Chin. Ann. Math.
28B(2), 2007, 235–242
DOI: 10.1007/s11401-005-0052-9

Chinese Annals of Mathematics, Series B

Quasi-convex Functions in Carnot Groups***

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Abstract In this paper, the authors introduce the concept of h-quasiconvex functions on Carnot groups G. It is shown that the notions of h-quasiconvex functions and h-convex sets are equivalent and the L^∞ estimates of first derivatives of h-quasiconvex functions are given. For a Carnot group G of step two, it is proved that h-quasiconvex functions are locally bounded from above. Furthermore, the authors obtain that h-convex functions are locally Lipschitz continuous and that an h-convex function is twice differentiable almost everywhere.

Keywords h-Quasiconvex function, Carnot group, Lipschitz continuity
2000 MR Subject Classification 43A80, 26B25

1 Introduction

Convex functions have played a very important role in PDEs, especially in fully nonlinear elliptic PDEs in Euclidean space $\mathbb{R}^n$ (see [4, 5]), while quasiconvex functions have many interesting properties similar to convex functions in $\mathbb{R}^n$ and play an important role in mathematical programming (see [1, 9, 10, 13, 16]). Recently, some interesting properties and notions of convex functions on the Carnot groups have been investigated by Danielli-Garofalo-Nhieu, and Lu-Manfredi-Stroffolini and others (see [2, 6, 7, 14, 15, 19–21]). In this paper, motivated by ideas from the papers indicated above, we introduce the concept of h-quasiconvex functions in a Carnot group G, and give some interesting properties similar to those of h-convex functions on G. In particular, we show that notions of h-quasiconvex functions and h-convex sets are equivalent, give L^∞ estimates of first derivatives of h-quasiconvex functions, and prove that h-quasiconvex functions on a Carnot group G of step two are locally bounded from above. Furthermore, for a Carnot group G of step two, we obtain that h-convex functions are locally Lipschitz continuous and that a h-convex function is twice differentiable almost everywhere.

We begin by recalling some basic facts about Carnot groups (see [12, 17, 18]). A Carnot group G is a stratified, nilpotent Lie group of step r, with Lie algebra $\mathcal{G}= V_1 \oplus V_2 \oplus \cdots \oplus V_r$. This means that $[V_1, V_j] = V_{j+1}$ for $j = 1, 2, \cdots, r-1$, whereas $[V_1, V_r] = \{0\}$. We assume that a scalar product $\langle \cdot, \cdot \rangle$ is given on $\mathcal{G}$ for which the V_j's are mutually orthogonal. Let

Manuscript received February 23, 2005. Revised March 23, 2006. Published online March 5, 2007.
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***Project supported by the Science Foundation for Pure Research of Natural Sciences of the Education Department of Hunan Province (No. 2004c251), the Hunan Provincial Natural Science Foundation of China (No. 05JJ30006) and the National Natural Science Foundation of China (No. 10471063).

$m_j = \dim V_j,\ j = 1, \cdots, r$, and denote by $N = m_1 + \cdots + m_r$ the topological dimension of G. The notation $\{X_{j,1}, \cdots, X_{j,m_j}\},\ j = 1, \cdots, r$, will indicate a fixed orthonormal basis of the j-th layer V_j. Element of V_j are assigned the formal degree j. As a rule, we will use letters g, g', g_0, p, q for points in G, and use the letter e for the group identity in G, whereas we will reserve the letters Z, Z', Z_0, for elements of the Lie algebra $\mathcal{G}$. We will denote by $L_{g_0}(g) = g_0 g$ the left-translations on G by an element $g_0 \in G$. Recall that the exponential map $\exp : \mathcal{G} \to G$ is a global analytic diffeomorphism [22], which allows to define analytic maps $\xi_i : G \to V_i$ for $i = 1, 2, \cdots, r$, by letting $g = \exp(\xi_1(g) + \cdots + \xi_r(g))$ for $g \in G$. The mapping $\xi : G \to \mathcal{G}$ defined by $\xi(g) = \xi_1(g) + \cdots + \xi_r(g)$ is the inverse of the exponential mapping. The stratification of the Lie algebra allows us to define a natural family of non-isotropic dilation $\Delta_\lambda : \mathcal{G} \to \mathcal{G}$ as follows

$$\Delta_\lambda \xi(g) = \lambda \xi_1(g) + \lambda^2 \xi_2(g) + \cdots + \lambda^r \xi_r(g). \tag{1.1}$$

Therefore the exponential map induces a group of dilations on G via the formula

$$\delta_\lambda(g) = \exp \circ \Delta_\lambda \circ \exp^{-1}(g), \quad g \in G. \tag{1.2}$$

For $g \in G$, the projection of the exponential coordinates of g onto the layer $V_j,\ j = 1, \cdots, r$, are defined as follows

$$x_{j,s}(g) = \langle \xi_j(g), X_{j,s} \rangle, \quad s = 1, \cdots, m_j. \tag{1.3}$$

In the sequel it will be convenient to have a separate notation for the first two layers V_1 and V_2. For simplicity, we set $m = m_1, k = m_2$, and indicate

$$\{X_1, \cdots, X_m\} = \{X_{1,1}, \cdots, X_{1,m}\}, \quad \{Y_1, \cdots, Y_k\} = \{X_{2,1}, \cdots, X_{2,k}\}. \tag{1.4}$$

We indicate with

$$x_i(g) = \langle \xi_1(g), X_i \rangle, \quad i = 1, \cdots, m, \quad y_s(g) = \langle \xi_2(g), Y_s \rangle, \quad s = 1, \cdots, k, \tag{1.5}$$

the projections of the exponential coordinates of g onto V_1 and V_2. Let $x(g) = (x_1(g), \cdots, x_m(g)),\ y(g) = (y_1(g), \cdots, y_k(g))$. We will often identify $g \in G$ with its exponential coordinates

$$g = (x(g), y(g), \cdots), \tag{1.6}$$

where the dots indicate the $(N - (m + k))$-dimensional vector

$$(x_{3,1}(g), \cdots, x_{3,m_3}(g), \cdots, x_{r,1}(g), \cdots, x_{r,m_r}(g)).$$

When G is a group of step two, (1.6) simply becomes $g = (x(g), y(g))$. Such identification of G with its Lie algebra is justified by the Baker-Campbell-Hausdorff formula (see, e.g. [22])

$$\exp Z \exp Z' = \exp\Big(Z + Z' + \frac{1}{2}[Z, Z'] + \frac{1}{12}\{[Z, [Z, Z']] - [Z', [Z, Z']]\} + \cdots\Big), \tag{1.7}$$

where $Z, Z' \in \mathcal{G}$ and the dots indicate a finite linear combination of terms containing commutators of order three and higher.

We denote by X and Y the system of left-invariant vector fields on G defined by

$$X_i(g) = (L_g)_*(X_i), \quad i = 1, \cdots, m, \quad Y_s(g) = (L_g)_*(Y_s), \quad s = 1, \cdots, k,$$

where $(L_g)_*$ denotes the differential of L_g. System X defines a basis for the so-called horizontal subbundle HG of the tangent bundle TG. For a given function $f : G \to R$, the action of X_j on f is specified by the equation

$$X_j f(g) = \lim_{t\to 0} \frac{f(g\exp(tX_j)) - f(g)}{t} = \frac{d}{dt} f(g\exp(tX_j))\Big|_{t=0}.$$

The sub-Laplacian associated with a basis X is the second-order partial differential operator on G given by

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2. \tag{1.8}$$

We denote by dg the bi-invariant Haar measure on G obtained by pushing forward Lebesgue measure on $\mathcal{G}$ via the exponential map. One has $dg(\delta_\lambda(g)) = \lambda^Q dg(g)$, so that the number

$$Q = m_1 + 2m_2 + \cdots + rm_r$$

plays the role of a dimension with respect to the group dilations. For this reason Q is called the homogeneous dimension of G. Such a number is larger than the topological dimension N of G defined above.

The Euclidean distance to the origin $|\cdot|$ on $\mathcal{G}$ induces a homogeneous pseudo-norm $|\cdot|_{\mathcal{G}}$ on $\mathcal{G}$ and (via the exponential map) one on the group G in the following way (see [11]). For $\xi \in \mathcal{G}$, with $\xi = \xi_1 + \cdots + \xi_r,\ \xi_i \in V_i$, we let

$$|\xi|_{\mathcal{G}} = \Big(\sum_{i=1}^{r} (|\xi_i|)^{2r!/i}\Big)^{2r!}, \tag{1.9}$$

and then define a pseudo-norm on G by the equation

$$N(g) = N_G(g) = |\xi|_{\mathcal{G}}, \quad \text{if } g = \exp\xi. \tag{1.10}$$

The function N is usually referred to as non-isotropic gauge. It defines a pseudo-distance on G

$$d(g, g') = N(g^{-1}g'). \tag{1.11}$$

This is called the gauge pseudo-distance, and it is equivalent to the Carnot-Carathéodory distance $\rho(\cdot\,,\cdot)$ generated by the system X (see [3]). We let $B(g,R) = \{g' \in G \mid d(g,g') < R\}$.

For a given open set $\Omega \subset G$, the classe $\Gamma^k(\Omega)$ represents the collection of all functions having continuous derivatives up to order k with respect to the vector fields $X_1, X_2, \cdots, X_m$.

Given a point $g_0 \in G$, the horizotal plane through g_0 as the m-dimensional embedded submanifold of G given by

$$H_{g_0} = L_{g_0}(\exp(V_1 \times \{0\})),$$

where 0 denotes the $(N-m)$-dimensional zero vector in $\mathcal{G}$, with $N = \dim V_1 + \cdots + \dim V_r$.

This paper, except for the introduction, is divided into two sections. In Section 2 we give the definition of h-quasiconvex functions and the main results. We will give the proofs of main theorems in Section 3.

2 Definitions and Main Results

We first introduce the definition of h-quasiconvexity which generalizes the notion of h-convexity in the Carnot group G.

Given two points $g, g' \in G$, for $\lambda \in [0,1]$ we will denote

$$g_\lambda = g_\lambda(g; g') \stackrel{\text{def}}{=} g\delta_\lambda(g^{-1}g') \tag{2.1}$$

the twisted "convex combination" of g and g' based at g. Note (2.1), and we recall the following notion of h-convexity in [14], which is called weak H-convexity in [6]: A subset Ω of a Carnot group G is called h-convex if for any $g \in \Omega$, and every $g' \in \Omega \cap H_g$, we have $g_\lambda \in \Omega$ for every $\lambda \in [0,1]$. Furthermore, given an h-convex open set $\Omega \subset G$, a function $u : \Omega \to \mathbb{R}$ is called h-convex if for any $g \in \Omega$, and for every $g' \in \Omega \cap H_g$, we have for every $\lambda \in [0,1]$

$$u(g_\lambda) \le (1-\lambda)u(g) + \lambda u(g').$$

Throughout the paper, the open set Ω will be assumed to be h-convex. Similarly to [20], we give the following definition of quasiconvex functions.

Definition 2.1 *Let G be a Carnot group and $\Omega \subset G$. A function $u : \Omega \to \mathbb{R}$ is called h-quasiconvex if for any $g \in \Omega$, and for every $g' \in \Omega \cap H_g$, we have for every $\lambda \in [0,1]$*

$$u(g_\lambda) \le \max\{u(g), u(g')\}.$$

Here, g_λ is as in (2.1).

Example 2.1 Let $\mathbb{H}^1$ be the Heisenberg group with coordinates (x, y, z), the group operation $(x, y, z)(x', y', z') = (x + x', y + y', z + z' + 2(x'y - xy'))$ and Lie algebra $\mathcal{G}$ generated by the left-invariant vector fields

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}, \quad X_3 = \frac{\partial}{\partial z}.$$

Let $u(g) = -xy$ for $g \in \mathbb{H}^{+1} = \{(x, y, z) \mid x \ge 0, y \ge 0\}$. It is immediate to check that $\mathbb{H}^{+1}$ is h-convex. From the arithmetic-geometric mean inequality, if $u(g) \le u(g')$ for any $g \in \mathbb{H}^{+1}$, and for every $g' \in \mathbb{H}^{+1} \cap H_g$, we have $u(g_\lambda) = u(g\delta_\lambda(g^{-1}g')) \le u(g)$ for every $\lambda \in [0,1]$. Therefore $u(g)$ is h-quasiconvex. It is easy to see $X_iX_iu(g) = 0$ $(i = 1,2)$ and $X_iX_ju(g) = -1$ $(i, j = 1, 2, i \ne j)$. From [6, Theorem 5.11] we derive that $u(g)$ is not h-convex. Hence for $g \in \mathbb{H}^{+1}$ the function $u(g) = -xy$ provides us an example of h-quasiconvexity which is not h-convex.

Our main results are the following three theorems.

Theorem 2.1 *Let G be a Carnot group and $\Omega \subset G$. If $u : \Omega \to \mathbb{R}$ is h-quasiconvex, $g \in G, r > 0$, and $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing, then $u \circ L_g : L_{g^{-1}}(\Omega) \to \mathbb{R}$, $u \circ \delta_r : \delta_{\frac{1}{r}}(\Omega) \to \mathbb{R}$ and $f \circ u$ are h-quasiconvex. Furthermore, if $\{u_\alpha : \Omega \to \mathbb{R}\}_{\alpha \in A}$ is an arbitrary family of h-quasiconvex functions, then $\sup\limits_{\alpha \in A} u_\alpha$ is h-quasiconvex. Finally, a function $u : G \to \mathbb{R}$ is h-quasiconvex if and only if for any $a \in \mathbb{R}$ the level sets $\Omega_a = \{g \in G \mid u(g) \le a\}$ are h-convex.*

Theorem 2.2 *Let $\Omega \subset G$ be an open set and $u : \Omega \to \mathbb{R}$ be an h-quasiconvex and continuous function with $u(q) \le u(p)$ for any $p \in \Omega$ and $q \subset \Omega \cap H_p$. Let B_R be a ball such that $B_{3R} \subset \Omega$.*

Then u is locally Lipschitz and we have the bound

$$\|Xu\|_{L^\infty(B_R)} \le C\|u\|_{L^\infty(B_{3R})}, \tag{2.2}$$

where C is a constant independent of u and R.

Theorem 2.3 *Let G be a Carnot group of step two and $\Omega \subset G$. If $u : \Omega \to \mathbb{R}$ is an h-quasiconvex function, then it is locally bounded from above, and furthermore if $u : \Omega \to \mathbb{R}$ is an h-convex function, then it is locally Lipschitz.*

Remark 2.1 Applying Theorem 2.3 and [7, Theorem 1.1], we can prove that if G is a Carnot group of step two and $u : G \to \mathbb{R}$ is an h-convex function, then for every $i, j = 1, \cdots, m$ the horizontal second derivatives X_iX_ju exist a.e. in G. This result is the version of the Busemann-Feller-Alexandrov theorem for the class of h-convex functions in Carnot groups of step two. In addition, by the idea similar to that in [7, Theorem 4.3], we can derive an interesting property of sub-Laplacian as follows: Let G be a Carnot group G of step two, and $\Omega \subset G$ be a bounded open set. If $u \in \Gamma^3(\Omega)$, then for any $D \Subset \Omega$ we have for some constant $C > 0$ depending on G, Ω and D

$$\int_D \mathcal{L}u dg \le C(osc_\Omega u). \tag{2.3}$$

It is worth noting at this point that any convexity conditon is not needed for the functions in (2.3), while the functions in [7, Theorem 4.3] are $2h$-convex.

3 The Proofs of Theorems

Proof of Theorem 2.1 Without loss of generality, we assume that $\Omega = G$. For every $p \in G$, any $g \in G$, and every $g' \in H_g$, noting $pg' \in H_{pg}$, we have for every $\lambda \in [0, 1]$

$$u(pg\delta_\lambda(g^{-1}g')) = u(pg\delta_\lambda((pg)^{-1}(pg'))) \le \max\{u(pg), u(pg')\},$$

which establishes the first part of Theorem 2.1.

From the fact that $g' \in H_g$ if and only if $\delta_r g' \in H_{\delta_r g}$ and the equation

$$\delta_r(g_\lambda) = (\delta_r g)(\delta_r(\delta_\lambda(g^{-1}g')) = (\delta_r g)\delta_\lambda((\delta_r g)^{-1}\delta_r g')$$

we obtain the second part of Theorem 2.1. From Definition 2.1 and the notion of h-convexity in [6], we can derive the remaining parts of Theorem 2.1 .

Proof of Theorem 2.2 The proof is similar to that of [20, Theorem 2.10], but the proof of Theorem 2.2 is more complicated. For the convenience of reader, we include a complete proof here. Given a kernel $K \in C_0^\infty(G)$, $K \ge 0$, $\mathrm{supp} K \subset \overline{B}(e, 1)$, we have $\int_G K(g)dg = 1$. Consider the corresponding approximation to the identity $\{K_\epsilon = \epsilon^{-Q} K \circ \delta_{1/\epsilon}\}_{\epsilon>0}$ associated with it. Let $u_\epsilon = K_\epsilon \star u$. Then $u_\epsilon \in C^\infty(\Omega_\epsilon)$, where $\Omega_\epsilon = \{p \in \Omega \mid \mathrm{dist}(p, \partial\Omega) > \epsilon\}$. By the hypothesis $u \in C(\Omega)$, we have $u_\epsilon \to u$ uniformly on compact subsets of Ω. Moreover, we have that the group convolution preserves h-quasiconvexity using the proof similar to that of (4.7) in [20]. Let p, q be two points in Ω_ϵ, we introduce the function $\varphi : [0, 1] \to \mathbb{R}$ defined by

$$\varphi(\lambda) = u_\epsilon(p\delta_\lambda(p^{-1}q)). \tag{3.1}$$

Clearly, $\varphi(0) = u_\epsilon(p)$. By the hypothesis and Definition 2.1, for any $p \in \Omega_\epsilon$ and $q \in \Omega_\epsilon \cap H_p$, we derive

$$\varphi'(0) = \lim_{\lambda \to 0^+} \frac{u_\epsilon(p\delta_\lambda(p^{-1}q)) - u_\epsilon(p)}{\lambda} \leq 0. \tag{3.2}$$

For a Carnot group G of step two, let $g = \exp(\xi(g))$, $\xi(g) = \xi_1(g) + \xi_2(g)$, $\xi_1(g) = \sum_{\alpha=1}^{m} x_\alpha(g) e_\alpha$, and $\xi_2(g) = \sum_{\beta=1}^{k} y_\beta(g)\epsilon_\beta$. For $g = p, p^{-1}q$, from the definition (1.2), using (1.7) we obtain for the function $\varphi(\lambda)$ defined in (3.1)

$$\begin{aligned}
\varphi(\lambda) &= u_\epsilon\Big(\exp\Big(\xi_1(p) + \lambda\xi_1(p^{-1}q) + \xi_2(p) + \lambda^2\xi_2(p^{-1}q) + \frac{\lambda}{2}[\xi_1(p), \xi_1(p^{-1}q)]\Big)\Big) \\
&= u_\epsilon\Big(x_1(p) + \lambda x_1(p^{-1}q), \cdots, x_m(p) + \lambda x_m(p^{-1}q), \\
&\qquad y_1 + \lambda^2 y_1(p^{-1}q) + \frac{\lambda}{2} \sum_{\alpha,\alpha'=1}^{m} b^1_{\alpha\alpha'} x_\alpha(p) x_{\alpha'}(p^{-1}q), \\
&\qquad \cdots, \\
&\qquad y_k + \lambda^2 y_k(p^{-1}q) + \frac{\lambda}{2} \sum_{\alpha,\alpha'=1}^{m} b^k_{\alpha\alpha'} x_\alpha(p) x_{\alpha'}(p^{-1}q)\Big).
\end{aligned} \tag{3.3}$$

Differentiating (3.3) with respect to λ, and setting $\lambda = 0$, we have

$$\varphi'(0) = \sum_{\alpha=1}^{m} \Big[\frac{\partial u_\epsilon}{\partial x_\alpha}(p) + \frac{1}{2} \sum_{\beta=1}^{k} \sum_{\alpha'=1}^{m} b^\beta_{\alpha'\alpha} x_{\alpha'}(p) \frac{\partial u_\epsilon}{\partial y_\beta}(p)\Big] x_\alpha(p^{-1}q) = \langle X u_\epsilon(p), \zeta\rangle, \tag{3.4}$$

where $\zeta = \xi_1(q) - \xi_1(p)$ and in the last equality we have used [6, Lemma 5.3]. Similarly, for a Carnot group G of higher step, we have that (3.4) holds also by using [6, (5.11)].

Now we fix a gauge ball $B(p_0, R) \subset B(p_0, 3R) \subset \Omega_\epsilon$. For any fixed $p \in B(p_0, R)$ and every $q \in H_p$, when $q \neq p$, from (3.2) and (3.4), we get

$$\Big\langle X u_\epsilon(p), \frac{\zeta}{\|\zeta\|}\Big\rangle \leq 0. \tag{3.5}$$

Passing to the supremum on all $q \in \partial B(p, R) \cap H_p$ in (3.5) for $\epsilon > 0$ small enough, and noting that $p \in B(p_0, R)$ is arbitrary, we can derive

$$\|X u_\epsilon\|_{L^\infty(B(p_0,R))} \leq 0. \tag{3.6}$$

From [6, (3.6), Theorem 2.5], letting $\epsilon \to 0$, we obtain for $p, q \in B(p_0, R)$,

$$|u(p) - u(q)| \leq C \|u\|_{L^\infty(B(p_0,3R))} d(p, q), \tag{3.7}$$

where $C > 0$ is an absolute constant. Thus u is locally Lipschitz. Applying [6, Theorem 2.4], from (3.7) we drive that weak derivatives $X_1 u, \cdots, X_m u$ exist dg-a.e. in Ω and belong to $L^\infty_{\mathrm{loc}}(\Omega)$. Furthermore, we obtain (2.2).

This completes the proof of Theorem 2.2.

Proof of Theorem 2.3 The proof of the first part of Theorem 2.3 is similar to that of [21, Theorem 1.4] once we have the following step I. We sketch its proof. Without loss of generality, we assume that $\Omega = G$, and Lie algebra of Carnot group G is $\mathcal{G}= V_1 \oplus V_2$, with $m = \dim V_1$ and $k = \dim V_2$. The proof of the first part of Theorem 2.3 is divided into two steps:

Step I We first prove that given a Carnot group G, a function $u : G \to \mathbb{R}$ is h-quasiconvex if and only if for any $p \in G$ and every $q \in H_p$ the restriction of the composition $u \circ \exp$ to the segment $[\xi(p), \xi(q)]$ is a quasiconvex function. Here $p = \exp(\xi(p)), q = \exp(\xi(q))$ and $[\xi(p), \xi(q)]$ denotes the convex closure of the $\{\xi(p), \xi(q)\}$ in the Euclidean sense.

By [6, Proposition 4.4] and Definition 2.1, it is easy to prove the sufficiency for the above conclusion. Next we prove the necessity. Suppose that $u : G \to \mathbb{R}$ is h-quasiconvex. For any $p \in G$ and every $q \in H_p$, using [6, Proposition 4.4], we easily find $\xi(p\delta_\lambda(p^{-1}q)) = (1-\lambda)\xi(p) + \lambda\xi(q) \in [\xi(p), \xi(q)]$. Therefore for any $\xi(p_1), \xi(p_2) \in [\xi(p), \xi(q)]$, where $p_i = \exp(\xi(p_i)) \in G$ $(i = 1, 2)$, we have $\xi(p_i) = \xi(p\delta_{\lambda_i}(p^{-1}q))$ for $i = 1, 2$. Thus we have for $i = 1, 2$,

$$p_i = p\delta_{\lambda_i}(p^{-1}q). \tag{3.8}$$

Now we show that $p_2 \in H_{p_1}$. From [6, Proposition 4.2], i.e., $p_1^{-1}p_2 \in H_e$, this translates into

$$\xi_i(p_1^{-1}p_2) = 0 \in V_i, \quad i = 2, \cdots, r. \tag{3.9}$$

From (1.1) and (1.2), for any $g \in G$ we derive for every $\lambda \in [0, 1]$

$$\delta_\lambda(g) = \exp(\lambda\xi_1(g) + \lambda^2\xi_2(g) + \cdots + \lambda^r\xi_r(g)). \tag{3.10}$$

Using the hypothesis $q \in H_p$ and [6, Proposition 4.2], we have $p^{-1}q \in H_e$. Thus using (3.10), we derive for every $\lambda \in [0, 1]$

$$\delta_\lambda(p^{-1}q) = \exp(\lambda\xi_1(p^{-1}q)). \tag{3.11}$$

Noticing $(\exp(\lambda\xi_1(p^{-1}q))^{-1} = \exp(-\lambda\xi_1(p^{-1}q))$, from (3.8), (3.11) and Baker-Campbell-Hausdorff formula, we have

$$p_1^{-1}p_2 = \exp(-\lambda_1\xi_1(p^{-1}q))\exp(\lambda_2\xi_1(p^{-1}q)) = \exp[(\lambda_2 - \lambda_1)\xi_1(p^{-1}q)]. \tag{3.12}$$

From (3.12), we get that (3.9) is valid. Thus $p_2 \in H_{p_1}$. Again using [6, Proposition 4.4], for any $\xi(p_1), \xi(p_2) \in [\xi(p), \xi(q)]$ and every $\lambda \in [0, 1]$, we have

$$u(\exp((1-\lambda)\xi(p_1) + \lambda\xi(p_2))) = u(\exp(\xi(p_1\delta_\lambda(p_1^{-1}p_2)))) \le \max\{u(\exp(\xi(p_1))), u(\exp(\xi(p_2)))\}.$$

This proves that restriction of $u \circ \exp$ to the segment $[\xi(p), \xi(q)]$ is a quasiconvex function.

Step II From [6, Proposition 4.3], [20, Lemma 3.1], the result proved in the step I and the invariance of h-quasiconvexity by left translations and dilations, using the method similar to that of the first part of [21, Theorem 1.4], we can derive the first part of Theorem 2.3.

From the first part of Theorem 2.3, we have that h-convex function u on Carnot group G of the step two is locally bounded from above. Therefore, it is locally Lipschitz using Theorem 3.18 due to Magnani in [15].

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