Chin. Ann. Math.
28B(3), 2007, 265–282
DOI: 10.1007/s11401-006-0423-x

Chinese Annals of Mathematics, Series B

Inverse Piston Problem for the System of One-Dimensional Isentropic Flow

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Abstract In this paper, the authors consider the inverse piston problem for the system of one-dimensional isentropic flow and obtain that, under suitable conditions, the piston velocity can be uniquely determined by the initial state of the gas on the right side of the piston and the position of the forward shock.

Keywords Inverse piston problem, System of one-dimensional isentropic flow, Shock
2000 MR Subject Classification 35L45, 35L60, 35R30, 74J40

1 Introduction and Main Result

Suppose that the piston originally located at the origin at $t = 0$ moves with the speed $v = \phi(t)$ $(t \geq 0)$ in a tube, the length of which is assumed to be infinite, and that the gas on the right side of the piston possesses an isentropic state at $t = 0$. In order to determine the state of the gas on the right side of this piston, in Lagrangian representation this piston problem reduces to the mixed initial-boundary value problem for the system

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau)}{\partial x} = 0 \end{cases} \tag{1.1}$$

with the initial data

$$t = 0: \ \tau = \tau_0^+(x) (> 0), \ u = u_0^+(x), \quad \forall\, x \geq 0 \tag{1.2}$$

and the boundary condition

$$x = 0: \ u = \phi(t), \quad \forall\, t \geq 0, \tag{1.3}$$

where τ is the specific volume, u is the velocity and $p = p(\tau)$ is the pressure. For polytropic gases

$$p = p(\tau) = A\tau^{-\gamma}, \quad \forall\, \tau > 0, \tag{1.4}$$

where $\gamma > 1$ is the adiabatic exponent and A is a positive constant.

Manuscript received October 10, 2006. Published online April 30, 2007.
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Suppose that

$$\phi(0) > u_0^+(0). \tag{1.5}$$

The motion of the piston produces a forward shock $x = x_2(t)$ passing through the origin at least for a short time T_0 (see [2, 6, 7]), such that the corresponding piecewise C^1 solution on the domain

$$D(T_0) = \{(t,x) \mid 0 \le t \le T_0,\ x \ge 0\} \tag{1.6}$$

is written as

$$(\tau, u) = \begin{cases} (\tau_0(t,x), u_0(t,x)), & 0 \le x \le x_2(t), \\ (\tau_+(t,x), u_+(t,x)), & x \ge x_2(t), \end{cases} \tag{1.7}$$

where $(\tau_0(t,x), u_0(t,x))$, $(\tau_+(t,x), u_+(t,x)) \in C^1$ satisfy system (1.1) in the classical sense on their domains respectively and verify the Rankine-Hugoniot conditions

$$\begin{cases} [\tau]x_2'(t) + [u] = 0, \\ [u]x_2'(t) - [p(\tau)] = 0 \end{cases} \tag{1.8}$$

and the entropy condition

$$\begin{cases} \lambda_1(\tau_0(t,x_2(t))) < x_2'(t) < \lambda_2(\tau_0(t,x_2(t))), \\ x_2'(t) > \lambda_2(\tau_+(t,x_2(t))) \end{cases} \tag{1.9}$$

on $x = x_2(t)$, in which $[\tau] = \tau_+(t,x_2(t)) - \tau_0(t,x_2(t))$, etc. and

$$-\lambda_1(\tau) = \lambda_2(\tau) = \sqrt{-p'(\tau)}. \tag{1.10}$$

Introducing the Riemann invariants

$$\begin{cases} r = \dfrac{1}{2}\Big(u - \displaystyle\int_\tau^\infty \sqrt{-p'(\eta)}d\eta\Big) = \dfrac{1}{2}u - \dfrac{\sqrt{A\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}}, \\ s = \dfrac{1}{2}\Big(u + \displaystyle\int_\tau^\infty \sqrt{-p'(\eta)}d\eta\Big) = \dfrac{1}{2}u + \dfrac{\sqrt{A\gamma}}{\gamma-1}\tau^{-\frac{\gamma-1}{2}} \end{cases} \tag{1.11}$$

as new unknown functions, (1.1)–(1.3) can be reduced to the following problem

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r,s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + \mu(r,s)\dfrac{\partial s}{\partial x} = 0, \end{cases} \tag{1.12}$$

$$t = 0: \quad (r,s) = (r_0^+(x), s_0^+(x)), \quad \forall\, x \ge 0, \tag{1.13}$$

$$x = 0: \quad s = -r + \phi(t), \quad \forall\, t \ge 0, \tag{1.14}$$

where

$$\begin{cases} r_0^+(x) = \dfrac{1}{2}u_0^+(x) - \dfrac{\sqrt{A\gamma}}{\gamma-1}(\tau_0^+(x))^{-\frac{\gamma-1}{2}}, \\ s_0^+(x) = \dfrac{1}{2}u_0^+(x) + \dfrac{\sqrt{A\gamma}}{\gamma-1}(\tau_0^+(x))^{-\frac{\gamma-1}{2}} \end{cases} \tag{1.15}$$

with

$$s_0^+(x) - r_0^+(x) > 0, \quad \forall\, x \geq 0 \tag{1.16}$$

and

$$-\lambda(r,s) = \mu(r,s) = \sqrt{-p'(\tau(s-r))} = \frac{(\frac{\gamma-1}{2})^{\frac{\gamma+1}{\gamma-1}}}{(A\gamma)^{\frac{1}{\gamma-1}}}(s-r)^{\frac{\gamma+1}{\gamma-1}}. \tag{1.17}$$

In the special case that the piston moves with a constant speed u_p and the initial state is a constant state (τ_+, u_+) $(\tau_+ > 0)$ with

$$u_p > u_+, \tag{1.18}$$

(1.13) and (1.14) become, respectively,

$$t = 0: \;\; (r,s) = (r_+, s_+), \quad \forall\, x \geq 0 \tag{1.19}$$

and

$$x = 0: \;\; s = -r + u_p, \quad \forall\, t \geq 0, \tag{1.20}$$

where

$$\begin{cases} r_+ = \dfrac{1}{2}u_+ - \dfrac{\sqrt{A\gamma}}{\gamma-1}(\tau_+)^{-\frac{\gamma-1}{2}}, \\ s_+ = \dfrac{1}{2}u_+ + \dfrac{\sqrt{A\gamma}}{\gamma-1}(\tau_+)^{-\frac{\gamma-1}{2}} \end{cases} \tag{1.21}$$

with

$$s_+ - r_+ > 0$$

and

$$u_p > r_+ + s_+, \tag{1.22}$$

and the solution to the previous problem is the typical forward shock

$$(r,s) = \begin{cases} (r_0, s_0), & 0 \leq x \leq Vt, \\ (r_+, s_+), & x \geq Vt, \end{cases} \tag{1.23}$$

where V is the speed of propagation of the typical forward shock

$$V = G(r_+, s_+, r_0, s_0) \tag{1.24}$$

satisfying the entropy condition

$$\begin{cases} \lambda(r_0, s_0) < V < \mu(r_0, s_0), \\ V > \mu(r_+, s_+), \end{cases} \tag{1.25}$$

in which r_0, s_0 and (1.24) are uniquely determined by

$$r_0 + s_0 = u_p \tag{1.26}$$

and the Rankine-Hugoniot conditions

$$(r_0+s_0)-(r_++s_+)=\sqrt{-(p(\tau(s_0-r_0))-p(\tau(s_+-r_+)))(\tau(s_0-r_0)-\tau(s_+-r_+))}, \tag{1.27}$$

$$V=\sqrt{-\frac{p(\tau(s_0-r_0))-p(\tau(s_+-r_+))}{\tau(s_0-r_0)-\tau(s_+-r_+)}} \tag{1.28}$$

(see [1, 2]).

As a perturbation of the simplest piston problem mentioned above, in [2, 7], the piston problem (1.1)–(1.3) is globally studied and we have the following

Theorem 1.1 *Suppose that $\tau_0^+(x)$, $u_0^+(x)$ and $\phi(t)\in C^1$ and*

$$\tau_0^+(0)=\tau_+,\quad u_0^+(0)=u_+,\quad \phi(0)=u_p. \tag{1.29}$$

Suppose furthermore that

$$|\tau_0^+(x)-\tau_+|,\ |u_0^+(x)-u_+|\le\varepsilon,\quad \forall\, x\ge 0, \tag{1.30}$$

$$|\phi(t)-\phi(0)|\le\varepsilon,\quad \forall\, t\ge 0, \tag{1.31}$$

$$|\tau_0^{+'}(x)|,\ |u_0^{+'}(x)|\le\frac{\eta}{1+x},\quad \forall\, x\ge 0, \tag{1.32}$$

$$|\phi'(t)|\le\frac{\eta}{1+t},\quad \forall\, t\ge 0, \tag{1.33}$$

where $\varepsilon>0$ and $\eta>0$ are suitably small. Then, the piston problem (1.1)–(1.3) *admits a unique global piecewise C^1 solution*

$$(\tau(t,x),u(t,x))=\begin{cases}(\tau_0(t,x),u_0(t,x)), & 0\le x\le x_2(t),\\ (\tau_+(t,x),u_+(t,x)), & x\ge x_2(t)\end{cases} \tag{1.34}$$

on the domain

$$D=\{(t,x)\mid t\ge 0,\ x\ge 0\}. \tag{1.35}$$

This solution, containing only one forward shock $x=x_2(t)$ passing through the origin with $x_2'(0)=V$, satisfies the following estimates: on the domain

$$D_+=\{(t,x)\mid t\ge 0,\ x\ge x_2(t)\}, \tag{1.36}$$

we have

$$|\tau_+(t,x)-\tau_+|,\ |u_+(t,x)-u_+|\le K_1\varepsilon, \tag{1.37}$$

$$\Big|\frac{\partial\tau_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial\tau_+}{\partial t}(t,x)\Big|,\ \Big|\frac{\partial u_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial u_+}{\partial t}(t,x)\Big|\le\frac{K_2\eta}{1+t}; \tag{1.38}$$

on the domain

$$D_-=\{(t,x)\mid t\ge 0,\ 0\le x\le x_2(t)\}, \tag{1.39}$$

we have

$$|\tau_0(t,x)-\tau_0|,\ |u_0(t,x)-u_0|\le K_3\varepsilon, \tag{1.40}$$

$$\Big|\frac{\partial\tau_0}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial\tau_0}{\partial t}(t,x)\Big|,\ \Big|\frac{\partial u_0}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial u_0}{\partial t}(t,x)\Big|\le\frac{K_4\eta}{1+t}. \tag{1.41}$$

Besides,

$$|x_2'(t)-V|\le K_5\varepsilon,\quad \forall\, t\ge 0, \tag{1.42}$$

$$|x_2''(t)|\le\frac{K_6\eta}{1+t},\quad \forall\, t\ge 0. \tag{1.43}$$

Here and henceforth, K_i $(i=1,2,\cdots)$ are positive constants independent of ε and η.

In this paper, we consider the corresponding inverse piston problem: supposing that we know the original state $(\tau_0^+(x),u_0^+(x))$ of the gas on the right side of this piston and the position of the forward shock $x=x_2(t)\in C^2$ with

$$x_2(0)=0, \tag{1.44}$$

$$x_2'(0)=V, \tag{1.45}$$

can we determine the piston velocity $v=\phi(t)$? As in [3], this problem can be easily solved in the local sense. In this paper we will globally give an affirmative answer to this problem. We have

Theorem 1.2 *Suppose that the position of the forward shock $x=x_2(t)\in C^2$ $(t\ge 0)$ with* (1.44)–(1.45) *is prescribed and, for suitably small $\varepsilon>0$ and $\eta>0$, we have*

$$|x_2'(t)-V|\le\varepsilon,\quad \forall\, t\ge 0, \tag{1.46}$$

$$|x_2''(t)|\le\frac{\eta}{1+t},\quad \forall\, t\ge 0, \tag{1.47}$$

where V satisfies (1.24)–(1.25). *Then, for any given $\tau_0^+(x)$ and $u_0^+(x)\in C^1$ $(x\ge 0)$ satisfying*

$$\tau_0^+(0)=\tau_+,\quad u_0^+(0)=u_+, \tag{1.48}$$

$$|\tau_0^+(x)-\tau_+|,\ |u_0^+(x)-u_+|\le\varepsilon,\quad \forall\, x\ge 0, \tag{1.49}$$

$$|\tau_0^{+'}(x)|,\ |u_0^{+'}(x)|\le\frac{\eta}{1+x},\quad \forall\, x\ge 0, \tag{1.50}$$

we can uniquely determine the piston velocity $v=\phi(t)$ $(t\ge 0)$ with

$$\phi(0)=u_p, \tag{1.51}$$

$$|\phi(t)-u_p|\le K_7\varepsilon,\quad \forall\, t\ge 0, \tag{1.52}$$

$$|\phi'(t)|\le\frac{K_8\eta}{1+t},\quad \forall\, t\ge 0, \tag{1.53}$$

where u_p is the same as in (1.18), *such that by Theorem* 1.1 *the corresponding direct piston problem* (1.1)–(1.3) *admits a unique global piecewise C^1 solution $(\tau(t,x),u(t,x))$ in which the forward shock passing through the origin is just $x=x_2(t)$.*

Remark 1.1 The inverse piston problem under consideration can be regarded as a perturbation of the simplest inverse piston problem: to determine the constant piston velocity under the condition that the constant speed of propagation of the forward typical shock V and the constant initial state (τ_+, u_+) of the gas on the right side of the piston are given.

Theorem 1.2 will be proved in Section 2. Then the corresponding discussion in Eulerian representation will be given in Section 3.

2 Proof of Theorem 1.2

By (1.15) and (1.21), it follows from (1.48)–(1.50) that

$$r_0^+(0) = r_+, \quad s_0^+(0) = s_+, \tag{2.1}$$

$$|r_0^+(x) - r_+|,\ |s_0^+(x) - s_+| \le C_1\varepsilon, \quad \forall\, x \ge 0, \tag{2.2}$$

$$|r_0^{+'}(x)|,\ |s_0^{+'}(x)| \le \frac{C_2\eta}{1+x}, \quad \forall\, x \ge 0. \tag{2.3}$$

Here and henceforth, C_i $(i = 1, 2, \cdots)$ are positive constants independent of ε and η.

From Lemma 6.1 in Chapter 6 of [2], we have

Lemma 2.1 *Suppose that* (2.1)–(2.3) *hold for suitably small* $\varepsilon > 0$ *and* $\eta > 0$*. Then the Cauchy problem*

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r,s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + \mu(r,s)\dfrac{\partial s}{\partial x} = 0, \\ t = 0:\ (r,s) = (r_0^+(x), s_0^+(x)), \quad x \ge 0 \end{cases} \tag{2.4}$$

admits a unique global C^1 *solution* $(r,s) = (\widetilde{r}_+(t,x), \widetilde{s}_+(t,x))$ *on the domain*

$$\widehat{D}_+ = \{(t,x) \mid t \ge 0,\ x \ge \xi t\}, \tag{2.5}$$

where ξ *is a constant satisfying*

$$\xi > \mu(r_+, s_+). \tag{2.6}$$

Moreover, we have

$$\widetilde{s}_+(t,x) - \widetilde{r}_+(t,x) > 0, \quad \forall\, (t,x) \in \widehat{D}_+, \tag{2.7}$$

$$|\widetilde{r}_+(t,x) - r_+|,\ |\widetilde{s}_+(t,x) - s_+| \le K_9\varepsilon, \quad \forall\, (t,x) \in \widehat{D}_+, \tag{2.8}$$

$$\Big|\frac{\partial \widetilde{r}_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial \widetilde{r}_+}{\partial t}(t,x)\Big|,\ \Big|\frac{\partial \widetilde{s}_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial \widetilde{s}_+}{\partial t}(t,x)\Big| \le \frac{K_{10}\eta}{1+t}, \quad \forall\, (t,x) \in \widehat{D}_+. \tag{2.9}$$

Proof of Theorem 1.2

Step 1 We first solve Cauchy problem (2.4) on the domain D_+ defined by (1.36).

Let

$$\xi = \frac{1}{2}(V + \mu(r_+, s_+)). \tag{2.10}$$

Noting (1.25), we have

$$V > \xi > \mu(r_+, s_+). \tag{2.11}$$

Hence, for suitably small $\varepsilon > 0$, by (1.44)–(1.46) we have

$$D_+ \subseteq \widehat{D}_+.$$

Hence, by Lemma 2.1, Cauchy problem (2.4) admits a unique global C^1 solution $(r, s) = (r_+(t, x), s_+(t, x))$ on the domain D_+ and we have

$$s_+(t, x) - r_+(t, x) > 0, \quad \forall\, (t, x) \in D_+, \tag{2.12}$$

$$|r_+(t, x) - r_+|,\ |s_+(t, x) - s_+| \le C_3\varepsilon, \quad \forall\, (t, x) \in D_+, \tag{2.13}$$

$$\Big|\frac{\partial r_+}{\partial x}(t, x)\Big|,\ \Big|\frac{\partial r_+}{\partial t}(t, x)\Big|,\ \Big|\frac{\partial s_+}{\partial x}(t, x)\Big|,\ \Big|\frac{\partial s_+}{\partial t}(t, x)\Big| \le \frac{C_4\eta}{1 + t}, \quad \forall\, (t, x) \in D_+. \tag{2.14}$$

Then, we obtain the value of (r, s) on the right side of $x = x_2(t)$

$$(r, s) = (\widetilde{r}_+(t), \widetilde{s}_+(t)) = (r_+(t, x_2(t)), s_+(t, x_2(t))), \quad \forall\, t \ge 0, \tag{2.15}$$

and we have

$$\widetilde{r}_+(0) = r_+, \quad \widetilde{s}_+(0) = s_+, \tag{2.16}$$

$$\widetilde{s}_+(t) - \widetilde{r}_+(t) > 0, \quad \forall\, t \ge 0, \tag{2.17}$$

$$|\widetilde{r}_+(t) - r_+|,\ |\widetilde{s}_+(t) - s_+| \le C_3\varepsilon, \quad \forall\, t \ge 0. \tag{2.18}$$

Besides, noting (1.46), we also have

$$\Big|\frac{d\widetilde{r}_+(t)}{dt}\Big|,\ \Big|\frac{d\widetilde{s}_+(t)}{dt}\Big| \le \frac{C_5\eta}{1 + t}, \quad \forall\, t \ge 0. \tag{2.19}$$

Step 2 By the Rankine-Hugoniot conditions, we now find the value of (r, s) on the left side of $x = x_2(t)$.

On the forward shock $x = x_2(t)$, the Rankine-Hugoniot conditions are

$$\begin{cases} [\tau(s - r)]x_2'(t) + [r + s] = 0, \\ [r + s]x_2'(t) - [p(\tau(s - r))] = 0, \end{cases} \tag{2.20}$$

where $[\tau] = \tau(t, x_2(t) + 0) - \tau(t, x_2(t) - 0)$, etc. Denote the value of (r, s) on the left side of $x = x_2(t)$ as $(r, s) = (\widetilde{r}_-(t), \widetilde{s}_-(t))$ and $x_2'(t) = d$. (2.20) can be rewritten as

$$\begin{cases} (\tau(\widetilde{s}_- - \widetilde{r}_-) - \tau(\widetilde{s}_+ - \widetilde{r}_+))d + (\widetilde{r}_- + \widetilde{s}_- - \widetilde{r}_+ - \widetilde{s}_+) = 0, \\ (\widetilde{r}_- + \widetilde{s}_- - \widetilde{r}_+ - \widetilde{s}_+)d - (p(\tau(\widetilde{s}_- - \widetilde{r}_-)) - p(\tau(\widetilde{s}_+ - \widetilde{r}_+))) = 0. \end{cases} \tag{2.21}$$

Similarly to [4, Lemma 3.1] (also see [3]), in a neighbourhood of (r_+, s_+, r_0, s_0, V), (2.21) can be rewritten as

$$\begin{cases} \widetilde{r}_- = g(\widetilde{r}_+, \widetilde{s}_+, d), \\ \widetilde{s}_- = h(\widetilde{r}_+, \widetilde{s}_+, d), \end{cases} \tag{2.22}$$

where $g(\cdot), h(\cdot) \in C^2$ and

$$\begin{cases} r_0 = g(r_+, s_+, V), \\ s_0 = h(r_+, s_+, V). \end{cases} \tag{2.23}$$

Hence, the value of (r, s) on the left side of $x = x_2(t)$ can be uniquely determined as

$$\begin{cases} r = \widetilde{r}_-(t) = g(\widetilde{r}_+(t), \widetilde{s}_+(t), x_2'(t)), \\ s = \widetilde{s}_-(t) = h(\widetilde{r}_+(t), \widetilde{s}_+(t), x_2'(t)). \end{cases} \tag{2.24}$$

Moreover, noting (1.44)–(1.47), (2.16), (2.18)–(2.19) and (2.23), we have

$$\widetilde{r}_-(0) = r_0, \quad \widetilde{s}_-(0) = s_0, \tag{2.25}$$

$$|\widetilde{r}_-(t) - r_0|,\ |\widetilde{s}_-(t) - s_0| \le C_6\varepsilon, \quad \forall\, t \ge 0, \tag{2.26}$$

$$\Big|\frac{d\widetilde{r}_-(t)}{dt}\Big|,\ \Big|\frac{d\widetilde{s}_-(t)}{dt}\Big| \le \frac{C_7\eta}{1+t}, \quad \forall\, t \ge 0. \tag{2.27}$$

Besides, noting (1.25) and (1.45)–(1.46), we have

$$\lambda(\widetilde{r}_-(t), \widetilde{s}_-(t)) < x_2'(t) < \mu(\widetilde{r}_-(t), \widetilde{s}_-(t)). \tag{2.28}$$

Step 3 We finally solve the generalized Cauchy problem

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r, s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + \mu(r, s)\dfrac{\partial s}{\partial x} = 0, \\ x = x_2(t): \ (r, s) = (\widetilde{r}_-(t), \widetilde{s}_-(t)), \quad t \ge 0 \end{cases} \tag{2.29}$$

on the domain D_- defined by (1.39).

Due to (2.28), the generalized Cauchy problem (2.29) always admits a unique local C^1 solution $(r, s) = (r_-(t, x), s_-(t, x))$ (see [6]). For the time being we assume that on any existence domain of $(r, s) = (r_-(t, x), s_-(t, x))$, we have

$$|r_-(t, x) - r_0|,\ |s_-(t, x) - s_0| \le \delta, \tag{2.30}$$

where $\delta > 0$ is suitably small. At the end of the proof, we will explain that this hypothesis is reasonable.

Let

$$\begin{cases} \overline{t} = x_2(t) - x, \\ \overline{x} = t. \end{cases} \tag{2.31}$$

Then, by (1.25), (1.44)–(1.46) and (2.30), the generalized Cauchy problem (2.29) on the domain D_- reduces to the following Cauchy problem

$$\begin{cases} \dfrac{\partial \overline{r}}{\partial \overline{t}} + \overline{\lambda}(\overline{x}, \overline{r}, \overline{s}) \dfrac{\partial \overline{r}}{\partial \overline{x}} = 0, \\ \dfrac{\partial \overline{s}}{\partial \overline{t}} + \overline{\mu}(\overline{x}, \overline{r}, \overline{s}) \dfrac{\partial \overline{s}}{\partial \overline{x}} = 0, \\ \overline{t} = 0: \ (\overline{r}, \overline{s}) = (\widetilde{r}_-(\overline{x}), \widetilde{s}_-(\overline{x})), \quad \forall\, \overline{x} \geq 0 \end{cases} \tag{2.32}$$

on the domain $\overline{D}_- = \{(\overline{t}, \overline{x}) \mid \overline{t} \geq 0,\ \overline{x} \geq \theta(\overline{t})\}$, where

$$(\overline{r}(\overline{t}, \overline{x}), \overline{s}(\overline{t}, \overline{x})) = (r_-(\overline{x}, x_2(\overline{x}) - \overline{t}), s_-(\overline{x}, x_2(\overline{x}) - \overline{t})), \tag{2.33}$$

$$\overline{\lambda}(\overline{x}, \overline{r}, \overline{s}) = \frac{1}{x_2'(\overline{x}) - \lambda(\overline{r}, \overline{s})}, \tag{2.34}$$

$$\overline{\mu}(\overline{x}, \overline{r}, \overline{s}) = \frac{1}{x_2'(\overline{x}) - \mu(\overline{r}, \overline{s})} \tag{2.35}$$

and $\overline{x} = \theta(\overline{t}) \in C^2$ with $\theta(0) = 0$ is determined by

$$x_2(\overline{x}) = \overline{t}. \tag{2.36}$$

Besides, by (1.25), (1.45)–(1.46) and (2.30), we have

$$\frac{1}{x_2'(\overline{x})} > \overline{\lambda}(\overline{x}, \overline{r}, \overline{s}) > \overline{\mu}(\overline{x}, \overline{r}, \overline{s}), \tag{2.37}$$

and it follows from (2.26)–(2.27) that

$$|\widetilde{r}_-(\overline{x}) - r_0|,\ |\widetilde{s}_-(\overline{x}) - s_0| \leq C_8 \varepsilon, \quad \forall\, \overline{x} \geq 0, \tag{2.38}$$

$$|\widetilde{r}_-'(\overline{x})|,\ |\widetilde{s}_-'(\overline{x})| \leq \frac{C_9 \eta}{1 + \overline{x}}, \quad \forall\, \overline{x} \geq 0. \tag{2.39}$$

Obviously, problem (2.32) admits a unique local C^1 solution $(\overline{r}, \overline{s}) = (\overline{r}(\overline{t}, \overline{x}), \overline{s}(\overline{t}, \overline{x})) = (r_-(\overline{x}, x_2(\overline{x}) - \overline{t}), s_-(\overline{x}, x_2(\overline{x}) - \overline{t}))$ on the domain $\overline{D}_-(\delta_0) = \{(\overline{t}, \overline{x}) \mid 0 \leq \overline{t} \leq \delta_0,\ \overline{x} \geq \theta(\overline{t})\}$, where $\delta_0 > 0$ is a small number (see [6]). Since the system in (2.32) depends explicitly on x, in order to get the global existence of C^1 solution on $\overline{D}_-$, we need a uniform a priori estimate on the C^1 norm of C^1 solution $(\overline{r}(\overline{t}, \overline{x}), \overline{s}(\overline{t}, \overline{x}))$ on any existence domain $\overline{D}_-(T)$.

Noting (2.26), we have

$$|\overline{r}(\overline{t}, \overline{x}) - r_0|,\ |\overline{s}(\overline{t}, \overline{x}) - s_0| \leq C_6 \varepsilon, \quad \forall\, (\overline{t}, \overline{x}) \in \overline{D}_-(T). \tag{2.40}$$

In what follows, we want to get a uniform a priori estimate on the C^0 norm of $\frac{\partial \overline{r}}{\partial \overline{x}}$, $\frac{\partial \overline{r}}{\partial \overline{t}}$, $\frac{\partial \overline{s}}{\partial \overline{x}}$ and $\frac{\partial \overline{s}}{\partial \overline{t}}$ on $\overline{D}_-(T)$. For this purpose, instead of the usual Lax transformation, we introduce

$$w = e^{q(\overline{r}, \overline{s})} \frac{\partial \overline{r}}{\partial \overline{t}}, \tag{2.41}$$

where $q(\overline{r}, \overline{s}) \in C^1$ satisfies

$$\frac{\partial q}{\partial \overline{s}} = \frac{1}{\lambda(\overline{r}, \overline{s}) - \mu(\overline{r}, \overline{s})} \frac{\partial \lambda}{\partial \overline{s}}. \tag{2.42}$$

By (2.32)–(2.35), it is easy to get

$$\begin{cases} \dfrac{\partial w}{\partial \overline{t}} + \overline{\lambda}(\overline{x}, \overline{r}, \overline{s}) \dfrac{\partial w}{\partial \overline{x}} = \dfrac{\partial \lambda(\overline{r}, \overline{s})}{\partial \overline{r}} \overline{\lambda}(\overline{x}, \overline{r}, \overline{s}) e^{-q(\overline{r}, \overline{s})} w^2, \\ \overline{t} = 0: \ \ w = -e^{q(\widetilde{r}_-(\overline{x}), \widetilde{s}_-(\overline{x}))} \overline{\lambda}(\overline{x}, \widetilde{r}_-(\overline{x}), \widetilde{s}_-(\overline{x})) \widetilde{r}'_-(\overline{x}), \quad \forall \, \overline{x} \geq 0. \end{cases} \tag{2.43}$$

By (2.37), each characteristic passing through any given point $(\overline{t}, \overline{x}) = (0, \beta)$ $(\beta \geq 0)$ intersects the boundary $\overline{x} = \theta(\overline{t})$ $(\overline{t} \geq 0)$ of $\overline{D}_-$ in a finite time. Let $\overline{x} = \overline{x}_1(\overline{t}, \beta)$ be the forward characteristic passing through a point $(0, \beta)$ and $(\overline{T}, \overline{x}_1(\overline{T}, \beta))$ be the intersection point of $\overline{x} = \overline{x}_1(\overline{t}, \beta)$ with $\overline{x} = \theta(\overline{t})$.

Noting (1.44), (1.46), (2.30) and (2.33), for suitably small $\varepsilon > 0$ and $\delta > 0$, we have

$$\frac{\overline{T}}{V - \frac{1}{4}\lambda(r_0, s_0)} \leq \overline{x}_1(\overline{T}, \beta) = \beta + \int_0^{\overline{T}} \overline{\lambda}(\overline{x}, \overline{r}, \overline{s})(\tau, \overline{x}_1(\tau, \beta)) d\tau \leq \beta + \frac{\overline{T}}{V - \frac{1}{2}\lambda(r_0, s_0)}. \tag{2.44}$$

Hence

$$\overline{T} \leq M_0 \beta, \tag{2.45}$$

where

$$M_0 = \frac{(4V - \lambda(r_0, s_0))(2V - \lambda(r_0, s_0))}{-2\lambda(r_0, s_0)} > 0. \tag{2.46}$$

Noting (2.32), on $\overline{x} = \overline{x}_1(\overline{t}, \beta)$ we have

$$\overline{r}(\overline{t}, \overline{x}) = \overline{r}(\overline{t}, \overline{x}_1(\overline{t}, \beta)) = \widetilde{r}_-(\beta), \tag{2.47}$$

$$\overline{s}(\overline{t}, \overline{x}) = \overline{s}(\overline{t}, \overline{x}_1(\overline{t}, \beta)) = \widetilde{s}_-(\alpha(\overline{t}, \beta)), \tag{2.48}$$

where $\alpha(\overline{t}, \beta)$ is the $\overline{x}$-coordinate of the intersection point of the backward characteristic passing through $(\overline{t}, \overline{x}_1(\overline{t}, \beta))$ with the $\overline{x}$ axis. Then, it follows from (2.43) that on $\overline{x} = \overline{x}_1(\overline{t}, \beta)$ we have

$$w(\overline{t}, \overline{x}) = w(\overline{t}, \overline{x}_1(\overline{t}, \beta)) = \frac{-e^{q(\widetilde{r}_-(\beta), \widetilde{s}_-(\beta))} \overline{\lambda}(\beta, \widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) \widetilde{r}'_-(\beta)}{1 + B}, \tag{2.49}$$

where

$$\begin{aligned} B = \int_0^{\overline{t}} & \frac{\partial \lambda}{\partial \overline{r}} (\widetilde{r}_-(\beta), \overline{s}(\tau, \overline{x}_1(\tau, \beta))) \overline{\lambda}(\beta, \widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) \overline{\lambda}(\overline{x}_1(\tau, \beta), \widetilde{r}_-(\beta), \overline{s}(\tau, \overline{x}_1(\tau, \beta))) \\ & \cdot \widetilde{r}'_-(\beta) e^{q(\widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) - q(\widetilde{r}_-(\beta), \overline{s}(\tau, \overline{x}_1(\tau, \beta)))} d\tau. \end{aligned}$$

Hence, by (2.41), we get

$$\frac{\partial \overline{r}}{\partial \overline{t}}(\overline{t}, \overline{x}_1(\overline{t}, \beta)) = \frac{-e^{q(\widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) - q(\widetilde{r}_-(\beta), \overline{s}(\overline{t}, \overline{x}_1(\overline{t}, \beta)))} \overline{\lambda}(\beta, \widetilde{r}_-(\beta), \widetilde{s}_-(\beta)) \widetilde{r}'_-(\beta)}{1 + B}. \tag{2.50}$$

By (1.46), (2.38) and (2.47)–(2.48) and noting (1.25), on $\overline{x} = \overline{x}_1(\overline{t}, \beta)$ we have

$$\frac{1}{2}(V - \lambda(r_0, s_0)) < x'_2(\overline{x}) - \lambda(\overline{r}, \overline{s}) < 2(V - \lambda(r_0, s_0)). \tag{2.51}$$

Then, noting (2.38) and (2.47)–(2.48), we get

$$e^{2|q|}|\overline{\lambda}| \le M_1, \quad \Big|\frac{\partial \lambda}{\partial \overline{r}}\Big|\overline{\lambda}^2 e^{2|q|} \le M_2, \tag{2.52}$$

where M_1 and M_2 are two positive constants independent of ε and η.

We choose $\eta > 0$ so small that

$$C_9 M_0 M_2 \eta < \frac{1}{2}. \tag{2.53}$$

Then, it follows from (2.50) that

$$\Big|\frac{\partial \overline{r}}{\partial \overline{t}}(\overline{t}, \overline{x}_1(\overline{t}, \beta))\Big| \le M_1 \frac{C_9 \eta}{1+\beta}\Big(1 - M_2 \frac{C_9\eta}{1+\beta}\overline{T}\Big)^{-1}. \tag{2.54}$$

Thus, noting (2.45) and (2.53), we get

$$\Big|\frac{\partial \overline{r}}{\partial \overline{t}}(\overline{t}, \overline{x}_1(\overline{t}, \beta))\Big| \le C_9 M_1 \frac{\eta}{1+\beta}(1 - C_9 M_0 M_2 \eta)^{-1} \le 2C_9 M_1 \frac{\eta}{1+\beta} \le \frac{C_{10}\eta}{1+\overline{t}}, \quad 0 \le \overline{t} \le T. \tag{2.55}$$

Hence, noting (2.34) and (2.51), we get

$$\Big|\frac{\partial \overline{r}}{\partial \overline{x}}(\overline{t}, \overline{x}_1(\overline{t}, \beta))\Big| \le \frac{C_{11}\eta}{1+\overline{t}}, \quad 0 \le \overline{t} \le T. \tag{2.56}$$

Finally, we obtain

$$\Big|\frac{\partial \overline{r}}{\partial \overline{t}}(\overline{t}, \overline{x})\Big|, \ \Big|\frac{\partial \overline{r}}{\partial \overline{x}}(\overline{t}, \overline{x})\Big| \le \frac{C_{12}\eta}{1+\overline{t}}, \quad \forall\, (\overline{t}, \overline{x}) \in \overline{D}_-(T). \tag{2.57}$$

Similarly, we have

$$\Big|\frac{\partial \overline{s}}{\partial \overline{t}}(\overline{t}, \overline{x})\Big|, \ \Big|\frac{\partial \overline{s}}{\partial \overline{x}}(\overline{t}, \overline{x})\Big| \le \frac{C_{13}\eta}{1+\overline{t}}, \quad \forall\, (\overline{t}, \overline{x}) \in \overline{D}_-(T). \tag{2.58}$$

Thus, we get a unique global C^1 solution $(\overline{r}, \overline{s}) = (\overline{r}(\overline{t}, \overline{x}), \overline{s}(\overline{t}, \overline{x}))$ to (2.32) on $\overline{D}_-$. Noting (2.31), for the generalized Cauchy problem (2.29), we obtain the unique global C^1 solution

$$(r, s) = (r_-(t, x), s_-(t, x)) = (\overline{r}(x_2(t) - x, t), \overline{s}(x_2(t) - x, t)) \tag{2.59}$$

on the domain D_-. Noting (2.40) and (2.59) we immediately obtain that

$$|r_-(t, x) - r_0|, \ |s_-(t, x) - s_0| \le C_{14}\varepsilon, \quad \forall\, (t, x) \in D_-, \tag{2.60}$$

which also implies that hypothesis (2.30) is reasonable. Besides, noting (1.46), it follows from (2.57)–(2.58) that

$$\Big|\frac{\partial r_-}{\partial t}(t, x)\Big|, \ \Big|\frac{\partial r_-}{\partial x}(t, x)\Big|, \ \Big|\frac{\partial s_-}{\partial t}(t, x)\Big|, \ \Big|\frac{\partial s_-}{\partial x}(t, x)\Big| \le \frac{C_{15}\eta}{1+t}, \quad \forall\, (t, x) \in D_-. \tag{2.61}$$

Hence, we get the piston velocity

$$\phi(t) = r_-(t, 0) + s_-(t, 0), \quad \forall t \ge 0. \tag{2.62}$$

Moreover, noting (2.25) and (1.26), we see that (1.51)–(1.53) hold.

Theorem 1.2 is then proved.

3 Related Problems in Eulerian Representation

In Eulerian representation, the system of one-dimensional isentropic flow is written as

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial (\rho u)}{\partial x} = 0, \\ \dfrac{\partial (\rho u)}{\partial t} + \dfrac{\partial (\rho u^2 + p(\rho))}{\partial x} = 0, \end{cases} \tag{3.1}$$

where ρ is the density, u is the velocity and $p = p(\rho)$ is the pressure. For polytropic gases

$$p = p(\rho) = A\rho^\gamma, \quad \forall\, \rho > 0, \tag{3.2}$$

where $\gamma > 1$ is the adiabatic exponent and A is a positive constant. In this situation, the corresponding piston problem asks us to solve the following mixed initial-boundary value problem for system (3.1) with the initial data

$$t = 0: \ \rho = \rho_0^+(x)(> 0), \ u = u_0^+(x), \quad \forall\, x \ge 0 \tag{3.3}$$

and the boundary condition

$$x = f(t): \ u = \varphi(t), \quad \forall\, t \ge 0 \tag{3.4}$$

with

$$f(t) = \int_0^t \varphi(\xi) d\xi. \tag{3.5}$$

Suppose that

$$\varphi(0) > u_0^+(0). \tag{3.6}$$

The motion of the piston produces a forward shock $x = x_f(t)$ passing through the origin at least for a short time T_1 (see [6]), such that the corresponding piecewise C^1 solution on the domain

$$\Omega(T_1) = \{(t, x) \mid 0 \le t \le T_1, x \ge f(t)\} \tag{3.7}$$

is written as

$$(\rho, u) = \begin{cases} (\rho_0(t, x), u_0(t, x)), & f(t) \le x \le x_f(t), \\ (\rho_+(t, x), u_+(t, x)), & x \ge x_f(t), \end{cases} \tag{3.8}$$

where $(\rho_0(t, x), u_0(t, x))$, $(\rho_+(t, x), u_+(t, x)) \in C^1$ satisfy system (3.1) in the classical sense on their domains respectively and verify the Rankine-Hugoniot conditions

$$\begin{cases} [\rho] x_f'(t) - [\rho u] = 0, \\ [\rho u] x_f'(t) - [\rho u^2 + p(\rho)] = 0 \end{cases} \tag{3.9}$$

and the entropy condition

$$\begin{cases} \lambda_1(\rho_0(t, x_f(t))) < x_f'(t) < \lambda_2(\rho_0(t, x_f(t))), \\ x_f'(t) > \lambda_2(\rho_+(t, x_f(t))) \end{cases} \tag{3.10}$$

on $x = x_f(t)$, in which $[\rho] = \rho_+(t, x_f(t)) - \rho_0(t, x_f(t))$, etc. and

$$-\lambda_1(\rho) = \lambda_2(\rho) = \sqrt{p'(\rho)}. \tag{3.11}$$

In the special case that the piston moves with a constant speed u_p and the initial state is a constant state (ρ_+, u_+) $(\rho_+ > 0)$ with $u_p > u_+$, the solution to the previous problem is the typical forward shock (see [1])

$$(\rho, u) = \begin{cases} (\rho_0, u_p), & u_p t \le x \le Ut, \\ (\rho_+, u_+), & x \ge Ut, \end{cases} \tag{3.12}$$

where U, the speed of propagation of the typical forward shock, and ρ_0 are determined by Rankine-Hugoniot conditions

$$\begin{cases} (\rho_0 u_p - \rho_+ u_+)^2 = (\rho_0 - \rho_+)(\rho_0 u_p^2 + p(\rho_0) - \rho_+ u_+^2 - p(\rho_+)), \\ U = \dfrac{\rho_0 u_p - \rho_+ u_+}{\rho_0 - \rho_+}. \end{cases} \tag{3.13}$$

As a global perturbation of the simplest piston problem mentioned above, for the piston problem (3.1) and (3.3)–(3.4) we have the following

Theorem 3.1 *Suppose that* $\rho_0^+(x)$, $u_0^+(x) \in C^1$ *and* $f(t) \in C^2$ *and*

$$\rho_0^+(0) = \rho_+, \quad u_0^+(0) = u_+, \quad \varphi(0) = u_p. \tag{3.14}$$

Suppose furthermore that

$$|\rho_0^+(x) - \rho_+|,\ |u_0^+(x) - u_+| \le \varepsilon, \quad \forall\, x \ge 0, \tag{3.15}$$

$$|\varphi(t) - u_p| \le \varepsilon, \quad \forall\, t \ge 0, \tag{3.16}$$

$$|\rho_0^{+'}(x)|,\ |u_0^{+'}(x)| \le \frac{\eta}{1+x}, \quad \forall\, x \ge 0, \tag{3.17}$$

$$|\varphi'(t)| \le \frac{\eta}{1+t}, \quad \forall\, t \ge 0, \tag{3.18}$$

where $\varepsilon > 0$ *and* $\eta > 0$ *are suitably small. Then, the piston problem* (3.1) *and* (3.3)–(3.4) *admits a unique global piecewise* C^1 *solution*

$$(\rho(t,x), u(t,x)) = \begin{cases} (\rho_0(t,x), u_0(t,x)), & f(t) \le x \le x_f(t), \\ (\rho_+(t,x), u_+(t,x)), & x \ge x_f(t) \end{cases} \tag{3.19}$$

on the domain

$$\Omega = \{(t,x) \mid t \ge 0,\ x \ge f(t)\}. \tag{3.20}$$

This solution, containing only one forward shock $x = x_f(t)$ *passing through the origin with* $x_f'(0) = U$*, satisfies the following estimates: on the domain*

$$\Omega_+ = \{(t,x) \mid t \ge 0,\ x \ge x_f(t)\}, \tag{3.21}$$

we have

$$|\rho_+(t,x)-\rho_+|,\ |u_+(t,x)-u_+|\le K_{11}\varepsilon, \tag{3.22}$$

$$\Big|\frac{\partial\rho_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial\rho_+}{\partial t}(t,x)\Big|,\ \Big|\frac{\partial u_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial u_+}{\partial t}(t,x)\Big|\le\frac{K_{12}\eta}{1+t}; \tag{3.23}$$

on the domain

$$\Omega_-=\{(t,x)\mid t\ge 0,\ f(t)\le x\le x_f(t)\}, \tag{3.24}$$

we have

$$|\rho_0(t,x)-\rho_0|,\ |u_0(t,x)-u_0|\le K_{13}\varepsilon, \tag{3.25}$$

$$\Big|\frac{\partial\rho_0}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial\rho_0}{\partial t}(t,x)\Big|,\ \Big|\frac{\partial u_0}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial u_0}{\partial t}(t,x)\Big|\le\frac{K_{14}\eta}{1+t}. \tag{3.26}$$

Besides,

$$|x_f'(t)-U|\le K_{15}\varepsilon,\quad \forall\, t\ge 0, \tag{3.27}$$

$$|x_f''(t)|\le\frac{K_{16}\eta}{1+t},\quad \forall\, t\ge 0. \tag{3.28}$$

Proof Take the Lagrange coordinates $(\widetilde{t},m)$:

$$\begin{cases} m=\displaystyle\int_{(0,0)}^{(t,x)}\rho dx-\rho u dt,\\ \widetilde{t}=t \end{cases} \tag{3.29}$$

as new variables. Problem (3.1) and (3.3)–(3.4) reduces to (1.1)–(1.3) in which (t,x) is replaced by $(\widetilde{t},m)$ and

$$\tau_0^+(m)=\frac{1}{\rho_0^+(x(m))},\quad u_0^+(m)=u_0^+(x(m)), \tag{3.30}$$

$$\phi(\widetilde{t})=\varphi(\widetilde{t}), \tag{3.31}$$

where $x=x(m)$ is determined by

$$m=\int_0^x\rho_0^+(\xi)\ d\xi.$$

By (3.14)–(3.18), it is easy to see that

$$\tau_0^+(0)=\frac{1}{\rho_+}=\tau_+,\quad u_0^+(0)=u_+,\quad \phi(0)=u_p, \tag{3.32}$$

$$|\tau_0^+(m)-\tau_+|,\ |u_0^+(m)-u_+|\le C_{16}\varepsilon,\quad \forall\, m\ge 0, \tag{3.33}$$

$$|\phi(\widetilde{t})-\phi(0)|\le\varepsilon,\quad \forall\,\widetilde{t}\ge 0, \tag{3.34}$$

$$|\tau_0^{+'}(m)|,\ |u_0^{+'}(m)|\le\frac{C_{17}\eta}{1+m},\quad \forall\, m\ge 0, \tag{3.35}$$

$$|\phi'(\widetilde{t})|\le\frac{\eta}{1+\widetilde{t}},\quad \forall\,\widetilde{t}\ge 0. \tag{3.36}$$

By Theorem 1.1 we obtain that problem (1.1)–(1.3) corresponding to problem (3.1) and (3.3)–(3.4) admits a unique global piecewise C^1 solution

$$(\tau(\widetilde{t},m),u(\widetilde{t},m))=\begin{cases}(\tau_0(\widetilde{t},m),u_0(\widetilde{t},m)), & 0\le m\le m_2(\widetilde{t}),\\ (\tau_+(\widetilde{t},m),u_+(\widetilde{t},m)), & m\ge m_2(\widetilde{t})\end{cases} \tag{3.37}$$

on the domain

$$\{(\widetilde{t},m)\mid \widetilde{t}\ge 0,\ m\ge 0\}. \tag{3.38}$$

This solution, containing only one forward shock $m=m_2(\widetilde{t})$ passing through the origin with $m_2'(0)=V$, where

$$V=\rho_+(U-u_+), \tag{3.39}$$

satisfies the following estimates: on the domain

$$\{(\widetilde{t},m)\mid \widetilde{t}\ge 0,\ m\ge m_2(\widetilde{t})\}, \tag{3.40}$$

we have

$$|\tau_+(\widetilde{t},m)-\tau_+|,\ |u_+(\widetilde{t},m)-u_+|\le C_{18}\varepsilon, \tag{3.41}$$

$$\Big|\frac{\partial\tau_+}{\partial m}(\widetilde{t},m)\Big|,\ \Big|\frac{\partial\tau_+}{\partial\widetilde{t}}(\widetilde{t},m)\Big|,\ \Big|\frac{\partial u_+}{\partial m}(\widetilde{t},m)\Big|,\ \Big|\frac{\partial u_+}{\partial\widetilde{t}}(\widetilde{t},m)\Big|\le\frac{C_{19}\eta}{1+\widetilde{t}}; \tag{3.42}$$

on the domain

$$\{(\widetilde{t},m)\mid \widetilde{t}\ge 0,\ 0\le m\le m_2(\widetilde{t})\}, \tag{3.43}$$

we have

$$|\tau_0(\widetilde{t},m)-\tau_0|,\ |u_0(\widetilde{t},m)-u_0|\le C_{20}\varepsilon, \tag{3.44}$$

$$\Big|\frac{\partial\tau_0}{\partial m}(\widetilde{t},m)\Big|,\ \Big|\frac{\partial\tau_0}{\partial\widetilde{t}}(\widetilde{t},m)\Big|,\ \Big|\frac{\partial u_0}{\partial m}(\widetilde{t},m)\Big|,\ \Big|\frac{\partial u_0}{\partial\widetilde{t}}(\widetilde{t},m)\Big|\le\frac{C_{21}\eta}{1+\widetilde{t}}. \tag{3.45}$$

Besides,

$$|m_2'(\widetilde{t})-V|\le C_{22}\varepsilon,\quad \forall\,\widetilde{t}\ge 0, \tag{3.46}$$

$$|m_2''(\widetilde{t})|\le\frac{C_{23}\eta}{1+\widetilde{t}},\quad \forall\,\widetilde{t}\ge 0. \tag{3.47}$$

Using the inverse transformation of (3.29)

$$\begin{cases}x=\displaystyle\int_{(0,0)}^{(\widetilde{t},m)}\tau dm+ud\widetilde{t},\\ t=\widetilde{t},\end{cases} \tag{3.48}$$

we get

$$m=m(t,x). \tag{3.49}$$

Then, by means of $(\rho(t,x),u(t,x))=\big(\frac{1}{\tau(t,m(t,x))},u(t,m(t,x))\big)$, it is easy to see that the original piston problem (3.1) and (3.3)–(3.4) admits a unique global piecewise C^1 solution

$$\begin{aligned}&(\rho(t,x),u(t,x))\\ =&\begin{cases}(\rho_0(t,x),u_0(t,x))=\Big(\dfrac{1}{\tau_0(t,m(t,x))},u_0(t,m(t,x))\Big), & f(t)\le x\le x_f(t),\\ (\rho_+(t,x),u_+(t,x))=\Big(\dfrac{1}{\tau_+(t,m(t,x))},u_+(t,m(t,x))\Big), & x\ge x_f(t)\end{cases}\end{aligned}\tag{3.50}$$

on the domain (3.20), where

$$x_f(t)=\int_0^t(\tau(\sigma,m_2(\sigma))m_2'(\sigma)+u(\sigma,m_2(\sigma)))d\sigma;\tag{3.51}$$

moreover, (3.21)–(3.28) hold.

This proves Theorem 3.1.

For the global inverse piston problem, we have

Theorem 3.2 *Suppose that the position of the forward shock $x=x_f(t)\in C^2$ $(t\ge 0)$ with*

$$x_f(0)=0,\tag{3.52}$$

$$x_f'(0)=U\tag{3.53}$$

is prescribed and, for suitably small $\varepsilon>0$ and $\eta>0$, we have

$$|x_f'(t)-U|\le\varepsilon,\quad\forall\, t\ge 0,\tag{3.54}$$

$$|x_f''(t)|\le\frac{\eta}{1+t},\quad\forall\, t\ge 0.\tag{3.55}$$

Then, for any given $\rho_0^+(x)$ and $u_0^+(x)\in C^1$ $(x\ge 0)$ satisfying

$$\rho_0^+(0)=\rho_+,\quad u_0^+(0)=u_+,\tag{3.56}$$

$$|\rho_0^+(x)-\rho_+|,\ |u_0^+(x)-u_+|\le\varepsilon,\quad\forall\, x\ge 0,\tag{3.57}$$

$$|{\rho_0^+}'(x)|,\ |{u_0^+}'(x)|\le\frac{\eta}{1+x},\quad\forall\, x\ge 0,\tag{3.58}$$

we can uniquely determine the piston velocity $v=\varphi(t)$ $(t\ge 0)$ with

$$\varphi(0)=u_p,\tag{3.59}$$

$$|\varphi(t)-u_p|\le K_{17}\varepsilon,\quad\forall\, t\ge 0,\tag{3.60}$$

$$|\varphi'(t)|\le\frac{K_{18}\eta}{1+t},\quad\forall\, t\ge 0,\tag{3.61}$$

where u_p is the same as in (3.12)*, such that by Theorem* 3.1 *the corresponding direct piston problem* (3.1) *and* (3.3)–(3.4) *admits a unique global piecewise C^1 solution $(\rho(t,x),u(t,x))$ in which the forward shock passing through the origin is just $x=x_f(t)$.*

Proof First, we solve Cauchy problem (3.1) and (3.3) on the domain Ω_+ defined by (3.21). By (3.56)–(3.58), just as we did in Lagrangian representation (cf. Lemma 2.1), Cauchy problem

(3.1) and (3.3) admits a unique global C^1 solution $(\rho, u) = (\rho_+(t,x), u_+(t,x))$ on the domain Ω_+ and we have

$$|\rho_+(t,x) - \rho_+|,\ |u_+(t,x) - u_+| \le C_{24}\varepsilon, \quad \forall\,(t,x) \in \Omega_+, \tag{3.62}$$

$$\Big|\frac{\partial \rho_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial \rho_+}{\partial t}(t,x)\Big|,\ \Big|\frac{\partial u_+}{\partial x}(t,x)\Big|,\ \Big|\frac{\partial u_+}{\partial t}(t,x)\Big| \le \frac{C_{25}\eta}{1+t}, \quad \forall\,(t,x) \in \Omega_+. \tag{3.63}$$

By the Lagrange transformation (3.29), the forward shock in Eulerian representation $x = x_f(t)$ reduces to the forward shock $m = m_2(\widetilde{t})$ in Lagrangian representation with

$$m_2(\widetilde{t}) = \int_0^{\widetilde{t}} \rho_+(\sigma, x_f(\sigma))(x_f'(\sigma) - u_+(\sigma, x_f(\sigma)))d\sigma. \tag{3.64}$$

Noting (3.52)–(3.55) and (3.62)–(3.63), we have

$$m_2(0) = 0, \tag{3.65}$$

$$m_2'(0) = V, \tag{3.66}$$

$$|m_2'(\widetilde{t}) - V| \le C_{26}\varepsilon, \quad \forall\,\widetilde{t} \ge 0, \tag{3.67}$$

$$|m_2''(\widetilde{t})| \le \frac{C_{27}\eta}{1+\widetilde{t}}, \quad \forall\,\widetilde{t} \ge 0, \tag{3.68}$$

where V is given by (3.39). Besides, $\tau_0^+(m)$ and $u_0^+(m)$ defined by (3.30) satisfy

$$\tau_0^+(0) = \tau_+, \quad u_0^+(0) = u_+, \tag{3.69}$$

$$|\tau_0^+(m) - \tau_+|,\ |u_0^+(m) - u_+| \le C_{28}\varepsilon, \quad \forall\, m \ge 0, \tag{3.70}$$

$$|\tau_0^{+'}(m)|,\ |u_0^{+'}(m)| \le \frac{C_{29}\eta}{1+m}, \quad \forall\, m \ge 0. \tag{3.71}$$

Thus, the inverse piston problem in Eulerian representation reduces to the corresponding one in Lagrangian representation. By Theorem 1.2, in Lagrangian representation we can uniquely determine the piston velocity $v = \phi(\widetilde{t})$ $(\widetilde{t} \ge 0)$ with (1.51)–(1.53) such that the corresponding direct piston problem (1.1)–(1.3), where (t,x) is replaced by $(\widetilde{t}, m)$, admits a unique global piecewise C^1 solution $(\tau(\widetilde{t}, m), u(\widetilde{t}, m))$ in which the forward shock passing through the origin is just $m = m_2(\widetilde{t})$.

Using the inverse transformation (3.48), in Eulerian representation we get the piston path

$$x = f(t) = \int_0^t \phi(\xi)d\xi. \tag{3.72}$$

Then, by (1.51)–(1.53), the piston velocity $\varphi(t)$, which is nothing but $\phi(t)$, satisfies (3.59)–(3.61). Thus, by Theorem 3.1 the corresponding direct piston problem (3.1) and (3.3)–(3.4) admits a unique global piecewise C^1 solution $(\rho(t,x), u(t,x))$ in which the forward shock passing through the origin is just $x = x_f(t)$.

Theorem 3.2 is then proved.

Remark 3.1 The corresponding consideration on the one-dimensional gas dynamics system (containing three equations) can be found in [5].

References

[1] Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves, Interscience, New York, 1948.

[2] Li, T. T., Global Classical Solutions for Quasilinear Hyperbolic Systems, Research in Applied Mathematics **32**, John Wiley/Masson, 1994.

[3] Li, T. T., Exact shock reconstruction, *Inverse Problems*, **21**, 2005, 673–684.

[4] Li, T. T. and Wang, L. B., Global exact shock reconstruction, *Discrete Contin. Dyn. Syst.*, **15**, 2006, 597–609.

[5] Li, T. T. and Wang, L. B., Existence and uniqueness of global solution to an inverse piston problem, *Inverse Problems*, **23**, 2007, 683–694.

[6] Li, T. T. and Yu, W. C., Boundary Value Problems for Quasilinear Hyperbolic Systems, Duke University Mathematics, Series V, 1985.

[7] Li, T. T. and Zhao, Y. C., Global shock solutions to a class of piston problem for the system of one dimensional isentropic flow, *Chin. Ann. Math.*, **12B**(4), 1991, 495–499.