)$ is invertible if and only if

$$-\epsilon \xi_{i+1} + (1 - 2x_i)\xi_i = \eta_i \tag{2.2}$$

possesses a unique bounded solution for any given $\eta = \{\eta_i\}_{i \geq 0} \in l_\infty$. But (2.2) has a solution

$$\xi_i = \sum_{N \geq 0} \epsilon^N \Big(\prod_{k=0}^{N} (1 - 2x_{i+k})^{-1} \Big) \eta_{i+N} = \epsilon^{-1} \sum_{N \geq 0} ((f_\mu^{N+1})'(x_i))^{-1} \eta_{i+N}, \tag{2.3}$$

which is bounded for every $i \geq 0$ because (2.3) is a geometric series coming from the expanding property that $|(1 - 2x)\epsilon^{-1}| = |f'_\mu(x)| > 1 + C$ for some positive constant C and for all x belonging to $f_\mu^{-1}([0, 1])$ (e.g. [7, p. 37, 18, p. 32]). Furthermore, notice that

$$\xi_{i+N} = \xi_i \epsilon^{-N} \prod_{k=0}^{N-1} (1 - 2x_{i+k}) = (f_\mu^N)'(x_i)\xi_i, \quad \forall\, i \geq 0,\ N \geq 1$$

is a homogeneous solution of (2.2); then by the same property, we see $\{\xi_i\}$ is unbounded unless ξ is identical to $\mathbf{0}$. This means solution (2.3) is the only bounded solution. Therefore $D_{\mathbf{x}}F(\mathbf{x},\epsilon)$ is invertible and the theorem follows from the implicit function theorem.

Since the set $\Lambda_{\frac{1}{\epsilon}}$ consists of those points in the interval $[0,1]$ whose orbits are bounded, in the light of Theorem 2.2, we infer that

$$\Lambda_{\frac{1}{\epsilon}} = \bigcup_{\mathbf{x}^\dagger\in\Sigma_2} x_0^*(\epsilon;\mathbf{x}^\dagger)$$

and that the bijectivity of h in the commutative diagram in the introduction section can be constructed by the composition of mappings

$$\mathbf{x}^\dagger \mapsto \mathbf{x}^*(\epsilon;\mathbf{x}^\dagger) \mapsto x_0^*(\epsilon;\mathbf{x}^\dagger). \tag{2.4}$$

The projection $\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger) \mapsto x_0^*(\epsilon;\mathbf{x}^\dagger)$ certainly is continuous; we can therefore accomplish the embedding simply by showing the continuation $\mathbf{x}^\dagger \mapsto \mathbf{x}^*(\epsilon;\mathbf{x}^\dagger)$ is continuous with the product topology. Then h is a continuous bijection from the compact set Σ_2 into the interval $[0,1]$ and hence is a homeomorphism. To show h is a conjugacy, suppose an initial point x_0 which determines the itinerary $h^{-1}(x_0)$; then we must have $\sigma\circ h^{-1}(x_0) = h^{-1}(x_1) = h^{-1}\circ f_\mu(x_0)$.

Lemma 2.1 *With the product topology, the mapping $\mathbf{x}^\dagger \mapsto \mathbf{x}^*(\epsilon;\mathbf{x}^\dagger)$ is continuous.*

This lemma is a consequence of the following result.

Proposition 2.2 *Suppose $F(\mathbf{x},\epsilon)=\mathbf{0}$, $F(\mathbf{y},\epsilon)=\mathbf{0}$ and suppose the corresponding itinerary sequences are such that $x_i^\dagger = y_i^\dagger$ for $i\le N$. Providing $\epsilon < \frac{1}{2+\sqrt{5}}$ then there exists $\lambda > 1$ such that $|x_i - y_i| < \lambda^{-(N-i)}|x_N - y_N|$ for $i \le N$.*

To prove the proposition, notice that the condition that itinerary sequences of $\mathbf{x}$ and $\mathbf{y}$ agree for $i\le N$ guarantees two facts that $(f_\mu^{N-i})'(z)$ have the same sign for all z lying between x_i and y_i and that $|(f_\mu^{N-i})'(z)| > (1+C)^{N-i}$ for some $C>0$. The reason is that both x_i and y_i are continued respectively from $x_i^\dagger$ and $y_i^\dagger$, but these continuations cannot cross the middle point $x=\frac{1}{2}$. Then by invoking the mean value theorem, we get $|f_\mu^{N-i}(x_i) - f_\mu^{N-i}(y_i)| > (1+C)^{N-i}|x_i - y_i|$ and the proposition follows.

Having the proposition, we can easily derive the lemma. For instance, we suppose the product topology is induced by the metric, e.g. [18, p. 37],

$$d(\mathbf{x},\mathbf{y}) = \sum_{i\ge 0}\frac{|x_i - y_i|}{3^i};$$

then the proposition implies

$$d(\mathbf{x},\mathbf{y}) < \frac{\lambda^{-N-1} - 3^{-N-1}}{\lambda^{-1} - 3^{-1}} + \sum_{i>N}\frac{1}{3^i},$$

but we already presumed that $\frac{1}{3^{N+1}} \le d(\mathbf{x}^\dagger,\mathbf{y}^\dagger) \le \frac{1}{2\cdot 3^N}$.

Theorem 2.2 indicates that $\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger)$ depends continuously on ϵ in l_∞ (with the uniform topology) for every $\mathbf{x}^\dagger$, and therefore also depends continuously on ϵ with the product topology since the latter topology is weaker than the former one. Hence we conclude Λ_μ is continuously

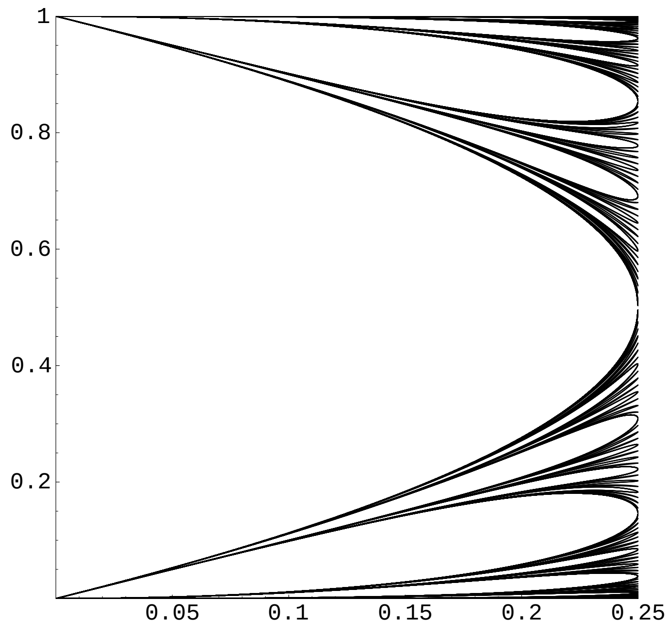

Figure 1 Depiction of Λ_μ versus $\frac{1}{\mu}$ for $4 < \mu < \infty$: horizontal axis—$\frac{1}{\mu}$, vertical axis—Λ_μ. The Cantor set Λ_μ is approximated by the set $f_\mu^{-8}(0)$, which constitutes 256 points for a given μ.

dependent upon μ. Figure 1 illustrates this fact. Notice in the figure that Λ_μ collapses to two points $\{0\}$ and $\{1\}$ when μ approaches infinity (i.e., ϵ approaches zero). The map h constructed by the composition (2.4) is no longer a conjugacy because $\mathbf{x}^*(\epsilon;\mathbf{x}^\dagger) \mapsto x_0^*(\epsilon;\mathbf{x}^\dagger)$ is not one-to-one but infinite-to-one when $\epsilon = 0$. Its being infinite-to-one is because equation (2.1) is not a dynamical system any more, thus the initial data $x_0^*(0;\mathbf{x}^\dagger) \equiv x_0^\dagger$ cannot determine the "orbit" $\mathbf{x}^*(0;\mathbf{x}^\dagger) \equiv \mathbf{x}^\dagger$. Any $\mathbf{x}^\dagger$ with $x_0^\dagger$ as its zeroth element will be mapped to $x_0^\dagger$ by h when $\epsilon = 0$. Hence the image of Σ_2 under h when $\epsilon = 0$ consists of only the two points $\{0\}$ and $\{1\}$. Consequently, $\lim_{\epsilon\to 0} \Lambda_{\frac{1}{\epsilon}}$ is not a Cantor set any more but two points. We write down what we have proved in this section, by exploiting the anti-integrable limit of the logistic maps.

Theorem 2.3 *When μ varies from $2+\sqrt{5}$ to infinity, Λ_μ form a continuous family of Cantor sets. For a given μ, the restriction of f_μ to Λ_μ is topologically conjugate to the Bernoulli shift with two symbols.*

Remark 2.1 The presented approach can be easily extended to the embedding of the Bernoulli shift with $K+1$ symbols into K-modal maps such as $f_\nu(x) = \nu x(a_1 - x)(a_2 - x)\cdots(a_{K-1} - x)(1 - x)$ for sufficiently large ν and $0 < a_1 < a_2 < \cdots < a_{K-1} < 1$.

The proofs of Theorem 2.2 and Proposition 2.2 rely on the property that $|f'_\mu(x)| > 1$ for all $x \in f_\mu^{-1}([0,1])$ when $\mu > 2+\sqrt{5}$. The case $4 < \mu \le 2+\sqrt{5}$ is subtler because for these parameter values there are points x with $|f'_\mu(x)| \le 1$; thus we need an additional property that the absolute values of the derivative of f_μ being less than or equal to one do not happen too often. It will be sufficient to obtain the desired property if we can show that for every $x \in \Lambda_\mu$ there is an integer $N_x \ge 1$ such that $|(f_\mu^{N_x})'(x)| > 1$. By virtue of the compactness of Λ_μ, it will then imply that Λ_μ is uniformly hyperbolic and vice versa (see [2, 11, 14] and [17, p. 220]). Recall that an invariant set Λ is said to be uniformly hyperbolic for a C^1-function $f : \mathbb{R} \hookleftarrow$ if there are constants $C > 0$ and $\lambda > 1$ such that $|(f^n)'(x)| \geq C\lambda^n$ for all $x \in \Lambda$ and $n \geq 1$. Therefore, the uniform hyperbolicity of an orbit $\mathbf{x}$ is sufficient to ensure the invertibility of $D_{\mathbf{x}}F(\mathbf{x}, \epsilon)$. For other proofs of the hyperbolicity, we refer the reader to the textbook of Robinson [18, pp. 33–37], where the Poincaré metric and the Schwarz lemma of complex analysis are employed, and to the pedagogical note of Glendinning [10] (see also [19]), in which an idea similar to the standard topological conjugacy between the logistic map with μ equal to 4 and the tent map with slope two are used.

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