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Products of Involutions in Steinberg Group over Skew Fields***

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Abstract Consider the stable Steinberg group $\mathrm{St}(K)$ over a skew field K. An element x is called an involution if $x^2 = 1$. In this paper, an involution is allowed to be the identity. The authors prove that an element A of $\mathrm{GL}_n(K)$ up to conjugation can be represented as BC, where B is lower triangular and C is simultaneously upper triangular. Furthermore, B and C can be chosen so that the elements in the main diagonal of B are $\beta_1, \beta_2, \cdots, \beta_n$, and of C are $\gamma_1, \gamma_2, \cdots, \gamma_n c_n$, where $c_n \in [K^*, K^*]$ and $\prod\limits_{j=1}^{n} \overline{\beta_j \gamma_j} = \det A$. It is also proved that every element δ in $\mathrm{St}(K)$ is a product of 10 involutions.

Keywords Steinberg group, Involution, Skew field
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1 Introduction

It is a classical problem in the research of classical groups to represent an element of a matrix group as a product of a special nature and to determine the smallest number of the factors in the representation (see [1–4]). It is known that every element of $\mathrm{SL}_n(F)(= E_n(F))$, the special linear group over a field, can be written as a product of at most four involutions for $n \geq 3$ (see [5]). The present note will consider the factorization of stable Steinberg groups over skew fields into involutions. Now we introduce some definitions and propositions that will be used in this paper (see [6, 7]).

Definition 1.1 *An element x of a group is called an involution if $x^2 = 1$. The Steinberg group* $\mathrm{St}_n(K)$ $(n \geq 3)$ *over an associative ring* (*with* 1) K *is the group with generators* $x_{ij}(r)$ $(r \in K,\ 1 \leq i \neq j \leq n)$, *and relations:*

(1) $x_{ij}(r) \cdot x_{ij}(s) = x_{ij}(r+s)$, $r, s \in K$;

(2) $[x_{ij}(r), x_{kl}(s)] = \begin{cases} x_{il}(rs), & j = k,\ i \neq l, \\ 1, & j \neq k,\ i \neq l. \end{cases}$

Let $\varphi_n : \mathrm{St}_n(K) \to E_n(K)$ (the elementary linear group) be the natural epimorphism mapping $x_{ij}(r)$ to $e_{ij}(r)$. Denote $K_{2,n}(K) = \ker \varphi_n$. Passing to the direct limit as $n \to \infty$, we obtain the stable Steinberg group $\mathrm{St}(K)$ and the epimorphism $\varphi : \mathrm{St}(K) \to E(K)$. Denote $K_2(K) = \ker \varphi$.

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Definition 1.2 $\mathrm{GL}(K)=\bigcup_{n\ge 1}\mathrm{GL}_n(K)$, $E(K)=\bigcup_{n\ge 1}E_n(K)$. *We define an injective homomorphism* $\tau_{n,m}:\mathrm{GL}_n(K)\to\mathrm{GL}_m(K)$ *as*

$$\tau_{n,m}(A)=\begin{pmatrix}A & \\ & I_{m-n}\end{pmatrix},$$

where $m\ge n$.

For $m\ge n$, define an injective homomorphism

$$f_{n,m}:\mathrm{St}_n(K)\to\mathrm{St}_m(K),$$
$$f_{n,m}(x_{ij}(a))=x_{ij}(a).$$

Then $f_n=f_m\cdot f_{n,m}$, and we can produce the commutative diagram

$$\begin{array}{ccccc}\mathrm{St}_n(K) & \xrightarrow{f_{n,m}} & \mathrm{St}_m(K) & \xrightarrow{f_m} & \mathrm{St}(K)\\ \downarrow\varphi_n & & \downarrow\varphi_m & & \downarrow\varphi\\ E_n(K) & \xrightarrow{\tau_{n,m}} & E_m(K) & \xrightarrow{\tau_m} & E(K)\end{array}$$

where $\tau_{n,m}(A)=\begin{pmatrix}A & \\ & I_{m-n}\end{pmatrix}$, $\tau_n=\tau_m\cdot\tau_{n,m}$. It is clear that $\mathrm{St}_m(K)\supseteq\mathrm{St}_n(K)$ as subgroups of $\mathrm{St}(K)$ and that $\mathrm{St}(K)=\bigcup_{n\ge 3}\mathrm{St}_n(K)$. It follows from the above commutative diagram that for $m\ge n$, $K_{2,m}(K)\supseteq K_{2,n}(K)$ as subgroups of $K_2(K)$. So $f_n:\mathrm{St}_n(K)\to\mathrm{St}(K)$ is the injection of $\mathrm{St}_n(K)$ into $\mathrm{St}(K)$. It is clear that $f_m(K_{2,m}(K))\supseteq f_n(K_{2,n}(K))$, and $K_2(K)=\bigcup_{n\ge 3}K_{2,n}(K)$. If K is a field, then $K_2(F)\cong K_{2,n}(F)$.

For any $u\in K^*$ (the set of units in K), define

$$w_{ij}(u)=x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u),\quad h_{ij}(u)=w_{ij}(u)w_{ij}(-1).$$

Proposition 1.1 (See [6, 7]) *Let* $w\in\mathrm{St}_n(K)$, $\varphi_n(w)=P(\pi)\mathrm{diag}(v_1,\cdots,v_n)$. *Then if* $\pi(i)=k$ *and* $\pi(j)=l$, *we have*

(1) $wx_{ij}(r)w^{-1}=x_{kl}(v_irv_j^{-1})$, $r\in K$,

(2) $ww_{ij}(u)w^{-1}=w_{kl}(v_iuv_j^{-1})$, $u\in K^*$,

(3) $wh_{ij}(u)w^{-1}=h_{kl}(v_iuv_j^{-1})h_{kl}(v_iv_j^{-1})^{-1}$.

Proposition 1.2 (See [6, 7]) *Let* $u,v\in K^*$. *We have*

(1) $w_{ij}(u)=w_{ji}(-u^{-1})$,

(2) $h_{ij}(u)h_{ji}(u)=1$, $h_{ij}(1)=1$,

(3) $[h_{ij}(u),h_{ik}(v)]=h_{ik}(uv)h_{ik}(u)^{-1}h_{ik}(v)^{-1}$.

Now, let K be a field, $u,v(\ne 0)\in K$. Define $\{u,v\}=h_{ik}(uv)h_{ik}(u)^{-1}h_{ik}(v)^{-1}$. By [6] and [7] we know that $K_2(K)$ is generated by symbol $\{u,v\}$ which is independent of the choice of indices i,k. For symbol $\{u,v\}$, we have

(1) $\{u,v\}^{-1}=\{v,u\}$,

(2) $\{u,1-u\}=\{u,-u\}=1$, $u\ne 0,1$,

(3) $\{u_1u_2,v\}=\{u_1,v\}\{u_2,v\}$, $\{u,v_1v_2\}=\{u,v_1\}\{u,v_2\}$.

2 Decomposition of Matrices over Skew Fields

In this section let K denote a skew field, $K^* = K\backslash\{0\}$, and $C = [K^*, K^*]$ be the commutator subgroup of K^*. Its factor commutator group $K^*/C = \overline{K}^*$ is abelian. To this group, we adjoin a zero element with obvious multiplication, and call the semi-group thus obtained $\overline{K}$. Every $a \neq 0$ of K has a canonical image $\overline{a}$ in $\overline{K}$, that is,

$$K \to \overline{K}, \quad a \to \overline{a}.$$

Then $\overline{ab} = \overline{a}\overline{b} = \overline{b}\overline{a} = \overline{ba}$ and $\overline{1}$ is the unit element of $\overline{K}$.

Dieudonne extended the theory of determinants to skew fields. There exists a homomorphism: $\mathrm{GL}_n(K)$ to $\overline{K}^* = K^*/C : A$ to $\det A$. The map is surjective and its kernel is $\mathrm{SL}_n(K)$. Thus we have that the formula $\det A = \overline{1}$ is equivalent to $A \in \mathrm{SL}_n(K)$. $\mathrm{SL}_n(K)$ is an invariant subgroup of $\mathrm{GL}_n(K)$, the kernel of the map $A \to \det A$, and the factor group is isomorphic to $\overline{K}^*$.

As usual $M_n(K)$ denotes the set of all $n \times n$ matrices over K and $\mathrm{GL}_n(K)$ denotes the group of all invertible $n \times n$ matrices. In this paper $T_{ij}(c)$, $i \neq j$, denotes the matrix whose element in the (i,j)-position is c and its elements in the other positions are the same as those in the identity matrix I, $E(i,j)$ denotes the matrix which is obtained by exchanging the i-row with the j-row of I, $D_i(c)$ is the matrix obtained from the identity by multiplying row i by c.

Lemma 2.1 *Let $n \geq 2$, $A = (a_{ij}) \in \mathrm{GL}_n(K)$. If A is not a central matrix, then A is similar to the matrix*

$$\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Proof Suppose first that A is the diagonal matrix $\mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})$. When there exist some i and j $(i \neq j)$ such that $a_{ii} \neq a_{jj}$, conjugating A by $E(1,i)$ and $E(2,j)$, we can suppose $a_{11} \neq a_{22}$. Then the element in the $(2,1)$-position of matrix

$$D_2((a_{11} - a_{22})^{-1})T_{21}(1)AT_{21}(-1)D_2(a_{11} - a_{22})$$

is 1. When $a_{11} = a_{ii} = a$ $(1 \leq i \leq n)$, $A = \mathrm{diag}(a, a, \cdots, a)$. By the hypothesis of this lemma there is an $x \neq 0$ such that $ax \neq xa$, so the element in the $(2,1)$-position of the matrix

$$D_2((xa - ax)^{-1})T_{21}(x)AT_{21}(-x)D_2(xa - ax)$$

is also 1.

If A is not a diagonal matrix, then there is some $a_{ij} \in K^*$, where $i \neq j$. Conjugating A by $D_2(a_{ij}^{-1})E(i,2)E(1,j)$, we can assume that a_{21} is 1. Thus the matrix

$$\prod_{\substack{i=1 \\ i\neq 2}}^{n} T_{i2}(-a_{i1})A\prod_{\substack{i=1 \\ i\neq 2}}^{n} T_{i2}(a_{i1})$$

has the desired property.

Lemma 2.2 *Suppose $n \geq 3$, $A = \left(\begin{smallmatrix} \beta_1\gamma_1 & Y \\ X & T \end{smallmatrix}\right) \in \mathrm{GL}_n(K)$, where $T \in M_{n-1}(K)$, $X = (1, 0, \cdots, 0)^T$. Let $Y_1 = Y + QT$, where $Q = (0, -1, 0, \cdots, 0)$. If $T - X\gamma_1^{-1}\beta_1^{-1}Y$ is a central matrix, then $T - X\gamma_1^{-1}\beta_1^{-1}Y_1$ is not a central matrix.*

Proof Let $T = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$, $Y = (y_2, y_3, \cdots, y_n)$. Then

$$T - X\gamma_1^{-1}\beta_1^{-1}Y = \begin{pmatrix} a_{22} - \gamma_1^{-1}\beta_1^{-1}y_2 & a_{23} - \gamma_1^{-1}\beta_1^{-1}y_3 & \cdots & a_{2n} - \gamma_1^{-1}\beta_1^{-1}y_n \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Since $T - X\gamma_1^{-1}\beta_1^{-1}Y$ is a central matrix, $a_{22} - \gamma_1^{-1}\beta_1^{-1}y_2 = a_{33} = \cdots = a_{nn}$, $a_{23} - \gamma_1^{-1}\beta_1^{-1}y_3 = 0$. Since $A \in \mathrm{GL}_n(K)$, $a_{ii} \neq 0$, $3 \leq i \leq n$. By computation we know that the element in the $(1,2)$-position of the matrix $T - X\gamma_1^{-1}\beta_1^{-1}Y_1$ is $a_{23} - \gamma_1^{-1}\beta_1^{-1}y_3 + \gamma_1^{-1}\beta_1^{-1}a_{33}$. But $a_{23} - \gamma_1^{-1}\beta_1^{-1}y_3 = 0$, so $T - X\gamma_1^{-1}\beta_1^{-1}Y_1$ is not a central matrix.

Now, we can extend the main theorem in [8] to get following theorem

Theorem 2.1 *Let A be a non-central invertible $n \times n$ matrix over a skew field K and β_j and γ_j, $1 \leq j \leq n$, be elements of K such that $\prod_{j=1}^{n} \overline{\beta_j\gamma_j} = \det A$. Then there exist $n \times n$ matrices B and C such that $PAP^{-1} = BC$, where B is lower triangular and C is simultaneously upper triangular, and P is in $\mathrm{GL}_n(K)$. Furthermore, B and C can be chosen so that the elements in the main diagonal of B are $\beta_1, \cdots, \beta_n$ and of C are $\gamma_1, \cdots, \gamma_{n-1}, \gamma_n c_n$, where $c_n \in [K^*, K^*]$.*

Proof We use induction on n. The result is vacuously true for $n = 1$. Now we assume that the conclusion of the theorem is true for all square matrices whose size is less than $n, n \geq 2$, and let A, β_j and γ_j be as in the statement of the theorem. Since A is not a central matrix, by Lemma 2.1, A is similar to the matrix

$$A_0 = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Let $P_0 = \begin{pmatrix} 1 & \beta_1\gamma_1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$. Then

$$P_0A_0P_0^{-1} = \begin{pmatrix} \beta_1\gamma_1 & b_{12} & \cdots & b_{1n} \\ 1 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

So A is similar to the matrix $A_1 = \begin{pmatrix} \beta_1\gamma_1 & Y \\ X & T \end{pmatrix}$, where $X = (1, 0, \cdots, 0)^T$.

In the case $n = 2$, we have that X, Y and T are just elements of K. Suppose $X = x$, $Y = y$, and $T = t$. Using the fact that $\det A = \overline{\beta_1\beta_2\gamma_1\gamma_2}$, we have $\det A = \overline{\beta_1}\overline{\gamma_1}(\overline{t - x\gamma_1^{-1}\beta_1^{-1}y})$, $\overline{\beta_2\gamma_2} = \overline{t - x\gamma_1^{-1}\beta_1^{-1}y}$, $t = \beta_2\gamma_2c_2 + x\gamma_1^{-1}\beta_1^{-1}y$ where $c_2 \in [K^*, K^*]$. We have

$$\begin{pmatrix} \beta_1\gamma_1 & y \\ x & t \end{pmatrix} = \begin{pmatrix} \beta_1 & 0 \\ x\gamma_1^{-1} & \beta_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & \beta_1^{-1}y \\ 0 & \gamma_2c_2 \end{pmatrix}.$$

This gives the conclusion of the theorem for $n = 2$.

We now assume that $n \geq 3$. If $T - X\gamma_1^{-1}\beta_1^{-1}Y$ is not a central matrix, then

$$A_1 = \begin{pmatrix} 1 & 0 \\ X\gamma_1^{-1}\beta_1^{-1} & I \end{pmatrix} \begin{pmatrix} \beta_1\gamma_1 & 0 \\ 0 & T - X\gamma_1^{-1}\beta_1^{-1}Y \end{pmatrix} \begin{pmatrix} 1 & \gamma_1^{-1}\beta_1^{-1}Y \\ & I \end{pmatrix}. \qquad (2.1)$$

Obviously we have $\det A_1 = \overline{\beta_1\gamma_1} \cdot \det(T - X\gamma_1^{-1}\beta_1^{-1}Y)$, and then $\det(T - X\gamma_1^{-1}\beta_1^{-1}Y) = \prod_{j=2}^{n} \overline{\beta_j\gamma_j}$. By the induction hypothesis

$$T - X\gamma_1^{-1}\beta_1^{-1}Y = P\begin{pmatrix} \beta_2 & & \\ \vdots & \ddots & \\ * & \cdots & \beta_n \end{pmatrix}\begin{pmatrix} \gamma_2 & \cdots & * \\ & \ddots & \vdots \\ & & \gamma_n c_n \end{pmatrix}P^{-1},$$

where $c_n \in [K^*, K^*]$. Then

$$\begin{aligned} A_1 &= \begin{pmatrix} 1 & 0 \\ X\gamma_1^{-1}\beta_1^{-1} & I \end{pmatrix}\begin{pmatrix} 1 & \\ & P \end{pmatrix}\begin{pmatrix} \beta_1 & & & \\ 0 & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & * & \cdots & \beta_n \end{pmatrix} \\ &\quad\times \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix}\begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix}\begin{pmatrix} 1 & \gamma_1^{-1}\beta_1^{-1}Y \\ & I \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ & P \end{pmatrix}\begin{pmatrix} 1 & 0 \\ P^{-1}X\gamma_1^{-1}\beta_1^{-1} & I \end{pmatrix}\begin{pmatrix} \beta_1 & & & \\ 0 & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ 0 & * & \cdots & \beta_n \end{pmatrix} \\ &\quad\times \begin{pmatrix} \gamma_1 & 0 & \cdots & 0 \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix}\begin{pmatrix} 1 & \gamma_1^{-1}\beta_1^{-1}YP \\ & I \end{pmatrix}\begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ & P \end{pmatrix}\begin{pmatrix} \beta_1 & & & \\ * & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & \beta_n \end{pmatrix}\begin{pmatrix} \gamma_1 & * & \cdots & * \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix}\begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix}. \end{aligned} \tag{2.2}$$

If $T - X\gamma_1^{-1}\beta_1^{-1}Y$ is a central matrix, then A_1 is similar to the matrix

$$A_2 = \begin{pmatrix} \beta_1\gamma_1 & Y_1 \\ X & T \end{pmatrix}\begin{pmatrix} 1 & -Q \\ 0 & I \end{pmatrix},$$

where $\left(\begin{smallmatrix} \beta_1\gamma_1 & Y_1 \\ X & T \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & Q \\ 0 & I \end{smallmatrix}\right) A_1$, $Q = (0, -1, 0, \cdots, 0)$.

By Lemma 2.2, $T - X\gamma_1^{-1}\beta_1^{-1}Y_1$ is not a central matrix. By (2.1) and (2.2), we have

$$\begin{aligned} A_2 &= \begin{pmatrix} 1 & 0 \\ X\gamma_1^{-1}\beta_1^{-1} & I \end{pmatrix}\begin{pmatrix} \beta_1\gamma_1 & 0 \\ 0 & T - X\gamma_1^{-1}\beta_1^{-1}Y_1 \end{pmatrix}\begin{pmatrix} 1 & \gamma_1^{-1}\beta_1^{-1}Y_1 \\ & I \end{pmatrix}\begin{pmatrix} 1 & -Q \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} 1 & \\ & P \end{pmatrix}\begin{pmatrix} \beta_1 & & & \\ * & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & \beta_n \end{pmatrix}\begin{pmatrix} \gamma_1 & * & \cdots & * \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix}\begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix}\begin{pmatrix} 1 & -Q \\ 0 & I \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 & \\ & P \end{pmatrix} \begin{pmatrix} \beta_1 & & & \\ * & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & \beta_n \end{pmatrix} \begin{pmatrix} \gamma_1 & * & \cdots & * \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix} \begin{pmatrix} 1 & -QP \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \\ & P \end{pmatrix} \begin{pmatrix} \beta_1 & & & \\ * & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & \beta_n \end{pmatrix} \begin{pmatrix} \gamma_1 & * & \cdots & * \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix} \begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix}.$$

Remark 2.1 Here we write PAP^{-1} as A since the problem that we consider is not affected by a similarity. We know that if the number of elements in a skew field is finite, then the skew field must be a finite field. Hence we have that any element of $\mathrm{SL}_n(K)$, where K is a finite field, is a product of at most four involutions and also these involutions are in $\mathrm{SL}_n(K)$. Thus we can assume that the skew field to be used here has infinitely elements, that is, $|K| = \infty$. Thus $|\mathrm{Cent}K| = \infty$ (see [9]). So by Theorem 2.1 and [10, pp. 207–209], for any A belonging to $\mathrm{SL}_n(K)$ and $A \neq aI$ where $a \in \mathrm{Cent}K^*$, or $A = aI$ where $a \notin \mathrm{Cent}K^*$, we have that there exist $\beta_i, \gamma_i \in K^*$ $(1 \leq j \leq n)$ such that $\beta_i \neq \beta_j, \gamma_i \neq \gamma_j$, and $\begin{pmatrix} \beta_i & \\ * & \beta_j \end{pmatrix}$ is similar to $\begin{pmatrix} \beta_i & \\ & \beta_j \end{pmatrix}$, $\begin{pmatrix} \gamma_i & * \\ & \gamma_j \end{pmatrix}$ is similar to $\begin{pmatrix} \gamma_i & \\ & \gamma_j c_n \end{pmatrix}$, when $i \neq j$, and $c_n \in [K^*, K^*]$. Thus

$$\begin{pmatrix} 1 & \\ & P^{-1} \end{pmatrix} A \begin{pmatrix} 1 & \\ & P \end{pmatrix} = \begin{pmatrix} \beta_1 & & & \\ * & \beta_2 & & \\ \vdots & \vdots & \ddots & \\ * & * & \cdots & \beta_n \end{pmatrix} \begin{pmatrix} \gamma_1 & * & \cdots & * \\ & \gamma_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \gamma_n c_n \end{pmatrix}$$

$$= Q \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_n \end{pmatrix} Q^{-1} H \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_n c_n \end{pmatrix} H^{-1}.$$

Further, by Theorem 2.1 and $|K| = \infty$, we can let $\beta_1, \cdots, \beta_n$ be $\delta_1, \delta_1^{-1}, \cdots, \delta_k, \delta_k^{-1}$, $\gamma_1, \cdots, \gamma_{n-2}$ be $\lambda_1, \lambda_1^{-1}, \cdots, \lambda_{k-1}, \lambda_{k-1}^{-1}$, and γ_{n-1}, γ_n be $xy, y^{-1}x^{-1}$, where $\delta_i, \lambda_i \neq 1$ and $c_n = xyx^{-1}y^{-1}$ when $n = 2k$. Thus we get

$$A = \begin{pmatrix} \delta_1 & & & & \\ & \delta_1^{-1} & & & \\ & & \ddots & & \\ & & & \delta_k & \\ & & & & \delta_k^{-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_1^{-1} & & & & & \\ & & \ddots & & & & \\ & & & \lambda_{k-1} & & & \\ & & & & \lambda_{k-1}^{-1} & & \\ & & & & & xy & \\ & & & & & & x^{-1}y^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \delta_1 & & & \\ \delta_1^{-1} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \delta_k \\ & & & \delta_k^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 1 & & \\ & & & 1 & 0 & & \\ & & & & & 0 & x \\ & & & & & x^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \lambda_1^{-1} & & & & & \\ \lambda_1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & \lambda_{k-1}^{-1} & & \\ & & & \lambda_{k-1} & 0 & & \\ & & & & & 0 & y^{-1} \\ & & & & & y & 0 \end{pmatrix}.$$

It is easy to see that the above four matrices are involutions.

If $n = 2k+1$, we can take β_{n+1} and γ_{n+1} both equal to one and other elements as the case of $n = 2k$. Then we can also prove our result as above case.

Next let us consider the case when A belongs to $\mathrm{SL}_n(K)$ and $A = aI$ where $a \in \mathrm{Cent}K^*$. If $a = 1$, then A is an involution. If $a \neq 1$, then under the mapping $\tau_{n,n+1}$ we have that matrix $\left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$ is a non-central invertible $(n+1)\times(n+1)$ matrix. Thus using the same method as above we know that $\left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$ is at most a product of four involutions. Up to now we have

Theorem 2.2 *Let $A \in \mathrm{SL}_{n+1}(K)$, A have the form $\left(\begin{smallmatrix} B & \\ & 1 \end{smallmatrix}\right)$, where $B \in \mathrm{SL}_n(K)$. Then A is the product of 4 involutions and these involutions are in $\mathrm{SL}_{n+1}(K)$.*

3 Generators of $K_2(K)$

In this section we shall find a set of generators for $K_2(K)$, where K is a skew field, in such a way that in the commutative case we just get the symbols $\{\alpha, \beta\}$ $(\alpha, \beta \in K^*)$.

Let K be a skew field. For $\alpha, \beta \in K^*$, write $c_{ijk}(\alpha, \beta) = [h_{ij}(\alpha), h_{ik}(\beta)]$, so $c_{ijk}(\alpha, \beta) = c_{ikj}(\beta, \alpha)^{-1}$. But

$$c_{ijk}(\alpha, \beta) = h_{ik}(\alpha\beta)h_{ik}(\alpha)^{-1}h_{ik}(\beta)^{-1}$$

whence $c_{ijk}(\alpha, \beta)$ is independent of the choice of j, and hence is independent of the choice of k also, we may write unambiguously

$$c_i(\alpha, \beta) = h_{ik}(\alpha\beta)h_{ik}(\alpha)^{-1}h_{ik}(\beta)^{-1}.$$

Now $w_{1i}(-1)h_{1k}(r)w_{1i}(-1)^{-1} = h_{ik}(r)$, since $h_{ik}(1) = 1$, and also

$$w_{1i}(-1)c_1(\alpha, \beta)w_{1i}(-1)^{-1} = c_i(\alpha, \beta).$$

Let $a_i(r) \in \mathrm{GL}(K)$ be the diagonal matrix with r in the ith diagonal position and all other entries on the diagonal equal to 1. Then $\varphi(c_i(\alpha, \beta)) = a_i([\alpha, \beta]) \in E(K)$. Since $a_i(r)a_i(\delta) = a_i(r\delta)$ $(r, \delta \in K^*)$, we deduce $a_i(r) \in E(K)$, for all $r \in K^{*\prime}$ (the derived subgroup of K^*). So for each $r \in K^{*\prime}$, we can choose $b_1(r) \in \mathrm{St}(K)$ with $\varphi((b_1(r)) = a_1(r)$. In particular, we insist that $b_1(1) = 1$, and we also insist that $b_1(r) \in H(K)$, for all $r \in K^{*\prime}$, which is possible since $c_1(\alpha, \beta) \in H(K)$, for all α, β. We then define

$$b_i(r) = w_{1i}(-1)b_1(r)w_{1i}(-1)^{-1}, \quad i \neq 1,\ r \in K^{*\prime},$$

so that $\varphi(b_i(r)) = a_i(r)$.

Since $a_i(r)a_i(\delta)a_i(r\delta)^{-1} = 1$, for all i and all $r, \delta \in K^{*\prime}$, we may define

$$d(r, \delta) = b_1(r)b_1(\delta)b_1(r\delta)^{-1} \in K_2(K).$$

Conjugating this by $w_{1i}(-1),\ i \neq 1$, shows that

$$d(r,\delta) = b_i(r)b_i(\delta)b_i(r\delta)^{-1} \quad \text{for any } i, \text{ and } r,\delta \in K^{*\prime}.$$

Note that $d(r,\delta) = 1$ if $r = 1$ or $\delta = 1$, since $b_i(1) = 1$, so the elements $d(r,\delta)$ are trivial if K^* is abelian.

Next, $\varphi(c_1(\alpha,\beta)) = a_1([\alpha,\beta]) = \varphi(b_1([\alpha,\beta]))$, so we may define

$$e(\alpha,\beta) = c_1(\alpha,\beta)b_1([\alpha,\beta])^{-1} \in K_2(K).$$

And conjugating this by $w_{1i}(-1), i \neq 1$, shows that

$$e(\alpha,\beta) = c_i(\alpha,\beta)b_i([\alpha,\beta])^{-1} \quad \text{for any } i \text{ and } r,\delta \in K^{*\prime}.$$

Note that if $[\alpha,\beta] = 1$, then $e(\alpha,\beta) = \{\alpha,\beta\}$. Recall that $C(K) = H(K) \bigcap K_2(K)$. (Here the definition and the relationship between $C(K), H(K)$ and $K_2(K)$ are the same as the definition and the relationship in [6, 7].) Now we get

Lemma 3.1 *Let K be a skew field. Then $C(K)$ is generated by all $e(\alpha,\beta)$, $d(r,\delta)$ $(\alpha,\beta \in K^*,\ r,\delta \in K^{*\prime})$.*

Proof Let A be the subgroup of $K_2(K)$ generated by all $e(\alpha,\beta)$, $d(r,\delta)$ $(\alpha,\beta \in K^*$, $r,\delta \in K^{*\prime})$, so $A \subseteq C(K)$. Since $K_2(K) = \operatorname{Cent}\operatorname{St}(K)$, A is a normal subgroup of $\operatorname{St}(K)$, and we may write

$$b_i(r)b_i(\delta) \equiv b_i(r\delta) \pmod A, \quad c_i(\alpha,\beta) \equiv b_i([\alpha,\beta]) \pmod A.$$

Now

$$\begin{aligned} h_{1i}(\lambda)c_1(\alpha,\beta)h_{1i}(\lambda)^{-1} &= h_{1i}(\lambda)h_{1k}(\alpha\beta)h_{1k}(\alpha)^{-1}h_{1k}(\beta)^{-1}h_{1i}(\lambda)^{-1} \\ &= h_{1k}(\lambda\alpha\beta)h_{1k}(\lambda)^{-1}h_{1k}(\lambda)h_{1k}(\lambda\alpha)^{-1}h_{1k}(\lambda)h_{1k}(\lambda\beta)^{-1} \\ &= h_{1k}(\lambda\alpha\beta)h_{1k}(\lambda\alpha)^{-1}h_{1k}(\beta)^{-1}h_{1k}(\beta)h_{1k}(\lambda)h_{1k}(\lambda\beta)^{-1} \\ &= c_1(\lambda\alpha,\beta)c_1(\lambda,\beta)^{-1}, \quad \lambda,\alpha,\beta \in K^*, 1, i, k \text{ distinct.} \end{aligned}$$

It follows that

$$\begin{aligned} h_{1i}(\lambda)b_1([\alpha,\beta])h_{1i}(\lambda)^{-1} &\equiv b_1([\lambda\alpha,\beta])b_1([\lambda,\beta]) \pmod A \\ &\equiv b_1([\lambda\alpha,\beta][\lambda,\beta]) \pmod A \\ &\equiv b_1(\lambda[\alpha,\beta]\lambda^{-1}). \end{aligned}$$

Now $H(K)$ is generated by all $h_{1i}(\lambda), \lambda \in K^*$, so write B for the subgroup of $\operatorname{St}(K)$ generated by A and all $b_1(r)$ $(r \in K^{*\prime})$. Then B is normalized by $H(K)$. We then have

$$h_{1j}(\alpha)h_{1k}(\beta) \equiv h_{1k}(\beta)h_{1j}(\alpha) \pmod B, \quad h_{1k}(\alpha)h_{1k}(\beta) \equiv h_{1k}(\beta\alpha) \pmod B,$$

whence, for any $w \in H(K)$, we can write

$$w \equiv h_{12}(\alpha_2)h_{13}(\alpha_3)\cdots h_{1n}(\alpha_n) \pmod B \quad \text{or} \quad w \equiv h_{12}(\alpha_2)h_{13}(\alpha_3)\cdots h_{1n}(\alpha_n)b_1(r) \pmod A,$$

where $\alpha_2, \alpha_3, \cdots, \alpha_n \in K^*$ and $r \in K^{*\prime}$. It follows that

$$\varphi(w) = \operatorname{diag}(\alpha, \alpha_2^{-1}, \alpha_3^{-1}, \cdots, \alpha_n^{-1}, 1, 1, \cdots),$$

where $\alpha = \alpha_2\alpha_3\cdots\alpha_n r$. If $w \in K_2(K)$, we deduce that $\alpha_2 = \alpha_3 = \cdots = \alpha_n = r = 1$, or in other words $C(K) \subseteq A$, as required.

Since K is a skew field for which $C(K) = K_2(K)$ (see [6]), we deduce

Theorem 3.1 *Let K be a skew field. Then $K_2(K)$ is generated by all $e(\alpha,\beta)$, $d(r,\delta)$ $(\alpha,\beta \in K^*,\ r,\delta \in K^{*\prime})$.*

Remark 3.1 Without loss of generality, we can assume that $b_1(r) = c_1(\alpha,\beta)$ if $r \neq 1$, and $b_1(r) = 1$ if $r = 1$. Then we have $e(\alpha,\beta) = 1$ if $r \neq 1$ and $e(\alpha,\beta) = \{\alpha,\beta\}$ if $r = 1$.

4 Decomposition of Steinberg Groups

Since $\varphi : \mathrm{St}(K) \to E(K)$ is surjective, there is an element $\rho \in \mathrm{St}(K)$ such that $\varphi(\rho) = P$ for any given matrix P. Now we have $K_2(K) = \ker\varphi$ and it is the center of the stable Steinberg group $\mathrm{St}(K)$ (see [7]). Thus for any $x \in \mathrm{St}(K)$, there exists an $n \in Z$ such that $\varphi(x) \in E_n(K) = \mathrm{SL}_n(K) = \tau_{n,n+m}(\mathrm{SL}_n K) \subseteq \mathrm{SL}_{n+m}(K)$. Then by Theorem 2.2, we have

$$\varphi(x) = H_1H_2H_3H_4,$$

where H_i is an involution in $\mathrm{SL}_{n+m}(K)$. Of course, they are in $\mathrm{SL}(K) = E(K)$. Hence if we find four involutions $\delta_i, 1 \leq i \leq 4$, in $\mathrm{St}(K)$ such that $\varphi(\delta_i) = H_i$, then we can get

$$x = \omega \cdot (\delta_1\delta_2\delta_3\delta_4),$$

where ω is in $\ker\varphi$ (the center of $\mathrm{St}(K)$).

We know that $H = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}$ is an involution in $\mathrm{SL}_3(K) \subseteq \mathrm{SL}(K)$, but we easily get an element $w_{12}(1)h_{13}(-1) \in St(K)$ such that $\varphi(w_{12}(1)h_{13}(-1)) = H$ and it is not an involution in $\mathrm{St}(K)$ (see [6]). So we must show that for those involutions H_i $(1 \leq i \leq 4)$ in $\mathrm{SL}(K)$ and $H_1H_2H_3H_4$, we could find involutions δ_i $(1 \leq i \leq 4)$ such that they are in $\mathrm{St}(K)$ and they satisfy $\varphi(\delta_1\delta_2\delta_3\delta_4) = H_1H_2H_3H_4$. On the other hand, if we have proved that ω is a product of at most 10 involutions, of course, these involutions must be in $\mathrm{St}(K)$ at the same time, which occur in the representation of ω commute with δ_i, then we can get our main result.

Here we will show that we can find involutions δ_i that satisfy the above conditions. By the proof of Theorem 2.2, we know that the involutions occurring in the representation of Theorem 2.2 arise in the decomposition of a permutation or in the factorization of a permutation matrix. Now we only consider the case that a permutation is written as a product of two involutions. In fact, a permutation S with order n can be written as a product of two involutions and these involutions are similar to the direct sum of involutions of the form $I_1 = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}_{4\times 4}$ and $I_2 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & -1 \end{pmatrix}_{3\times 3}$. Hence we only need to show the simple case, that is to say, we can assume that

$$S = PI_1P^{-1} \cdot QI_2Q^{-1}, \quad \text{or} \quad S = PI_1P^{-1} \cdot QI_1Q^{-1} \quad \text{and} \quad S = PI_2P^{-1} \cdot QI_2Q^{-1}.$$

But we can send $\mathrm{SL}_n(K)$ to $\mathrm{SL}_m(K)$ under $\tau_{n,m}$. So in $\mathrm{SL}_{n+2}(K)$, we have

$$\begin{aligned} S = &\begin{pmatrix} P & \\ & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_1 & \\ & -I_{2\times 2} \end{pmatrix} \begin{pmatrix} P^{-1} & \\ & I_{2\times 2} \end{pmatrix} \\ &\cdot \begin{pmatrix} Q & \\ & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_2 & \\ & -I_{2\times 2} \end{pmatrix} \begin{pmatrix} Q^{-1} & \\ & I_{2\times 2} \end{pmatrix} \end{aligned}$$

or

$$S = \begin{pmatrix} P & \\ & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_1 & \\ & -I_{2\times 2} \end{pmatrix} \begin{pmatrix} P^{-1} & \\ & I_{2\times 2} \end{pmatrix} \cdot \begin{pmatrix} Q & \\ & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_1 & \\ & -I_{2\times 2} \end{pmatrix} \begin{pmatrix} Q^{-1} & \\ & I_{2\times 2} \end{pmatrix},$$

and

$$S = \begin{pmatrix} P & \\ & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_2 & \\ & -I_{2\times 2} \end{pmatrix} \begin{pmatrix} P^{-1} & \\ & I_{2\times 2} \end{pmatrix} \cdot \begin{pmatrix} Q & \\ & I_{2\times 2} \end{pmatrix} \begin{pmatrix} I_2 & \\ & -I_{2\times 2} \end{pmatrix} \begin{pmatrix} Q^{-1} & \\ & I_{2\times 2} \end{pmatrix}.$$

But by Proposition 1.1 and Proposition 1.2, we have

$$\varphi(w_{12}(1)h_{14}(-1)w_{34}(1)h_{56}(-1)) = \begin{pmatrix} I_1 & \\ & -I_{2\times 2} \end{pmatrix},$$

$$\varphi(w_{12}(1)h_{13}(-1)h_{45}(-1)) = \begin{pmatrix} I_2 & \\ & -I_{2\times 2} \end{pmatrix},$$

where $w_{12}(1)h_{14}(-1)w_{34}(1)h_{56}(-1)$ and $w_{12}(1)h_{13}(-1)h_{45}(-1)$ are involutions in $\mathrm{St}(K)$.

So far, we have shown that there are involutions δ_i $(1 \le i \le 4)$ in $\mathrm{St}(K)$ such that $\varphi(\delta_1\delta_2\delta_3\delta_4) = H_1H_2H_3H_4$.

Now we want to prove that ω is a product of at most 10 involutions, of course, these involutions must be in $\mathrm{St}(K)$. At the same time, we can prove that those involutions which occur in the representation of ω commute with δ_i. In order to complete the proof of the main result, let us first prove a lemma.

Lemma 4.1 *Let K be a skew field. Then every element of $K_2(K)$ can be written as a product of at most ten involutions.*

Proof (1) Let us consider the special case, namely, the generator $e(u,v) = \{u,v\}$. By definition

$$\begin{aligned}\{u,v\} &= h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1} = h_{12}(uv)h_{21}(u)h_{21}(v)\\ &= w_{12}(uv)h_{13}(-1)h_{45}(-1)h_{54}(-1)h_{31}(-1)w_{12}(-1)\\ &\quad \cdot w_{21}(u)h_{31}(-1)h_{45}(-1)h_{54}(-1)h_{13}(-1)w_{21}(-1)w_{21}(v)w_{21}(-1).\end{aligned}$$

Since $h_{13}(-1)w_{12}(u)h_{13}(-1)^{-1} = w_{12}(-u)$, we have

$$(w_{12}(u)h_{13}(-1)h_{45}(-1))^2 = w_{12}(u)w_{12}(-u)(h_{13}(-1))^2(h_{45}(-1))^2 = \{-1,-1\}\{-1,-1\} = 1.$$

That is, $w_{12}(u)h_{13}(-1)h_{45}(-1)$ is an involution in $\mathrm{St}(K)$. Similarly, $h_{54}(-1)h_{31}(-1)w_{12}(-1)$ and $h_{54}(-1)h_{13}(-1)w_{21}(-1)w_{21}(v)w_{21}(-1)$ are involutions in $\mathrm{St}(K)$ respectively.

(2) General case of products of generators: Every element $\omega = e(u,v)$ of $K_2(K)$ can be written as $\omega = \prod_{i=1}^{k}\{u_i, v_i\}$. Since the definition of $\{u_i, v_i\}$ is independent of the indexes of h_{kl},

we can write $\{u_i, v_i\} = T_i^{(1)}T_i^{(2)}T_i^{(3)}T_i^{(4)}$, where

$$\begin{aligned}
T_i^{(1)} &= w_{5(i-1)+1,5(i-1)+2}(u_iv_i)h_{5(i-1)+1,5(i-1)+3}(-1)h_{5(i-1)+4,5(i-1)+5}(-1),\\
T_i^{(2)} &= h_{5(i-1)+5,5(i-1)+4}(-1)h_{5(i-1)+3,5(i-1)+1}(-1)w_{5(i-1)+1,5(i-1)+2}(-1),\\
T_i^{(3)} &= w_{5(i-1)+2,5(i-1)+1}(u_i)h_{5(i-1)+3,5(i-1)+1}(-1)h_{5(i-1)+4,5(i-1)+5}(-1),\\
T_i^{(4)} &= h_{5(i-1)+5,5(i-1)+4}(-1)h_{5(i-1)+1,5(i-1)+3}(-1)w_{5(i-1)+2,5(i-1)+1}(-1)\\
&\quad \cdot w_{5(i-1)+2,5(i-1)+1}(v_i)w_{5(i-1)+2,5(i-1)+1}(-1)
\end{aligned}$$

are all involutions in $\mathrm{St}(K)$. Note that when $j \neq i$, the involutory factors in the factorization of $\{u_j, v_j\}$ and $\{u_i, v_i\}$ are respectively exchangeable. So ω is a product of 4 involutions.

(3) Let us consider the special case, the generator

$$\begin{aligned}
d(r,\delta) &= b_1(r)b_1(\delta)b_1(r\delta)^{-1} = c_1(a,b)c_1(x,y)c_1(u,v)^{-1}\\
&= h_{12}(ab)h_{12}(a)^{-1}h_{12}(b)^{-1}\cdot h_{12}(xy)h_{12}(x)^{-1}h_{12}(y)^{-1}\cdot [h_{12}(uv)h_{12}(u)^{-1}h_{12}(v)^{-1}]^{-1}\\
&= h_{12}(ab)h_{21}(a)h_{21}(b)\cdot h_{12}(xy)h_{21}(x)h_{21}(y)\cdot h_{12}(v)h_{12}(u)h_{21}(uv)\\
&= w_{12}(ab)w_{12}(-1)w_{21}(a)w_{21}(-1)w_{21}(b)w_{21}(-1)\\
&\quad \cdot w_{12}(xy)w_{12}(-1)w_{21}(x)w_{21}(-1)w_{21}(y)w_{21}(-1)\\
&\quad \cdot w_{12}(v)w_{12}(-1)w_{12}(u)w_{12}(-1)w_{21}(uv)w_{21}(-1)\\
&= w_{12}(ab)w_{12}(-a)w_{21}(b)w_{12}(-1)[w_{21}(-1)]^2\\
&\quad \cdot w_{12}(xy)w_{12}(-1)w_{21}(x)w_{21}(-1)w_{21}(y)w_{21}(-1)\\
&\quad \cdot w_{12}(v)w_{12}(-1)w_{12}(u)w_{12}(-1)w_{21}(uv)w_{21}(-1)\\
&= w_{12}(ab)w_{12}(-a)w_{21}(b)w_{21}(-1)\cdot w_{12}(xy)w_{12}(-x)w_{21}(y)w_{21}(-1)\\
&\quad \cdot w_{12}(v)w_{21}(-u)w_{21}(uv)w_{12}(-1)\\
&= w_{12}(ab)w_{12}(-a)w_{21}(b)\cdot w_{21}(-xy)w_{21}(x)w_{12}(-y)[w_{21}(-1)]^2\\
&\quad \cdot w_{12}(v)w_{21}(-u)w_{21}(uv)w_{12}(-1)\\
&= w_{12}(ab)w_{12}(-a)w_{21}(b)\cdot w_{21}(-xy)w_{21}(x)w_{12}(-y)\\
&\quad \cdot w_{12}(v)w_{21}(-u)w_{21}(uv)[w_{21}(-1)]^2w_{12}(-1)\\
&= w_{12}(ab)w_{12}(-a)w_{21}(b)\cdot w_{21}(-xy)w_{21}(x)w_{12}(-y)\cdot w_{12}(v)w_{21}(-u)w_{21}(uv)w_{21}(-1).
\end{aligned}$$

In fact, $w_{12}(s)\ w_{12}(t) = w_{12}(s)\ h_{13}(-1)\ h_{45}(-1)\cdot h_{54}(-1)\ h_{31}(-1)\ w_{12}(t)$, $s,t \in K^*$. But $[w_{12}(s)\ h_{13}(-1)\ h_{45}(-1)]^2 = 1$, $[h_{54}(-1)\ h_{31}(-1)\ w_{12}(t)]^2 = 1$. Similarly, $w_{12}(s)\ w_{21}(t)$, $w_{21}(s)w_{12}(t)$ and $w_{21}(s)w_{21}(t)$ can be represented as products of 2 involutions. Thus $d(r,\delta)$ can be represented as products of 10 involutions.

(4) Every element $\omega = \prod_{i=1}^{k} d(r_i,\delta_i)$ can be written as $\prod_{i=1}^{k} b_j(\gamma_i)b_j(\delta_i)b_j(\gamma_i\delta_i)^{-1}$ since the definition of $d(r,\delta)$ is independent of the indexes of b_j (by the definitions of $b_j(\gamma)$ and $c_i(\alpha,\beta)$). Hence, using the same idea as (2), ω is a product of 10 involutions.

Theorem 4.1 *Let K be a skew field. Then every element of* $\mathrm{St}(K)$ *can be written as a product of at most* 10 *involutions.*

Proof We assume that $\xi \in \mathrm{St}(K)$. If $\xi \in K_2(K)$, then the consequence of theorem can be obtained by Lemma 4.1. Now suppose that $\xi \notin K_2(K)$. Then by the definition of $\mathrm{St}(K)$ there is a positive integer $n \geq 4$ and 4 involutions $H_1, H_2, H_3, H_4 \in E_n(K) = \mathrm{SL}_n(K)$ such that there are four involutions $\delta_1, \delta_2, \delta_3, \delta_4 \in \mathrm{St}_n(K)$ also, and $H_i = \varphi(\delta_i)$, such that $\varphi(\xi) = H_1H_2H_3H_4$. Thus we have

$$\varphi(\xi) = \varphi(\delta_1\delta_2\delta_3\delta_4), \quad \text{i.e.,} \quad \xi = \omega \cdot \delta_1\delta_2\delta_3\delta_4,$$

where $\omega \in K_2(K)$.

Let $\omega = \prod\limits_{i=1}^{t}\prod\limits_{j=1}^{s} e(a_i, b_i)d(r_j, \delta_j)$. Since the symbol $e(a_i, b_i)$ and $d(r_j, \delta_j)$ are independent of the indexes of h_{rk} occurring in the representation of $e(a_i, b_i)$ and the indexes of b_q occurring in the representation of $d(r_j, \delta_j)$, we can choose sufficient large r, k and q such that

$$\begin{aligned} e(a_i, b_i) &= h_{2n+5(i-1)+1,2n+5(i-1)+2}(a_ib_i)h_{2n+5(i-1)+1,2n+5(i-1)+2}(a_i)^{-1} \\ &\quad \cdot h_{2n+5(i-1)+1,2n+5(i-1)+2}(b_i)^{-1}, \\ d(r_j, \delta_j) &= b_{4n+5(i-1)+1}(\gamma_j)b_{4n+5(i-1)+1}(\delta_j) \cdot h_{4n+5(i-1)+1}(\gamma_j\delta_j)^{-1}. \end{aligned}$$

By Lemma 4.1, $\prod\limits_{i=1}^{t} e(a_i, b_i)$ is a product of 4 involutions T_1, T_2, T_3, T_4, but the indexes r, k of h_{rk}, occurring in the representations of T_i are larger than $2n$. Thus T_i commutes with δ_i and $\prod\limits_{j=1}^{s} d(r_j, \delta_j)$ is a product of 10 involutions S_i $(1 \leq i \leq 10)$, but the indexes q of b_q occurring in the representation of S_i are larger than $4n$. Thus T_i, S_i and δ_i commute with each other. So we have that

$$\begin{aligned} \xi &= (T_1T_2T_3T_4)(S_1S_2S_3S_4S_5S_6S_7S_8S_9S_{10})(\delta_1\delta_2\delta_3\delta_4) \\ &= (T_1S_1\delta_1)(T_2S_2\delta_2)(T_3S_3\delta_3)(T_4S_4\delta_4)S_5S_6S_7S_8S_9S_{10} \end{aligned}$$

is a product of ten involutions, and also we have that these involutions which appear in the above are in $\mathrm{St}(K)$.

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