Chin. Ann. Math.
28B(2), 2007, 225–234
DOI: 10.1007/s11401-005-0080-5

Chinese Annals of Mathematics, Series B

Augmented Spinor Space**

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Abstract In this paper, based on the Pauli matrices, a notion of augmented spinor space is introduced, and a uniqueness of such augmented spinor space of rank n is proved. It may be expected that this new notion of spaces can be used in mathematical physics and geometry.

Keywords Augmented spinor space, Pauli matrices, Jack-orientation, Super algebra, Jack map
2000 MR Subject Classification 53C80, 53C27

1 Pauli Matrices

In order to decompose the natural positive elliptic operator Δ into a square of a first-order operator

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_{2n}^2} = \Big(A_1\frac{\partial}{\partial x_1} + \cdots + A_{2n}\frac{\partial}{\partial x_{2n}}\Big)^2,$$

a set of equalities

$$\begin{cases} A_i^2 = -I, & i = 1, \cdots, 2n, \\ A_iA_j + A_jA_i = 0, & i \neq j \end{cases}$$

must be assumed, where I is a unit matrix. Dirac advised expressing the above equations by matrices, which are called Pauli matrices. About it we know the following basic theorem.

Theorem 1.1 (1) *There is at least one non-degenerated unitary* $(N \times N)$*-matrix system* $\{A_1, \cdots, A_{2n}\}$ *satisfying the above equations. Here 'non-degenerated' means that* $\{A_{i_1} \cdots A_{i_s} \mid s = 0, 1, \cdots, 2n; i_1 < \cdots < i_s\}$ *span a complex vector space of* $\dim 4^n$*, where* $N = 2^n$*.*

(2) *For another matrix system* $\{\widetilde{A}_1, \cdots, \widetilde{A}_{2n}\}$ *satisfying the above conditions, there exists a unitary* $(N \times N)$*-matrix* T*, such that*

$$\widetilde{A}_i = T \cdot A_i \cdot T^{-1}, \quad i = 1, \cdots, 2n.$$

Moreover, such a T *is unique up to a multiple* $e^{i\theta}$*, where* $\theta \in R$*.*

Remark 1.1 In physics, Pauli matrices are little different from those in the above theorem, they are defined by $A_iA_j + A_jA_i = 2\delta_{ij}I$, for $i, j = 1, \cdots, 2n$.

Manuscript received March 10, 2005. Published online March 5, 2007.
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**Project supported by the National Natural Science Foundation of China (No. 10131020).

If $C_{2n}(-1)$ is the Clifford algebra, which is an associative algebra with a unit 1 over the real number field R generated by $e_1, e_2, \cdots, e_{2n}$ subject to the relations : $e_ie_j + e_je_i = -2\delta_{ij}$, for $i, j = 1, \cdots, 2n$, then it is easy to see that Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2 (1) *There exists a non-degenerated $C_{2n}(-1)$-action on C^N, i.e.,*

$$C_{2n}(-1) \times C^N \longrightarrow C^N,$$

which satisfies

(i) $e_i(\delta_1, \cdots, \delta_N) = (\delta_1, \cdots, \delta_N)A_i$,

(ii) $\langle\langle e_i(x), e_i(y)\rangle\rangle = \langle\langle x, y\rangle\rangle$, *for* $x, y \in C^N$, *where* $\delta_i = (0, \cdots, 1, \cdots, 0)$, $\langle\langle\ ,\ \rangle\rangle$ *is the standard Hermitian product in* C^N.

(2) *For any two non-degenerated $C_{2n}(-1)$-action $\{e_i\}$, $\{\widetilde{e}_i\}$ on C^N, for $i = 1, \cdots, 2n$, there exists a complex linear transformation $T : C^N \longrightarrow C^N$ preserving the Hermitian product, i.e.,*

$$\langle\langle T(x), T(y)\rangle\rangle = \langle\langle x, y\rangle\rangle, \quad \text{for } x, y \in C^N,$$

and the following diagram is commutative

$$\begin{array}{ccc} C^N & \xrightarrow{e_i} & C^N \\ T\downarrow & & \downarrow T \\ C^N & \xrightarrow{\widetilde{e}_i} & C^N \end{array}$$

Moreover, such a transformation T is unique up to a multiple $e^{i\theta}$, where $\theta \in R$, $N = 2^n$.

We may express the above theorem in the following way.

Theorem 1.3 (1) *Let S be a complex vector space of* $\dim 2^n$ *with a Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$. Then there exists a non-degenerated $C_{2n}(-1)$-action on S, which preserves the Hermitian inner product.*

(2) *For any two vector spaces S_i with Hermitian inner products $\langle\langle\ ,\ \rangle\rangle^{(i)}$, $i = 1, 2$, and two $C_{2n}(-1)$-actions $\{e_i\}$, $\{\widetilde{e}_i\}$ on them respectively, there exists a complex linear transformation $T : S_1 \longrightarrow S_2$ preserving the Hermitian product, i.e.,*

$$\langle\langle T(u), T(v)\rangle\rangle^{(2)} = \langle\langle u, v\rangle\rangle^{(1)} \quad \text{for } u, v \in S_1.$$

At the same time, the following diagram is commutative

$$\begin{array}{ccc} S_1 & \xrightarrow{e_i} & S_1 \\ T\downarrow & & \downarrow T \\ S_2 & \xrightarrow{\widetilde{e}_i} & S_2 \end{array}$$

Moreover, such a transformation T is unique up to a multiple $e^{i\theta}$, where $\theta \in R$.

Definition 1.1 *The space S satisfying Theorem* 1.3(1) *is called a spinor space of* $\dim 2^n$ *(or rank n), whose elements are called spinors.*

Remark 1.2 Let S be a spinor space. Choose a unitary basis $\{U_1, \cdots, U_N\}$, and define matrices $[e_i]$ by

$$e_i(U_1, \cdots, U_N) = (U_1, \cdots, U_N)[e_i].$$

Then $\{[e_1], \cdots, [e_N]\}$ are Pauli matrices, where $N = 2^n$.

In this paper, we will add some other structures on a spinor space S, such that the isomorphism T in Theorem 1.3 is unique.

2 Augmented Spinor Space

Definition 2.1 *Let S be a spinor space, an anti-complex linear map $J : S \to S$ is called a Jack map if it satisfies the following two conditions*

(i) $\langle\langle Ju, Jv\rangle\rangle = \overline{\langle\langle u, v\rangle\rangle}$, *for* $u, v \in S$,

(ii) $e_i \cdot J = J \cdot e_i : S \to S$, *for* $i = 1, \cdots, 2n$.

In [1], we have proved the following result.

Theorem 2.1 *Let S be a spinor space. Then*

(i) *there exists a Jack map in S;*

(ii) *for any two Jack maps J_1, J_2, there always exists a real number θ such that $J_1 = e^{i\theta} J_2$. Moreover, $J^2 = (-1)^{\frac{n(n+1)}{2}}$.*

Definition 2.2 *For a spinor space S with a Jack map J, an element $\epsilon \in S$ is called a Jack-orientation, if it satisfies the following conditions*

$$\sqrt{-1} e_{2i} \cdot \epsilon = e_{2i-1} \cdot \epsilon, \quad \langle\langle \epsilon, \epsilon\rangle\rangle = 1, \quad J \cdot \epsilon = K_n \cdot \epsilon,$$

where

$$K_n = \begin{cases} e_1 e_3 \cdots e_{2n-1}, & \text{when } n \text{ is odd,} \\ (\sqrt{-1}\,)^n e_2 e_4 \cdots e_{2n}, & \text{when } n \text{ is even.} \end{cases}$$

Lemma 2.1 *Given a spinor space S with a Jack map J, there exist exactly two Jack-orientations; one is ϵ, and the other is $-\epsilon$.*

Definition 2.3 *A spinor space S, with a Jack map J and a Jack-orientation ϵ, is called an augmented spinor space, which is denoted by $\overrightarrow{S}$.*

Remark 2.1 The notion of augmented spinor space in this paper is different from the one in [1]; the latter does not contain the Jack-orientation.

Theorem 2.2 *Given two augmented spinor spaces $\overrightarrow{S_1}$ and $\overrightarrow{S_2}$, there exists only one isomorphism $\Phi : \overrightarrow{S_1} \longrightarrow \overrightarrow{S_2}$, such that*

(i) $\Phi \cdot e_i^{(1)} = e_i^{(2)} \cdot \Phi$, *for all* $i = 1, \cdots, 2n$,

(ii) $\langle\langle \Phi(\mu), \Phi(\nu)\rangle\rangle^{(2)} = \langle\langle \mu, \nu\rangle\rangle^{(1)}$, *for all* $\mu, \nu \in \overrightarrow{S_1}$,

(iii) $J^{(2)} \cdot \Phi = \Phi \cdot J^{(1)} : \overrightarrow{S_1} \to \overrightarrow{S_2}$,

(iv) $\Phi(\epsilon^{(1)}) = \epsilon^{(2)}$,

where $e_i^{(\alpha)}$, $\langle\langle\, , \,\rangle\rangle^{(\alpha)}$, $J^{(\alpha)}$ *and* $\epsilon^{(\alpha)}$ *are, respectively, Clifford action, Hermitian inner product, Jack map, and Jack-orientation of* $\overrightarrow{S}_\alpha$, *for* $\alpha = 1, 2$.

3 A Naive Model of Augmented Spinor Space

Because of Theorem 2.2, if rank n is fixed, then any two augmented spinor spaces differ by a unique complex linear isomorphism. Now let us try to show one of augmented spinor spaces, which is called a naive model.

Definition 3.1 *Let $\Lambda_C^*(n)$ be a Grassmann algebra over complex number field C with generators $\{1, \Omega_1, \ldots, \Omega_n\}$, i.e., it is an associative algebra with generators $\{1, \Omega_1, \ldots, \Omega_n\}$ subject to only relations $\Omega_i\Omega_j + \Omega_j\Omega_i = 0$, for $i, j = 1, \cdots, n$. Sometimes we denote $\Lambda_C^*(n)$ by $\Lambda_C^*(\Omega_1, \cdots, \Omega_n)$. In this space, we define following two complex linear maps ϵ_i and ι_i by*

$$\epsilon_i : \Lambda_C^*(\Omega_1, \cdots, \Omega_n) \to \Lambda_C^*(\Omega_1, \cdots, \Omega_n) : \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s} \to \Omega_i \wedge \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s},$$

$$\iota_i : \Lambda_C^*(\Omega_1, \cdots, \Omega_n) \to \Lambda_C^*(\Omega_1, \cdots, \Omega_n) : \Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s} \to \sum_{k=1}^{s} (-1)^{k-1} \delta_{ii_k} \Omega_{i_1} \wedge \cdots \wedge \widehat{\Omega}_{i_k} \wedge \cdots \wedge \Omega_{i_s}.$$

Proposition 3.1 *Let*

$$\mu_{2i-1} = \epsilon_i - \iota_i, \quad \mu_{2i} = -\sqrt{-1}(\epsilon_i + \iota_i) \quad \textit{for } i = 1, \cdots, n.$$

Then it is easy to see that

$$\mu_i\mu_j + \mu_j\mu_i = -2\delta_{ij}, \quad \forall\, i, j = 1, \cdots, 2n.$$

In other words, map

$$C_\mu : \mathrm{Hom}_C(\Lambda_C^*(\Omega_1, \cdots, \Omega_n)) \longrightarrow C_{2n}(-1) \otimes C : \mu_i \longmapsto e_i$$

is an algebra isomorphism.

In Grassmann algebra $\Lambda_C^*(\Omega_1, \cdots, \Omega_n)$, we will define some structures as follows.

(1) Define Hermitian product $\langle\!\langle\, , \rangle\!\rangle$ such that $\{\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s} \cdot 1 \mid 0 \le s \le n\}$ is a unitary basis. It is easy to see that Hermitian product has a property $\langle\!\langle e_i u, e_i v\rangle\!\rangle = \langle\!\langle u, v\rangle\!\rangle$, $u,\ v \in \Lambda_C^*(n)$.

(2) Define a map $f : \Lambda_C^*(\Omega_1, \cdots, \Omega_n) \longrightarrow \Lambda_C^*(\Omega_1, \cdots, \Omega_n)$ such that

$$f = \begin{cases} \mu_1\mu_3 \cdots \mu_{2n-1}, & \text{when } n \text{ is odd}, \\ (\sqrt{-1}\,)^n \mu_2\mu_4 \cdots \mu_{2n}, & \text{when } n \text{ is even}. \end{cases}$$

Let $[f]_{\overline{C}} : \Lambda_C^*(n) \to \Lambda_C^*(n)$ be an anti-complex linear map.

Lemma 3.1 *In Grassmann algebra $\Lambda_C^*(\Omega_1, \cdots, \Omega_n)$, we define a Hermitian product as above and $e_i = \mu_i$, $i = 1, \cdots, 2n$, $J = [f]_{\overline{C}}$, $\epsilon = 1 \in \Lambda_C^*(n)$. Then $\{\langle\!\langle, \rangle\!\rangle,\ e_i,\ J,\ \epsilon\}$ induces an augmented spinor space structure on $\Lambda_C^*(\Omega_1, \cdots, \Omega_n)$.*

Lemma 3.2 *Let $\overrightarrow{S}$ be an augmented spinor space. Define $\overrightarrow{S}_R$ to be a real space spanned by standard basis, i.e., $\overrightarrow{S}_R = \mathrm{span}_R\{e_{2i_1-1} \cdots e_{2i_s-1}(\epsilon) \mid s = 1, \cdots, n; i_1 < i_2 < \cdots < i_s\}$. Then*

$$\overrightarrow{S}_R = \{a \in \overrightarrow{S} \mid J(a) = K_n a\},$$

where

$$K_n = \begin{cases} e_1 e_3 \cdots e_{2n-1}, & \textit{when } n \textit{ is odd}, \\ (\sqrt{-1}\,)^n e_2 e_4 \cdots e_{2n}, & \textit{when } n \textit{ is even}. \end{cases}$$

The above Lemma 3.1 provides a naive augmented spinor structure on Grassmann algebra $\Lambda^*_C(n)$. Let us consider the augmented spinor structures on fixed Grassmann algebra $\Lambda^*_C(n)$. We denote the set of all such structures on $\Lambda^*_C(n)$ by $\mathcal{SP}(n)$. Due to the uniqueness in Theorem 2.2, any two augmented spinor structures on $\Lambda^*_C(n)$ differ by an element in $\mathrm{Hom}(\Lambda^*_C(n))$. More exactly, we have a lemma.

Lemma 3.3 *For any two* $\sigma_\alpha = \{\langle\!\langle\, , \rangle\!\rangle^{(\alpha)},\ e_i^{(\alpha)},\ J^{(\alpha)},\ \epsilon^{(\alpha)}\} \in \mathcal{SP}(n)$, $\alpha = 1, 2$, *by Theorem* 2.2 *there exists only one isomorphism* $\Phi : \Lambda^*_C(n) \longrightarrow \Lambda^*_C(n)$, *such that*

(i) $\Phi \cdot e_i^{(1)} = e_i^{(2)} \cdot \Phi$, *for all* $i = 1, \cdots, 2n$,

(ii) $\langle\!\langle \Phi(\mu), \Phi(\nu) \rangle\!\rangle^{(2)} = \langle\!\langle \mu, \nu \rangle\!\rangle^{(1)}$, *for all* $\mu, \nu \in \overrightarrow{S_1}$,

(iii) $J^{(2)} \cdot \Phi = \Phi \cdot J^{(1)} \overrightarrow{S_1} \to \overrightarrow{S_2}$,

(iv) $\Phi(\epsilon^{(1)}) = \epsilon^{(2)}$.

If we denote the relation among the above σ_1, σ_2, Φ *by*

$$\sigma_2 = \Phi * \sigma_1,$$

then given any two of σ_1, σ_2, Φ, *we can determine the third by the equality* $\sigma_2 = \Phi * \sigma_1$.

4 Tensor Product

Given two augmented spinor structures σ_1, σ_2 on $\Lambda^*_C(n)$ and $\Lambda^*_C(m)$ respectively, we try to build an augmented spinor structure on $\Lambda^*_C(n+m)$ as follows. For the simplicity of notations we denote $\Lambda^*_C(n)$ and $\Lambda^*_C(m)$ by $\overrightarrow{S_1}$, $\overrightarrow{S_2}$ respectively.

Firstly we introduce super algebra structures on $\overrightarrow{S_i}$, $i = 1, 2$, and then use these super structures to define the desired augmented spinor structure on

$$\overrightarrow{S_1} \otimes \overrightarrow{S_2} \equiv \Lambda^*_c(n+m).$$

Definition 4.1 *In an augmented spinor space* $\overrightarrow{S} \equiv \Lambda^*_C(n)$, *define a composition as* $\overrightarrow{S} = \overrightarrow{S}^+ + \overrightarrow{S}^-$, *where*

$$\overrightarrow{S}^+ = \{\alpha \in \overrightarrow{S} \mid (\widetilde{e})\alpha = \alpha\},$$
$$\overrightarrow{S}^- = \{\alpha \in \overrightarrow{S} \mid (\widetilde{e})\alpha = -\alpha\},$$

in which $\widetilde{e} = (\sqrt{-1}\,)^n e_1 e_2 \cdots e_{2n}$. *The above decomposition is called a superstructure on* $\overrightarrow{S}$. *The elements in* S^+ *and* S^- *are called even and odd elements, respectively.*

Definition 4.2 *For any augmented spinor space* $\overrightarrow{S}$, *define a superstructure of* $\mathrm{Hom}_C(\overrightarrow{S})$ *as*

$$\mathrm{Hom}_C(\overrightarrow{S}) = (\mathrm{Hom}_C(\overrightarrow{S}))^+ + (\mathrm{Hom}_C(\overrightarrow{S}))^-,$$

where

$$(\mathrm{Hom}_C(\overrightarrow{S}))^+ = \{f \in \mathrm{Hom}_C(\overrightarrow{S}) \mid \widetilde{e} f = f \widetilde{e}\},$$
$$(\mathrm{Hom}_C(\overrightarrow{S}))^- = \{f \in \mathrm{Hom}_C(\overrightarrow{S}) \mid \widetilde{e} f = -f \widetilde{e}\}.$$

Here $(\mathrm{Hom}_C(\overrightarrow{S}))^+$ *and* $(\mathrm{Hom}_C(\overrightarrow{S}))^-$ *are even and odd parts, respectively.*

Now we are going to define an augmented spinor structure on $\Lambda_C^*(n)\otimes\Lambda_C^*(m)$.

(1) Define a Hermitian product on spinor tensor space $\Lambda_C^*(n)\otimes\Lambda_C^*(m)$ by

$$\langle\!\langle u_1\otimes u_2, v_1\otimes v_2\rangle\!\rangle = \langle\!\langle u_1, v_1\rangle\!\rangle^{(1)}\langle\!\langle u_2, v_2\rangle\!\rangle^{(2)}, \quad \text{where } u_1, v_1\in\Lambda_C^*(n),\ u_2, v_2\in\Lambda_C^*(m).$$

(2) Define a super algebra structure on $\mathrm{Hom}_C(\Lambda_C^*(n))\otimes\mathrm{Hom}_C(\Lambda_C^*(m))$ as

$$(f_1\otimes g_1)\widehat{\cdot}(f_2\otimes g_2) = (-1)^{|f_2|\cdot|g_1|}(f_1f_2\otimes g_1g_2),$$

where

$$(-1)^{|f_2|\cdot|g_1|} = \begin{cases} 1, & \text{either } f_2 \text{ or } g_1 \text{ is even,}\\ -1, & \text{both } f_2 \text{ and } g_1 \text{ are odd.}\end{cases}$$

$(f_1\otimes g_1)\widehat{\cdot}(f_2\otimes g_2)$ is denoted by $(f_1\widehat{\otimes}g_1)\cdot(f_2\widehat{\otimes}g_2)$.

There is a super algebra isomorphism

$$\mathrm{Hom}_C(\Lambda_C^*(n))\otimes\mathrm{Hom}_C(\Lambda_C^*(m)) = \mathrm{Hom}_C(\Lambda_C^*(n)\otimes\Lambda_C^*(m))$$

such that

$$(e_i, e_j)\mapsto(e_i\widehat{\otimes}1)\cdot(1\widehat{\otimes}e_j).$$

From the above definitions, it is easy to check that

$$\begin{cases} (e_i\widehat{\otimes}1)(e_j\widehat{\otimes}1)+(e_j\widehat{\otimes}1)(e_i\widehat{\otimes}1) = -2\delta_{ij}, & \text{for } i,j=1,\cdots,2n,\\ (e_i\widehat{\otimes}1)(1\widehat{\otimes}e_j)+(1\widehat{\otimes}e_j)(e_i\widehat{\otimes}1) = 0, & \text{for } i=1,\cdots 2n,\quad j=1,\cdots 2m,\\ (1\widehat{\otimes}e_i)(1\widehat{\otimes}e_j)+(1\widehat{\otimes}e_j)(1\widehat{\otimes}e_i) = -2\delta_{ij}, & \text{for } i,j=1,\cdots 2m.\end{cases}$$

It means a Clifford algebra generated by $\{e_1\widehat{\otimes}1,\cdots,e_{2n}\widehat{\otimes}1, 1\widehat{\otimes}e_1,\cdots,1\widehat{\otimes}e_{2m}\}$, and a Clifford action on $\Lambda_C^*(n)\otimes\Lambda_C^*(m)$. We can check that the inner product $\langle\!\langle\,,\,\rangle\!\rangle$ is invariant under the action of the set $\{e_i\widehat{\otimes}1, 1\widehat{\otimes}e_j)\}$, i.e.,

$$\langle\!\langle(e_i\widehat{\otimes}1)(u_1\otimes u_2),(e_i\widehat{\otimes}1)(v_1\otimes v_2)\rangle\!\rangle = \langle\!\langle u_1\otimes u_2, v_1\otimes v_2\rangle\!\rangle,$$
$$\langle\!\langle(1\widehat{\otimes}e_j)(u_1\otimes u_2),(1\widehat{\otimes}e_j)(v_1\otimes v_2)\rangle\!\rangle = \langle\!\langle u_1\otimes u_2, v_1\otimes v_2\rangle\!\rangle,$$

for all $u_1, v_1\in\Lambda_C^*(n),\ u_2, v_2\in\Lambda_C^*(m)$.

(3) In Grassmann algebra, let K_1, K_2 be the Jack maps of $\Lambda_C^*(n), \Lambda_C^*(m)$ respectively. Then $H\cdot(K_1\widehat{\otimes}K_2)$ is a complex conjugate linear map, which is defined as a Jack map on $\Lambda_C^*(n)\otimes\Lambda_C^*(m)$, where H is a linear map.

(4) Define a Jack-orientation ϵ of $\Lambda_C^*(n)\otimes\Lambda_C^*(m)$ to be $\epsilon_1\otimes\epsilon_2$, where $\epsilon_1,\ \epsilon_2$ are the Jack-orientations of $\Lambda_C^*(n),\ \Lambda_C^*(m)$ respectively.

It is easy to see that the above (1)–(4) define an augmented spinor structure on

$$\Lambda_C^*(n)\otimes\Lambda_C^*(m) = \Lambda_C^*(n+m).$$

Theorem 4.1 *Given two augmented spinor structures* σ_1, σ_2 *on* $\Lambda_C^*(n),\ \Lambda_C^*(m)$*, respectively, the above construction shows an augmented spinor structure* σ *on* $\Lambda_C^*(n+m) = \Lambda_C^*(n)\otimes$

$\Lambda_C^*(m)$. In other words, if we denote the above augmented spinor structure σ by $\sigma_1 \times \sigma_2$, then we have a map

$$\mathcal{SP}(n) \times \mathcal{SP}(m) \to \mathcal{SP}(n+m) : (\sigma_1, \sigma_2) \mapsto \sigma_1 \times \sigma_2.$$

Theorem 4.2 *The naive augmented spinor spaces provide naive augmented structures $\sigma_1^0, \sigma_2^0, \sigma_3^0$ on*

$$\Lambda_C^*(n),\ \Lambda_C^*(m),\ \Lambda_C^*(n+m)$$

respectively. For any augmented spinor structures $\sigma_1, \sigma_2, \sigma_3$, by using Lemma 3.3 there are $\Phi_i \in \mathrm{Hom}(S_i)$ such that

$$\sigma_i = \Phi_i * \sigma_i^0, \quad \forall\, i = 1, 2, 3,$$

where

$$S_1 = \Lambda_C^*(n), \quad S_2 = \Lambda_C^*(m), \quad S_3 = \Lambda_C^*(n+m).$$

We denote $\Phi_i = \sigma_i/\sigma_i^0$. Then Theorem 4.1 gives a map

$$U(2^n) \times U(2^m) \to U(2^{n+m}) : (\sigma_1/\sigma_1^0, \sigma_2/\sigma_2^0) \mapsto (\sigma_1 \times \sigma_2)/\sigma_3^0,$$

where $U(2^n)$ is the set of the unitary homomorphisms in $\mathrm{Hom}(\Lambda_C^*(n))$.

5 Group Actions on $\overrightarrow{S}$

Now we define the subset Spin$(2n)$ in $C_{2n}(-1)$ by

$$\mathrm{Spin}(2n) = \{u_1 \cdots u_{2k} \mid \langle u_i, u_i \rangle = 1, u_i \in R^{2n}\},$$

where

$$R^{2n} = \mathrm{Span}_R\{e_1, \cdots, e_{2n}\}.$$

Let

$$\mathrm{Spin}^C(2n) = \{e^{i\theta} g_0 \mid g_0 \in \mathrm{Spin}(2n), e^{i\theta} \in U(1)\} \subset C_{2n}(-1) \otimes C.$$

Definition 5.1 *Let $\overrightarrow{S}$ be the naive augmented spinor space. Define a $\mathrm{Spin}^C(2n)$-group action*

$$\mathrm{Spin}^C(2n) \times \overrightarrow{S} \longrightarrow \overrightarrow{S}$$

by the restriction of the algebra action $(C_{2n}(-1) \otimes C) \times \overrightarrow{S} \longrightarrow \overrightarrow{S}$.

Proposition 5.1 *For an arbitrary $g = g_0 e^{i\theta} \in \mathrm{Spin}^C(2n)$, define a map*

$$g_* : \mathrm{Hom}_R(\overrightarrow{S}, \overrightarrow{S}) \longrightarrow \mathrm{Hom}_R(\overrightarrow{S}, \overrightarrow{S})$$

by

$$g_*(f) = g \cdot f \cdot g^{-1}, \quad \forall f \in \mathrm{Hom}_R(\overrightarrow{S}, \overrightarrow{S}).$$

Then we have

$$g_*(J) = e^{2i\theta} J.$$

Proof According to the definition of g_*, we have

$$g_*(J) = g \cdot J \cdot g^{-1} = g_0 e^{i\theta} J e^{-i\theta} g_0^{-1} = g_0 e^{2i\theta} g_0^{-1} J = e^{2i\theta} J.$$

So the Proposition is proved.

The next proposition gives the relations between $U(n)$ action and $\mathrm{Spin}^C(2n)$ action on the naive model $\Lambda^*_C(n)$.

Definition 5.2 *Let $i : U(n) \longrightarrow C_{2n}(-1) \otimes C$ be the composition of maps*

$$U(n) \xrightarrow{\lambda} \mathrm{Hom}_C(\Lambda^*_C(\Omega_1, \cdots, \Omega_n)) \xrightarrow{C_\mu} C_{2n}(-1) \otimes C,$$

where $\lambda : U(n) \longrightarrow \mathrm{Hom}_C(\Lambda^*_C(\Omega_1, \cdots, \Omega_n))$ is a complex linear map defined by

$$\lambda(A)(\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s}) = (\lambda(A)\Omega_{i_1}) \wedge \cdots \wedge (\lambda(A)\Omega_{i_s}), \quad \forall A = (A_{ij}) \in U(n),$$

where $(\lambda(A)\Omega_1, \cdots, \lambda(A)\Omega_n) = (\Omega_1, \cdots, \Omega_n)A$, and C_μ is the naive isomorphism in Proposition 3.1.

Obviously, the above definition gives an imbedding of $U(n)$ into $C_{2n}(-1) \otimes C$. It is notable here that $\mathrm{Spin}^C(2n)$ is also a subset of $C_{2n}(-1) \otimes C$.

Proposition 5.2 $U(n) \subset \mathrm{Spin}^C(2n)$.

Proof It is sufficient to check $T_1(U(n)) \subset T_1(\mathrm{Spin}^C(2n))$ in $T_1(C_{2n}(-1) \otimes C)$. From [1, Proposition 2.6], we know that

$$\begin{aligned} T_1(\mathrm{Spin}^C(2n)) &= \mathrm{Span}_R\{\sqrt{-1},\ e_i e_j \in C_{2n}(-1) \mid i < j\}, \\ T_1(U(n)) &= \mathrm{Span}_R\{\sqrt{-1}\,\theta_{ii},\ \sqrt{-1}(\theta_{ij} + \theta_{ji}),\ \Xi_{ij} \mid i < j\}, \end{aligned}$$

where $(\theta_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$, $(\Xi_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}$. Letting $h = (h_{ij}) \in T_1(U(n))$, we have

$$\begin{aligned} \lambda(e^{th})(\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s}) &= (e^{th}\Omega_{i_1}) \wedge \cdots \wedge (e^{th}\Omega_{i_s}), \\ \frac{d}{dt}\Big|_{t=0} (e^{th}\Omega_{i_1}) \wedge \cdots \wedge (e^{th}\Omega_{i_s}) &= \sum_{i,j=1}^{n} h_{ij}\epsilon_i\iota_j(\Omega_{i_1} \wedge \cdots \wedge \Omega_{i_s}). \end{aligned}$$

So

$$\begin{aligned} h = \lambda_*(h) &= \sum_{i,j} h_{ij}\epsilon_i\iota_j = \sum_{i,j} h_{ij} \frac{(\epsilon_i + \iota_i) + (\epsilon_i - \iota_i)}{2} \cdot \frac{(\epsilon_j + \iota_j) - (\epsilon_j - \iota_j)}{2} \\ &= \frac{1}{4}\sum_{i,j} h_{ij}(\sqrt{-1}\,e_{2i} + e_{2i-1}) \cdot (\sqrt{-1}\,e_{2j} - e_{2j-1}). \end{aligned}$$

If $h = \sqrt{-1}\,\theta_{ii},\ \sqrt{-1}(\theta_{ij} + \theta_{ji}),$ or Ξ_{ij}, then

$$h_{\alpha\beta} = \sqrt{-1}\,\delta_{i\alpha}\delta_{i\beta}, \quad \sqrt{-1}(\delta_{i\alpha}\delta_{j\beta} + \delta_{j\alpha}\delta_{i\beta}) \quad \text{or} \quad \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha},$$

respectively. Therefore

$$\begin{aligned}\sqrt{-1}\,\theta_{ii} &= h = \lambda_*(h) = \frac{1}{4}\sqrt{-1}(\sqrt{-1}\,e_{2i} + e_{2i-1})\cdot(\sqrt{-1}\,e_{2i} - e_{2i-1})\\ &= \frac{1}{2}(\sqrt{-1} - e_{2i-1}e_{2i}),\\ \sqrt{-1}(\theta_{ij}+\theta_{ji}) &= h = \lambda_*(h) = \frac{1}{2}(e_{2i}e_{2j-1} - e_{2i-1}e_{2j}),\\ \Xi_{ij} &= h = \lambda_*(h) = \frac{1}{2}(-e_{2i}e_{2j} - e_{2i-1}e_{2j-1}).\end{aligned}$$

It means

$$T_1(U(n)) = \mathrm{Span}_R\{\sqrt{-1}\,\theta_{ii},\ \sqrt{-1}(\theta_{ij}+\theta_{ji}),\ \Xi_{ij}\} \subset \mathrm{Span}_R\{\sqrt{-1},\ e_ie_j\} = T_1(\mathrm{Spin}^C(2n)),$$

so the proposition is proved.

6 Examples

We write Pauli matrices for $n = 1$, and naive augmented spinor space for $n = 2$.

When $n = 1$, the basis of $\Lambda^*_C(1)$ can be represented by $(1, \Omega)$. Then

$$\begin{aligned}e_1(1,\Omega) &= (\epsilon - \iota)(1,\Omega) = (\Omega, -1) = (1,\Omega)[e_1],\\ e_2(1,\Omega) &= -\sqrt{-1}(\epsilon+\iota)(1,\Omega) = (-\sqrt{-1}\,\Omega, -\sqrt{-1}) = (1,\Omega)[e_2],\end{aligned}$$

where

$$[e_1] = \begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix},\quad [e_2] = \begin{pmatrix} 0 & -\sqrt{-1}\\ -\sqrt{-1} & 0\end{pmatrix}.$$

The above $[e_1], [e_2]$ are the Pauli matrices for $n = 1$.

When $n = 2$, since

$$\Lambda^*_C(\Omega_1,\Omega_2) = \Lambda^*_C(\Omega_1)\otimes\Lambda^*_C(\Omega_2) = (1\otimes 1, 1\otimes\Omega_2, \Omega_1\otimes 1, \Omega_1\otimes\Omega_2) = (1,\Omega_2,\Omega_1,\Omega_1\wedge\Omega_2),$$

by the following equalities

$$\begin{aligned}e_1(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) &= (\epsilon_1 - \iota_1)(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) = (1,\Omega_2,\Omega_1,\Omega_1\Omega_2)[e_1],\\ e_2(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) &= -\sqrt{-1}(\epsilon_1 + \iota_1)(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) = (1,\Omega_2,\Omega_1,\Omega_1\Omega_2)[e_2],\\ e_3(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) &= (\epsilon_2 - \iota_2)(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) = (1,\Omega_2,\Omega_1,\Omega_1\Omega_2)[e_3],\\ e_4(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) &= -\sqrt{-1}(\epsilon_2 + \iota_2)(1,\Omega_2,\Omega_1,\Omega_1\Omega_2) = (1,\Omega_2,\Omega_1,\Omega_1\Omega_2)[e_4],\end{aligned}$$

we can determine the Pauli matrices

$$[e_1] = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix},\qquad [e_2] = \begin{pmatrix} 0 & 0 & -\sqrt{-1} & 0\\ 0 & 0 & 0 & -\sqrt{-1}\\ -\sqrt{-1} & 0 & 0 & 0\\ 0 & -\sqrt{-1} & 0 & 0\end{pmatrix},$$

$$[e_3] = \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix},\quad [e_4] = \begin{pmatrix} 0 & -\sqrt{-1} & 0 & 0\\ -\sqrt{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{-1}\\ 0 & 0 & \sqrt{-1} & 0\end{pmatrix}.$$

It is easy to check that

$$e_1 = e_1\widehat{\otimes}1, \quad e_2 = e_2\widehat{\otimes}1, \quad e_3 = 1\widehat{\otimes}e_3, \quad e_4 = 1\widehat{\otimes}e_4.$$

The Jack map in the augmented spinor space is

$$J(1, \Omega_2, \Omega_1, \Omega_1\Omega_2) = -e_2e_4(1, \Omega_2, \Omega_1, \Omega_1\Omega_2) = (1, \Omega_2, \Omega_1, \Omega_1\Omega_2)[J],$$

where

$$[J] = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Obviously, here $\{[J],\ [e_i],\ i = 1, 2, 3, 4\}$ satisfy the equations

$$\begin{cases} [e_i][e_j] + [e_j][e_i] = -2\delta_{ij}I, & \text{for } i, j = 1, 2, 3, 4, \\ [J] \cdot \overline{[e_i]} = [e_i] \cdot [J], & \text{for } i = 1, 2, 3, 4. \end{cases}$$

The Jack-orientation in the augmented spinor space is $\epsilon = 1 \ \in \ \Lambda^*_C(2)$.

Applications of the augmented spinor spaces will be introduced in other papers.

References

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