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Controllability of Non-densely Defined Neutral Functional Differential Systems in Abstract Space**

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Abstract In this paper, by means of Sadovskii fixed point theorem, the authors establish a result concerning the controllability for a class of abstract neutral functional differential systems where the linear part is non-densely defined and satisfies the Hille-Yosida condition. As an application, an example is provided to illustrate the obtained result.

Keywords Controllability, Non-densely defined, Integral solution, Sadovskii fixed point theorem
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1 Introduction and Preliminaries

In this paper, we study the controllability of semilinear functional differential systems defined non-densely. More precisely, we consider the controllability problem of the following neutral system on a general Banach space X (with the norm $\|\cdot\|$):

$$\begin{aligned}&\frac{d}{dt}[x(t)-g(t,x_t)] = A[x(t)-g(t,x_t)] + Cu(t) + F(t,x_t), \quad 0\le t\le a,\\ &x_0 = \phi \in C([-r,0];X),\end{aligned} \tag{1.1}$$

where the state variable $x(\cdot)$ takes values in Banach space X and the control function $u(\cdot)$ is given in $L^2([0,a];U)$, the Banach space of admissible control functions with U a Banach space. C is a bounded linear operator from U into X, the unbounded linear operators A is not defined densely on X, that is, $\overline{D(A)} \neq X$. And $F,\ g : [0,a]\times C([-r,0];X) \to X$ are appropriate functions to be specified later. Let $r > 0$ be a constant, we denote by $C([-r,0];X)$ the space of continuous functions from $[-r,0]$ to X with the sup-norm $\|\phi\|_C = \max\limits_{s\in[-r,0]} \|\phi(s)\|$, and for a function x we define $x_t \in C([-r,0];X)$ by $x_t(s) = x(t+s),\ s\in[-r,0]$.

The problem of controllability of linear and nonlinear systems represented by ODE in finite dimensional space has been extensively studied. Many authors have extended the controllability concept to infinite dimensional systems in Banach space with unbounded operators (see [1–11] and the references therein). Triggiani [5] established sufficient conditions for controllability of linear and nonlinear systems in Banach space. Exact controllability of abstract semilinear equations has been studied by Lasiecka and Triggian [6]. Quinn and Carmichael [7] have shown that the controllability problem in Banach space can be converted into a fixed pointed problem

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for a single-valued mapping. Kwun et al. [8] investigated the controllability and approximate controllability of delay Voltera systems by using a fixed point theorem. Recently Balachandran and his cooperators have studied the (local) controllability of abstract semilinear functional differential systems (see [9]) and the controllability of abstract integro-differential systems (see [10]). In paper [11] the author has extended the problem to neutral systems with unbounded delay.

In all these work the linear operator A is always defined densely in X and satisfies the Hille-Yosida condition so that it generates a C_0-semigroup or analytic semigroup. However, as indicated in [12], we sometimes need to deal with the non-densely defined operators, for example, when we look at a one-dimensional heat equation with Dirichlet condition on $[0,1]$ and consider $A = \frac{\partial^2}{\partial^2 x}$ in $C([0,1];R)$, in order to measure the solution in the sup-norm we take the domain

$$D(A) = \{x \in C^2([0,1];R); x(0) = x(1) = 0\},$$

then it is not dense in $C([0,1];R)$ with the sup-norm. The example presented in Section 3 shows the advantage of the non-densely defined operators in handling practical problems. See [12] for more examples and remarks concerning the non-densely defined operators.

Up to now, there are very few papers (see [13]) in this direction dealing with the controllability problems for the important case that the linear parts are defined non-densely. The purpose of this paper is just to investigate the controllability for the non-densely defined system (1.1). The obtained result can be regarded as a continuation and an extension of those for densely defined control systems.

Throughout this paper we will always suppose the following hypothesis for equation (1.1):

Hypothesis (H_0) *The operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, i.e., there exist $\overline{M} \geq 0$ and $w \in R$ such that $(w, +\infty) \subset \rho(A)$ and*

$$\sup\{(\lambda - w)^n \|R(\lambda, A)^n\|, n \in N, \lambda > w\} \leq \overline{M},$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$.

Remark 1.1 According to [14], if operator A satisfies the Hille-Yosida condition, then A generates a non-degenerate, locally Lipschitz continuous integrated semigroup. For the theory of integrated semigroup we refer the readers to paper [14] and [15]. Here, for the sake of brevity, we give directly the definition of integral solutions for equation (1.1) by virtue of this theory.

Definition 1.1 *Let $\phi \in C([-r,0];X)$. A function $x : [-r,a] \to X$ is said to be an integral solution of equation* (1.1) *on $[-r,a]$, if the following conditions hold:*

(i) *x is continuous on $[0,a]$;*

(ii) *$\int_0^t [x(s) - g(s,x_s)]ds \in D(A)$ on $[0,a]$;*

(iii)

$$x(t) = \begin{cases} \phi(0) - g(0,\phi) + g(t,x_t) + A\displaystyle\int_0^t [x(s) - g(s,x_s)]ds \\ + \displaystyle\int_0^t [Cu(s) + F(s,x_s)]ds, & t \geq 0, \\ \phi(t), & -r \leq t < 0. \end{cases}$$

Let A_0 be the part of A on $\overline{D(A)}$ defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},$$
$$A_0 x = Ax.$$

Then A_0 generates a C_0-semigroup $\{T_0(t)\}_{t\geq 0}$ on $\overline{D(A)}$ (see [16] for the theory of C_0-semigroup) and the integral solution in Definition 1.1 (if it exists) is given by

$$x(t) = \begin{cases} T_0(t)[\phi(0) - g(0,\phi)] + g(t, x_t) \\ + \lim\limits_{\lambda\to+\infty} \int_0^t T_0(t-s)B(\lambda)[Cu(s) + F(s, x_s)]ds, & t \geq 0, \\ \phi(t), & -r \leq t < 0, \end{cases} \tag{1.2}$$

where $B(\lambda) = \lambda R(\lambda, A)$.

Remark 1.2 We should point out here that, from Definition 1.1, it is not difficult to verify that if x is an integral solution of equation (1.1) on $[-r, a]$, then for all $t \in [0, a]$, $x(t) - g(t, x_t) \in \overline{D(A)}$. In particular, $\phi(0) - g(0, \phi) \in \overline{D(A)}$.

Now we give the definition of the controllability for non-densely defined system (1.1).

Definition 1.2 *The system* (1.1) *is said to be controllable on the interval* $[0, a]$, *if for every initial function* $\phi \in C([-r, 0]; X)$ *with* $\phi(0) - g(0, \phi) \in \overline{D(A)}$ *and* $x_1 \in \overline{D(A)}$, *there exists a control* $u \in L^2([0, a]; U)$ *such that the integral solution* $x(\cdot)$ *of* (1.1) *satisfies* $x(a) = x_1$.

2 Main Result

To consider the controllability of system (1.1) we impose the following assumptions on it.

(H_1) $F : [0, a] \times C([-r, 0]; X) \to X$ satisfies the following conditions:

(i) For each $t \in [0, a]$, the function $F(t, \cdot) : C([-r, 0]; X) \to X$ is continuous and for each $\phi \in C([-r, 0]; X)$ the function $F(\cdot, \phi) : [0, a] \to X$ is strongly measurable;

(ii) For each positive number k, there is a function $f_k \in L^1([0, a])$ such that

$$\sup_{\|\phi\|_C \leq k} \|F(t, \phi)\| \leq f_k(t) \quad \text{and} \quad \liminf_{k\to+\infty} \frac{1}{k} \int_0^a f_k(s)ds = \gamma < \infty.$$

(H_2) The function $g : [0, a] \times C([-r, 0]; X) \to X$ is Lipschitz continuous, that is, there is a constant L_0, $0 < L_0 < 1$, such that

$$\|g(t_2, \phi_2) - g(t_1, \phi_1)\| \leq L_0(|t_2 - t_1| + \|\phi_2 - \phi_1\|_C)$$

for $0 \leq t_1,\ t_2 \leq a,\ \phi_1,\ \phi_2 \in C([-r, 0]; X)$. We also assume that there is a constant $L > 0$ such that the inequality

$$\|g(t, \phi)\| \leq L(\|\phi\|_C + 1)$$

holds for all $t \in [0, a]$, $\phi \in C([-r, 0]; X)$.

(H_3) The operator $C : U \to X$ is bounded and linear. The linear operator $W : L^2([0, a]; U) \to \overline{D(A)}$ defined by

$$Wu = \lim_{\lambda\to+\infty} \int_0^a T_0(a-s)B(\lambda)Cu(s)ds$$

induces a bounded invertible operator $\widetilde{W}$ defined on $L^2([0,a];U)/\ker W$ (see Appendix for the construction of $\widetilde{W}^{-1}$).

Theorem 2.1 *Suppose that the C_0-semigroup $T_0(t)$ is compact. Let $\phi \in C([-r,0];X)$ with $\phi(0) - g(0,\phi) \in \overline{D(A)}$. If the assumptions* (H_0)–(H_3) *are satisfied, then the system* (1.1) *is controllable on interval* $[0,a]$ *provided that*

$$(1 + aM\overline{M}\|C\|\|\widetilde{W}^{-1}\|)(L + M\overline{M}\gamma) < 1, \tag{2.1}$$

where $M = \sup_{t\in[0,a]} \|T_0(t)\|$.

Proof By means of the assumption (H_3), for arbitrary function $x(\cdot)$ we define the control

$$\begin{aligned} u(t) = \widetilde{W}^{-1}\{&x_1 - T_0(a)[\phi(0) - g(0,\phi)] - g(a, x_a) \\ &- \lim_{\lambda\to+\infty}\int_0^a T_0(a-s)B(\lambda)F(s,x_s)ds\}(t). \end{aligned}$$

Using this control we will show that the operator S defined by

$$\begin{aligned} (Sx)(t) = {}& T_0(t)[\phi(0) - g(0,\phi)] + g(t,x_t) \\ &+ \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)[Cu(s) + F(s,x_s)]ds, \quad 0 \le t \le a \end{aligned}$$

has a fixed point $x(\cdot)$. Then from (1.2) $x(\cdot)$ is a integral solution of system (1.1), and it is easy to verify that

$$x(a) = (Sx)(a) = x_1,$$

which implies that the system is controllable. Subsequently we will prove that S has a fixed point by applying Sadovskii fixed point theorem.

Let $y(\cdot) : [-r,a] \to X$ be the function defined by

$$y(t) = \begin{cases} T_0((t)\phi(0), & t \ge 0, \\ \phi(t), & -r \le t < 0. \end{cases}$$

Then $y_0 = \phi$ and the map $t \to y_t$ is continuous. We can assume that

$$N = \sup\{\|y_t\|_C : 0 \le t \le a\}.$$

For each $z \in C([0,a];\overline{D(A)})$, $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} z(t), & 0 \le t \le a, \\ 0, & -r \le t < 0. \end{cases}$$

If $x(\cdot)$ satisfies (1.2), we can decompose it as $x(t) = z(t) + y(t)$, $0 \le t \le a$, which implies $x_t = \overline{z}_t + y_t$ for every $0 \le t \le a$ and the function $z(\cdot)$ satisfies

$$z(t) = -T_0(t)g(0,\phi) + g(t,x_t) + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)[Cu(s) + F(s,\overline{z}_s + y_s)]ds, \quad 0 \le t \le a.$$

Let P be the operator on $C([0,a];\overline{D(A)})$ defined by

$$(Pz)(t) = -T_0(t)g(0,\phi) + g(t,x_t) + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)[Cu(s) + F(s,\overline{z}_s + y_s)]ds.$$

Obviously that the operator S has a fixed point is equivalent to that P has one, so it turns out that we only need to prove that P has a fixed point.

For each positive integer k, let

$$B_k = \{z \in C([0,a];\overline{D(A)}) : z(0) = 0,\ \|z(t)\| \le k,\ 0 \le t \le a\}.$$

Then for each k, B_k is clearly a bounded closed convex set in $C([0,a];\overline{D(A)})$. Obviously, P is well defined on B_k. We claim that there exists a positive integer k such that $PB_k \subseteq B_k$. If it is not true, then for each positive number k, there is a function $z_k(\cdot) \in B_k$, but $Pz_k \notin B_k$, that is, $\|Pz_k(t)\| > k$ for some $t(k) \in [0,a]$, where $t(k)$ denotes t is dependent on k. However, on the other hand, we have

$$\begin{aligned}
k &< \|(Pz_k)(t)\| \\
&= \Big\| - T_0(t)g(0,\phi) + g(t,\overline{z}_{k,t}+y_t) + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)Cu_k(s)ds \\
&\quad + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)F(s,\overline{z}_{k,s}+y_s)ds\Big\| \\
&= \Big\| - T_0(t)g(0,\phi) + g(t,\overline{z}_{k,t}+y_t) + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)C\widetilde{W}^{-1}\Big\{x_1 - T_0(a)[\phi(0) \\
&\quad - g(0,\phi)] - g(a,\overline{z}_{k,a}+y_a) - \lim_{\lambda\to+\infty}\int_0^a T_0(a-\tau)B(\lambda)F(\tau,\overline{z}_{k,\tau}+y_\tau)d\tau\Big\}(s)ds \\
&\quad + \lim_{\lambda\to+\infty}\int_0^t T_0(t-s)B(\lambda)F(s,\overline{z}_{k,s}+y_s)ds\Big\|,
\end{aligned}$$

where u_k is the corresponding control of $x_k, x_k = z_k + y$. Since

$$\begin{aligned}
&\|B(\lambda)\| \le \frac{\lambda\overline{M}}{\lambda-\omega} \to \overline{M}, \quad \lambda \to +\infty, \\
&\int_0^t \|F(s,\overline{z}_{k,s}+y_s)\|ds \le \int_0^a f_{k+N}(s)ds,
\end{aligned}$$

there holds

$$\begin{aligned}
k &< M\|g(0,\phi)\| + L(k+N+1) + \int_0^t M\overline{M}\|C\|\|\widetilde{W}^{-1}\|\{\|x_1\| + M\|\phi(0)\| \\
&\quad + M\|g(0,\phi)\| + L(k+N+1) + \int_0^a M\overline{M}\|F(\tau,\overline{z}_{k,\tau}+y_\tau)\|d\tau\}(s)ds \\
&\quad + \int_0^t M\overline{M}\|F(s,\overline{z}_{k,s}+y_s)\|ds \\
&\le M\|g(0,\phi)\| + L(k+N+1) + aM\overline{M}\|C\|\|\widetilde{W}^{-1}\|\{\|x_1\| + M\|\phi(0)\| \\
&\quad + M\|g(0,\phi)\| + L(k+N+1) + M\overline{M}\int_0^a f_{k+N}(\tau)d\tau\} + M\overline{M}\int_0^a f_{k+N}(s)ds \\
&= M^* + (1 + aM\|C\|\|\widetilde{W}^{-1}\|)\Big(Lk + M\overline{M}\int_0^a f_{k+N}(s)ds\Big) \\
&= M^* + (1 + aM\overline{M}\|C\|\|\widetilde{W}^{-1}\|)\Big[Lk + (k+N)\frac{M\overline{M}}{k+N}\int_0^a f_{k+N}(s)ds\Big].
\end{aligned}$$

Dividing on both sides by k and taking the lower limit, we get

$$(1 + aM\overline{M}\|C\|\|\widetilde{W}^{-1}\|)(L + M\overline{M}\gamma) \geq 1.$$

This contradicts (2.1). Hence for some positive number k, $PB_k \subseteq B_k$.

Now we define the operators P_1, P_2 on B_k by

$$(P_1 z)(t) = -T((t)g(0, \phi) + g(t, \overline{z}_t + y_t)$$

and

$$(P_2 z)(t) = \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B(\lambda)[Cu_k(s) + F(s, \overline{z}_s + y_s)]ds$$

for $z \in B_k$ and $0 \leq t \leq a$, respectively. In order to apply Sadovskii fixed point theorem, we need to prove that P_1 verifies a contraction condition while P_2 is a compact operator.

From condition (H$_2$) it is obvious that P_1 verifies a contraction condition since $L_0 < 1$. In order to show the compactness of P_2, firstly we prove that P_2 is continuous on B_k. Let $\{z_n\} \subseteq B_k$ with $z_n \to z$ in B_k. Then for each $s \in [0, a]$, $\overline{z}_{n,s} \to \overline{z}_s$, and by H$_1$(i), we have

$$F(s, \overline{z}_{n,s} + y_s) \to F(s, \overline{z}_s + y_s), \quad n \to \infty.$$

Since

$$\|F(s, \overline{z}_{n,s} + y_s) - F(s, \overline{z}_s + y_s)\| \leq 2f_{k+N}(s),$$

by the dominated convergence theorem we have

$$\begin{aligned}\|P_2 z_n - P_2 z\| = &\sup_{0\leq t\leq a} \Big\| \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B(\lambda)C[u_n(s) - u(s)]ds \\ &+ \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B(\lambda)[F(s, \overline{z}_{n,s} + y_s) - F(s, \overline{z}_s + y_s)]ds \Big\| \\ &\to 0, \quad \text{as } n \to +\infty,\end{aligned}$$

i.e., P_2 is continuous.

Next we prove that the family $\{P_2 z : z \in B_k\}$ is a equicontinuous family of functions. To do this, let $\epsilon > 0$ small, $0 < t_1 < t_2$. Then

$$\begin{aligned}\|(P_2 z)(t_2) - (P_2 z)(t_1)\| \leq &\lim_{\lambda\to+\infty} \int_0^{t_1-\epsilon} \|T_0(t_2-s) - T_0(t_1-s)\|\|B(\lambda)\|\|C\|\|u(s)\|ds \\ &+ \lim_{\lambda\to+\infty} \int_{t_1-\epsilon}^{t_1} \|T_0(t_2-s) - T_0(t_1-s)\|\|B(\lambda)\|\|C\|\|u(s)\|ds \\ &+ \lim_{\lambda\to+\infty} \int_{t_1}^{t_2} \|T_0(t_2-s)\|\|B(\lambda)\|\|C\|\|u(s)\|ds \\ &+ \lim_{\lambda\to+\infty} \int_0^{t_1-\epsilon} \|T_0(t_2-s) - T_0(t_1-s)\|\|B(\lambda)\|\|F(s, \overline{z}_s + y_s)\|ds \\ &+ \lim_{\lambda\to+\infty} \int_{t_1-\epsilon}^{t_1} \|T_0(t_2-s) - T_0(t_1-s)\|\|B(\lambda)\|\|F(s, \overline{z}_s + y_s)\|ds \\ &+ \lim_{\lambda\to+\infty} \int_{t_1}^{t_2} \|T_0(t_2-s)\|\|B(\lambda)\|\|F(s, \overline{z}_s + y_s)\|ds.\end{aligned}$$

Noting that

$$\begin{aligned}\|u(s)\| &\le \|\widetilde{W}^{-1}\|\Big[\|x\| + M\|\phi(0) - g(0,\phi)\| + \|g(a, \overline{z}_a + y_a)\| \\ &\quad + \lim_{\lambda\to+\infty}\int_0^a \|T_0(a-\tau)\|\|B(\lambda)\|\|F(\tau, \overline{z}_\tau + y_\tau)\|d\tau\Big] \\ &\le \|\widetilde{W}^{-1}\|\Big[\|x_1\| + M\|\phi(0) - g(0,\phi)\| + L(k+N+1) + M\overline{M}\int_0^a f_{k+N}(\tau)d\tau\Big]\end{aligned}$$

and $f_{k+N} \in L^1$, we see that $\|(P_2 z)(t_2) - (P_2 z)(t_1)\|$ tends to zero independently of $z \in B_k$ as $t_2 - t_1 \to 0$ with ϵ sufficiently small since the compactness of $T_0(t)$ $(t > 0)$ imples the continuity of $T_0(t)$ $(t > 0)$ in t in the uniform operators topology. Hence, P_2 maps B_k into a equicontinuous family functions.

It remains to prove that $V(t) = \{(P_2 z)(t) : z \in B_k\}$ is relatively compact in X. Let $0 < t \le a$ be fixed and $0 < \epsilon < t$. For $z \in B_k$, we define

$$\begin{aligned}(P_{2,\epsilon} z)(t) &= \lim_{\lambda\to+\infty}\int_0^{t-\epsilon} T_0(t-s)B(\lambda)[Cu(s) + F(s, \overline{z}_s + y_s)]ds \\ &= T_0(\epsilon)\lim_{\lambda\to+\infty}\int_0^{t-\epsilon} T_0(t-\epsilon-s)B(\lambda)[Cu(s) + F(s, \overline{z}_s + y_s)]ds.\end{aligned}$$

Using the estimation on $\|u(s)\|$ as above and by the compactness of $T_0(t)$ $(t > 0)$, we obtain that $V_\epsilon(t) = \{(P_{2,\epsilon} z)(t) : z \in B_k\}$ is relatively compact in X for every ϵ, $0 < \epsilon < t$. Moreover, for every $z \in B_k$, we have

$$\begin{aligned}\|(P_2 z)(t) - (P_{2\epsilon} z)(t)\| &\le \lim_{\lambda\to+\infty}\int_{t-\epsilon}^t \|T_0(t-s)B(\lambda)[Cu(s) + F(s, \overline{z}_s + y_s)]\|ds \\ &\le \int_{t-\epsilon}^t M\overline{M}\Big\{\|C\|\|W^{-1}\|\Big[\|x_1\| + M\|\phi(0)\| \\ &\quad + M\overline{M}\int_0^a f_{k+N}(\tau)d\tau\Big] + f_{k+N}(s)\Big\}ds.\end{aligned}$$

Therefore there are relatively compact sets arbitrarily close to the set $V(t) = \{(P_2 z)(t) : z \in B_k\}$. Hence the set $V(t)$ is also relatively compact in X.

Thus, by Arzela-Ascoli theorem P_2 is a compact operator. These arguments enable us to conclude that $P = P_1 + P_2$ is a condense mapping on B_k, and by the fixed point theorem of Sadovskii there exists a fixed point $z(\,\cdot\,)$ for P on B_k. If we define $x(t) = \overline{z}(t) + y(t), -r \le t \le a$, it is easy to see that $x(\,\cdot\,)$ is an integral solution of (1.1) satisfying $x_0 = \phi$, $x(a) = x_1$, which shows system (1.1) is controllable. The proof is completed.

3 An Example

As an application of Theorem 2.1, we consider the following system

$$\begin{aligned}&\frac{\partial}{\partial t}[z(t,x) - h(t, z(t-r,x))] = \frac{\partial^2}{\partial x^2}[z(t,x) - h(t, z(t-r,x))] + Cu(t) \\ &\qquad\qquad + f(t, z(t-r,x)), \quad 0 \le t \le a,\ 0 \le x \le \pi, \\ &u(t,0) = u(t,\pi) = 0, \\ &u(\theta,x) = \phi(\theta,x), \quad -r \le \theta \le 0,\ 0 \le x \le \pi.\end{aligned} \tag{3.1}$$

To write system (3.1) in the form of (1.1), we choose $X = C([0, \pi])$ and consider the operator A defined by

$$Af = f''$$

with the domain

$$D(A) = \{f(\cdot) \in X : f'' \in X, f(0) = f(\pi) = 0\}.$$

We have $\overline{D(A)} = \{f(\cdot) \in X : f(0) = f(\pi) = 0\} \neq X$, and

$$\rho(A) \supset (0, +\infty),$$
$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0.$$

This implies that A satisfies the Hille-Yosida condition on X.

It is well known that A generates a compact C_0-semigroup $\{T_0(t)\}_{t\geq 0}$ on $\overline{D(A)}$ such that $\|T_0(t)\| \leq e^{-t}$ for $t \geq 0$.

In addition, we set that, for $0 \leq t \leq a$ and $\phi = \phi(\cdot, x) \in C([-r, 0]; X)$,

$$F(t, \phi) = f(t, \phi(-r, x)), \quad g(t, \phi) = h(t, \phi(-r, x)).$$

A case that the system (3.1) can be handled by using the classical semigroup theory is when the function is assumed to satisfy

$$f(t, 0) = h(t, 0) = 0 \quad \text{for all } 0 \leq t \leq a. \tag{3.2}$$

In this case, the functions F and g take their values in the space $\overline{D(A)}$ and the operator A generates a strongly continuous semigroup on $\overline{D(A)}$. However, here the integrated semigroup theory allows the ranges of F and g to be X without the condition (3.2). Now it is easy to adapt our previous result to obtain the controllability of system (3.1). We assume that

(i) For the function $f : [0, a] \times R \to R$ the following three conditions are satisfied:

(1) For each $t \in [0, a]$, $f(t, \cdot)$ is continuous.

(2) For each $z \in X$, $f(\cdot, z)$ is measurable.

(3) There are positive functions $h_1, h_2 \in L^1([0, a])$ such that

$$|f(t, z)| \leq h_1(t)\|z\| + h_2(t)$$

for all $(t, z) \in [0, a] \times C([0, a]; X)$. Clearly, these conditions ensure that F yields the condition (H_1) with $\gamma = \|h_1(\cdot)\|_{L^1}$.

(ii) The function $h : [0, a] \times R \to R$ is Lipschitz continuous such that condition (H_2) holds for positive constants $L_0 < 1$ and L.

(iii) $C : U \to X$ is a bounded linear operator.

(iv) The linear operator $W : U \to X$ defined by

$$Wu = \lim_{\lambda \to +\infty} \int_0^a T_0(t - s)Cu(s)ds$$

satisfies the condition (H_3). Thus, all the conditions of Theorem 2.1 are verified. Therefore, from Theorem 2.1, for any initial function ϕ with $\phi(0, 0) - g(0, \phi(\cdot 0)) = \phi(0, \pi) - g(0, \phi(\cdot \pi)) = 0$, the system (3.1) is controllable on $[0,a]$ provided that $(1+a\|C\|\|\widetilde{W}^{-1}\|)(L+\gamma)<1$ (here $\overline{M}=M=1$).

Appendix

Construction of $\widetilde{W}^{-1}$ (see [7]). Let

$$Y=\frac{L^2([0,a];U)}{\ker W}.$$

Since $\ker W$ is closed, Y is a Banach space under the norm

$$\|[u]\|_Y=\inf_{u\in[u]}\|u\|_{L^2}=\inf_{W\widetilde{u}=0}\|u+\widetilde{u}\|_{L^2},$$

where $[u]$ denotes the equivalence class of u.

Define $\widetilde{W};Y\to\overline{D(A)}$ by

$$\widetilde{W}[u]=Wu,\quad u\in[u].$$

Then $\widetilde{W}$ is one-to-one and

$$\|\widetilde{W}[u]\|_X\le\|W\|\|[u]\|_Y.$$

We claim that $V=\operatorname{Range}W$ is a Banach space with the norm

$$\|v\|_V=\|\widetilde{W}^{-1}v\|_Y.$$

This norm is equivalent to the graph norm on $D(\widetilde{W}^{-1})=\operatorname{Range}W$. $\widetilde{W}$ is bounded and since $D(\widetilde{W})=Y$ is closed, $\widetilde{W}^{-1}$ is closed, and so the above norm makes $\operatorname{Range}W=V$ a Banach space.

Moreover,

$$\|Wu\|_V=\|\widetilde{W}^{-1}Wu\|_Y=\|\widetilde{W}^{-1}\widetilde{W}[u]\|=\|[u]\|=\inf_{u\in[u]}\|u\|\le\|u\|,$$

so $W\in\mathcal{L}(L^2([0,a];U),V)$. Since $L^2([0,a];U)$ is reflexive and $\ker W$ is weakly closed, the infimum in the definition of the norm on Y is attained. For any $v\in V$, we can therefore choose a control $u\in L^2([0,a];U)$ such that $u=\widetilde{W}^{-1}v$.

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