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A Linear System Arising from a Polynomial Problem and Its Applications

Wen-Xiu MA* Boris SHEKHTMAN*

Abstract A linear system arising from a polynomial problem in the approximation theory is studied, and the necessary and sufficient conditions for existence and uniqueness of its solutions are presented. Together with a class of determinant identities, the resulting theory is used to determine the unique solution to the polynomial problem. Some homogeneous polynomial identities as well as results on the structure of related polynomial ideals are just by-products.

Keywords Linear system, Determinant identity, Polynomial ideal in two variables
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1 Introduction and Motivation

Let $m \geq 0$ be an integer, $\mathbb{F}$ be a field, and $\mathbb{F}[x, y]$ be the polynomial ring in two variables, x and y, over $\mathbb{F}$. We use the standard notation: $\mathbb{F}_m[x, y]$ denotes the set of homogeneous polynomials of degree m, and $\mathbb{F}_{\leq m}[x, y]$ denotes the space of all polynomials of degree at most m. Hence $\mathbb{F}_m[x, y] \subset \mathbb{F}_{\leq m}[x, y]$, and $\mathbb{F}_{\leq m}[x, y]$ is a linear subspace of $\mathbb{F}[x, y]$ with

$$M := \dim \mathbb{F}_{\leq m}[x, y] = \frac{(m+1)(m+2)}{2}.$$

Let $\mathfrak{J}_m$ be the set of all ideals in $\mathbb{F}[x, y]$ complemented to $\mathbb{F}_{\leq m}[x, y]$, i.e., $\mathfrak{J}_m$ is the set of all ideals $J \subset \mathbb{F}[x, y]$ such that

$$\mathbb{F}[x, y] = J \oplus \mathbb{F}_{\leq m}[x, y].$$

Note that every ideal $J \in \mathfrak{J}_m$ can be generated by a collection of $m + 2$ polynomials $x^i y^{m+1-i} - h_i$, $0 \leq i \leq m + 1$, where $h_i \in \mathbb{F}_{\leq m}[x, y]$, $0 \leq i \leq m + 1$. More precisely, we have that

$$J \in \mathfrak{J}_m \Rightarrow J = \langle x^i y^{m+1-i} - h_i : i = 0, \cdots, m + 1\rangle$$

for some polynomials $h_i \in \mathbb{F}_{\leq m}[x, y]$, $0 \leq i \leq m + 1$, where $\langle S\rangle$ denotes the ideal generated by the polynomials in S. Unfortunately, the converse is not true. Namely, not every collection of polynomials $h_i \in \mathbb{F}_{\leq m}[x, y]$, $0 \leq i \leq m + 1$, generates an ideal in $\mathfrak{J}_m$ this way. Actually, $m + 2$ polynomials $h_i \in \mathbb{F}_{\leq m}[x, y]$, $0 \leq i \leq m + 1$, generate an ideal $J = \langle x^i y^{m+1-i} - h_i : i =$

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*Department of Mathematics, University of South Florida, PHY114, 4202 E Fowler Avenue, Tampa, FL 33620-5700, USA. E-mail: mawx@math.usf.edu boris@math.usf.edu

$0, \cdots, m+1\rangle$ complemented to $\mathbb{F}_{\le m}[x,y]$ if and only if they satisfy the following equation

$$yh_{i+1} - xh_i = \sum_{j=0}^{m+1} c_{ij}(x^j y^{m+1-j} - h_j), \quad 0 \le i \le m \tag{1.1}$$

for some collection of constants $c_{ij} \in \mathbb{F}$.

Now a natural question about this crucial system (1.1) arises. For a given matrix $C = (c_{ij})$, where $c_{ij} \in \mathbb{F}$, does there exist any collection of polynomials $h_i \in \mathbb{F}_{\le m}[x,y]$, which solves (1.1)? If (1.1) has solutions for h_i, how many solutions and what solution formula can we have? This question, especially, how to present the solutions, is very interesting and challenging to us, though it looks simple. We here aim to discuss this question and give a complete answer to the question, together with two kinds of solution formulas for h_i.

To find solutions for the above polynomial problem, one may write

$$h_i = \sum_{n=0}^{m} h_i^{(n)}, \quad h_i^{(n)} = \sum_{j=0}^{n} h_{ij}^{(n)} x^j y^{n-j}, \quad 0 \le i \le m+1, \tag{1.2}$$

where the $h_{ij}^{(n)}$'s are constants in $\mathbb{F}$. Then upon equating the coefficients of monomials in (1.1), each set of $h_{ij}^{(n)}$ with fixed n will satisfy an over-determined linear system, with one exceptional case of $n = m$ in which $h_{ij}^{(m)}$ will satisfy a linear system with a square coefficient matrix. Therefore, the polynomial problem is transformed into a problem of how to determine solutions of the resulting linear systems, which are of the same type.

The polynomial problem mentioned above can also be used to classify ideal projections (see [1]) onto polynomials in two variables. The study of the problem itself was stimulated by Carl de Boor's conjecture, made at one of the Mid Southeast Chapter Fall Conferences in Gatlinburg, that every ideal projection is a limit of interpolating projections (see [3]).

In this paper, we would like to study the resulting linear system from the above polynomial problem. We will first develop a solution theory for the linear system in a general case, and then we apply the resulting solution theory to prove the existence and uniqueness of solutions for the polynomial problem, together with a set of determinant identities. We will also present two kinds of solution formulas for the unique solution of the polynomial problem, and a set of interesting homogenous polynomial identities. A few concluding remarks are given at the end of the paper.

2 Linear System

Let $m \ge n$ be two fixed non-negative integers. The system of linear equations that the constants $h_{ij}^{(n)}$ satisfy reads as

$$\begin{cases} x_{i+1,0} = d_{i0}, & 0 \le i \le m, \\ x_{i+1,j} - x_{i,j-1} = d_{ij}, & 0 \le i \le m,\ 1 \le j \le n, \\ -x_{in} = d_{i,n+1}, & 0 \le i \le m, \end{cases} \tag{2.1}$$

where $d_{ij} \in \mathbb{F},\ 0 \le i \le m,\ 0 \le j \le n+1$, are a set of given constants and $x_{ij},\ 0 \le i \le m+1,\ 0 \le j \le n$, are a set of unknowns. Obviously, there are totally $(m+1)(n+2)$ linear equations and $(m+2)(n+1)$ unknowns, and so the linear system (2.1) is over-determined when $m > n$. We show the existence of solutions and present the solution formula for (2.1) as follows.

Theorem 2.1 *Let $m \geq n$ be two non-negative integers. Suppose that $d_{ij} \in \mathbb{F}$, $0 \leq i \leq m$, $0 \leq j \leq n+1$, are a set of given constants. Then*

(a) *the system* (2.1) *of linear equations in x_{ij}, $0 \leq i \leq m+1$, $0 \leq j \leq n$, is consistent if and only if*

$$\sum_{k=0}^{n+1} d_{i+k,k} = 0, \quad 0 \leq i \leq m-n-1; \tag{2.2}$$

(b) *the linear system* (2.1) *has the unique solution given by*

$$\begin{cases} x_{i+1,j} = \displaystyle\sum_{k=0}^{j} d_{i-j+k,k}, & 0 \leq j \leq i \leq m, \\ x_{i,j-1} = -\displaystyle\sum_{k=j}^{n+1} d_{i-j+k,k}, & 0 \leq i < j \leq n+1, \end{cases} \tag{2.3}$$

if it is consistent.

Proof Step 1 The cases of $j = 0$ and $j = n+1$ of the solution in (2.3) are given by the first set of equations, and the last set of equations with $i \leq n$, of the linear system (2.1), respectively.

Step 2 Let $1 \leq j \leq i \leq m$. Then we have

$$x_{i+1,j} = \sum_{k=1}^{j}(x_{i-j+k+1,k} - x_{i-j+k,k-1}) + x_{i-j+1,0} = \sum_{k=1}^{j} d_{i-j+k,k} + d_{i-j,0} = \sum_{k=0}^{j} d_{i-j+k,k}.$$

Step 3 Let $0 \leq i < j \leq n$. Then we have

$$\begin{aligned} x_{i,j-1} &= -\sum_{k=j}^{n}(x_{i-j+k+1,k} - x_{i-j+k,k-1}) + x_{i-j+n+1,n} \\ &= -\sum_{k=j}^{n} d_{i-j+k,k} - d_{i-j+n+1,n+1} = -\sum_{k=j}^{n+1} d_{i-j+k,k}. \end{aligned}$$

Step 4 The above three steps imply that if there exists a solution of (2.1), then the solution must be determined by (2.3).

Step 5 Note that from the above deduction, we know that the solution (2.3) is equivalent to the first and second sets of equations, and the third set of equations with $i \leq n$, in (2.1). So what we have to check is the third set of equations with $i > n$ in (2.1). This requires the following compatibility conditions

$$\sum_{k=0}^{n} d_{i-n+k,k} = x_{i+1,n} = -d_{i+1,n+1}, \quad n \leq i \leq m-1,$$

the first of which is from the solution (2.3) and the second of which is exactly the third set of equations with $i > n$ in (2.1). Obviously, these conditions are equivalent to the ones in (2.2). The proof is finished.

If $n = m$, then there is no condition posed in (2.2). The system (2.1) is not over-determined, and there is the unique solution for whatever constants of d_{ij}.

Corollary 2.1 *If $n = m$, then the system* (2.1) *has the unique solution defined by* (2.3) *for all constants of* d_{ij}.

But if $n = m - 1$, we need to check one condition for the existence of solutions.

Corollary 2.2 *If $n = m-1$, then the system* (2.1) *has the unique solution defined by* (2.3) *iff* $\sum\limits_{k=0}^{n+1} d_{kk} = 0$.

If $n = 0$, no equation appears in the second set of equations in (2.1), but we need a set of conditions for d_{ij}.

Corollary 2.3 *If $n = 0$, then the system* (2.1) *has the unique solution*

$$x_{00} = -d_{01}, \quad x_{i+1,0} = d_{i0}, \quad 0 \le i \le m,$$

iff $d_{i0} + d_{i+1,1} = 0, \quad 0 \le i \le m-1.$

One can also observe the linear system (2.1) and the conditions in (2.2) in a geometric way. In Figures 1 and 2, the two rectangles on the (i, j) plane are divided into three areas. The

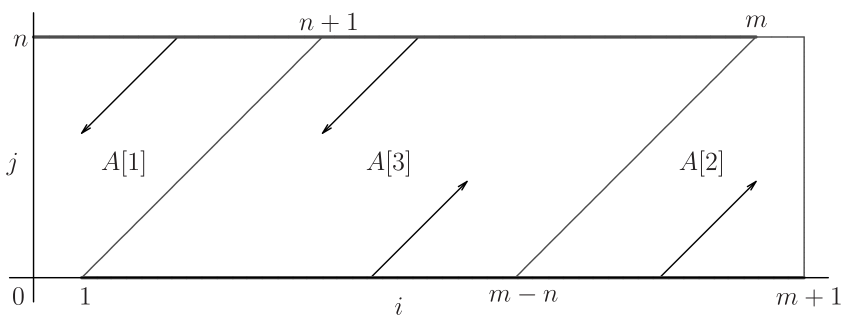

Figure 1 Different areas for the linear system

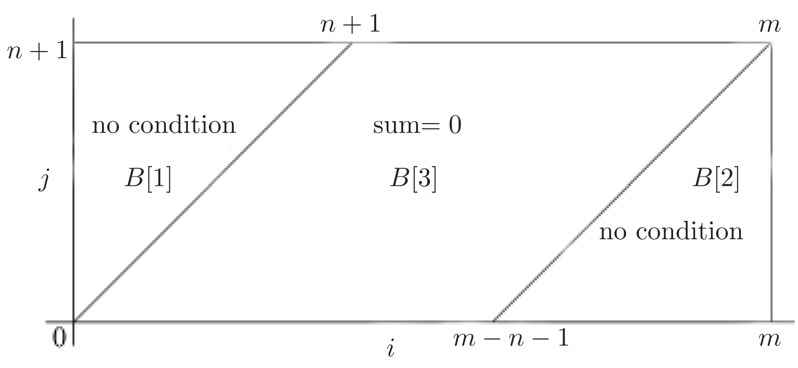

Figure 2 Different areas for the conditions of existence

three areas in Figure 1 correspond to different situations of the linear system (2.1). The first subsystem and the last subsystem in the linear system determine the boundary conditions at the integer points on the line $j = n$ from $i = 0$ to $i = m$ and on the line $j = 0$ from $i = 1$ to

$i = m+1$, respectively. The second subsystem determines the solutions x_{ij} at other integer points by going diagonally up or down on the lines $i = j+k$ for $-(n+1) \le k \le m+1$, one step each time from the two boundaries. In Area $A[1]$ and Area $A[2]$ of Figure 1, the solutions x_{ij} can be determined by the first and second formulas in (2.3), respectively. However, in Area $A[3]$ of Figure 1, both formulas work for determining x_{ij}, and thus there are conditions on the lines $i = j+k$ from $k=1$ to $k = m-n$, which are exactly the $m-n$ conditions in (2.2). In Figure 2, Areas $B[1]$, $B[2]$ and $B[3]$ correspond to Areas $A[1]$, $A[2]$ and $A[3]$ in Figure 1, respectively. Thus, there is no condition for d_{ij} in Areas $B[1]$ and $B[2]$. The conditions in (2.2) mean that the sum of d_{ij} at the integer points on each line $i = j+k$ for $0 \le k \le m-n-1$ in the rectangle is zero, which is the required conditions in Area $B[3]$ of Figure 2.

3 Applications

The linear system (2.1) is the condition that each set of $h_{ij}^{(n)}$ with fixed n needs to satisfy, and thus it can be used to prove the uniqueness of solutions for (1.1).

Theorem 3.1 *Let $m \ge 0$ be an integer. Suppose that $C = (c_{ij})_{(m+1)\times(m+2)}$ is a given matrix, where $c_{ij} \in \mathbb{F},\ 0 \le i \le m,\ 0 \le j \le m+1$; and*

$$A = \begin{bmatrix} -x & y & & & 0 \\ & -x & y & & \\ & & \ddots & \ddots & \\ 0 & & & -x & y \end{bmatrix}_{(m+1)\times(m+2)}. \tag{3.1}$$

Then there exists a unique collection of polynomials $h_i \in \mathbb{F}_{\le m}[x,y]$, $0 \le i \le m+1$, which solves the polynomial problem (1.1)*. Moreover, the unique collection can be given by*

$$h_i = x^i y^{m+1-i} - (-1)^i \det((A+C)_i), \quad 0 \le i \le m+1, \tag{3.2}$$

where $(A+C)_i$ denotes the matrix generated from $A+C$ by deleting its i-th column, or by (1.2) *with the coefficients being recursively defined by*

$$\begin{cases} h_{i+1,j}^{(m)} = \displaystyle\sum_{k=0}^{j} c_{i-j+k,k}, & 0 \le j \le i \le m, \\ h_{i,j-1}^{(m)} = -\displaystyle\sum_{k=j}^{m+1} c_{i-j+k,k}, & 0 \le i < j \le m+1, \end{cases} \tag{3.3}$$

$$\begin{cases} h_{i+1,j}^{(n)} = \displaystyle\sum_{k=0}^{j}\sum_{l=0}^{m+1} c_{i-j+k,l} h_{lk}^{(n+1)}, & 0 \le j \le i \le m, \\ h_{i,j-1}^{(n)} = -\displaystyle\sum_{k=j}^{n+1}\sum_{l=0}^{m+1} c_{i-j+k,l} h_{lk}^{(n+1)}, & 0 \le i < j \le n+1, \end{cases} \tag{3.4}$$

where $0 \le n \le m-1$.

Proof Existence and First Solution Formula Set

$$\widetilde{h}_i = x^i y^{m+1-i} - h_i, \quad 0 \le i \le m+1.$$

Then the original polynomial problem (1.1) becomes

$$y\widetilde{h}_{i+1} - x\widetilde{h}_i = -\sum_{j=0}^{m+1} c_{ij}\widetilde{h}_j, \quad 0 \le i \le m. \tag{3.5}$$

On the other hand, we obviously have

$$\det \begin{bmatrix} c_{i0} \ \ c_{i1} \ \ \cdots \ \ c_{ii} - x \ \ c_{i,i+1} + y \ \ \cdots \ \ c_{i,m+1} \\ A + C \end{bmatrix} = 0, \quad 0 \le i \le m, \tag{3.6}$$

where A is defined by (3.1) and $C = (c_{ij})_{(m+1)\times(m+2)}$. The Laplace expansions of those determinants about the first row yield

$$\sum_{j=0}^{m+1} c_{ij}\widetilde{r}_j - x\widetilde{r}_i + y\widetilde{r}_{i+1} = 0, \quad 0 \le i \le m, \tag{3.7}$$

if we set

$$\widetilde{r}_i = (-1)^i \det((A + C)_i), \quad 0 \le i \le m + 1, \tag{3.8}$$

where $(A + C)_i$ denotes the matrix generated from $A + C$ by deleting its i-th column. This implies that the polynomials $\widetilde{r}_i$ defined above are a solution to (3.5). Note that $x^i y^{m+1-i} - \widetilde{r}_i$, $0 \le i \le m + 1$, are all polynomials in $\mathbb{F}_{\le m}[x, y]$, and thus there is at least one solution to the polynomial problem (1.1), defined by (3.2).

Uniqueness and Second Solution Formula First, equating the coefficients of the monomials of the highest degree $m + 1$ in the polynomial problem (1.1) leads to

$$\begin{cases} h^{(m)}_{i+1,0} = c_{i0}, & 0 \le i \le m, \\ h^{(m)}_{i+1,j} - h^{(m)}_{i,j-1} = c_{ij}, & 0 \le i \le m,\ 1 \le j \le m, \\ -h^{(m)}_{im} = c_{i,m+1}, & 0 \le i \le m. \end{cases} \tag{3.9}$$

Second, equating the coefficients of the monomials of degree $1 \le n + 1 \le m$ in the polynomial problem (1.1) leads to

$$\begin{cases} h^{(n)}_{i+1,0} = d^{(n)}_{i0} := -\displaystyle\sum_{l=0}^{m+1} c_{il} h^{(n+1)}_{l0}, & 0 \le i \le m, \\ h^{(n)}_{i+1,j} - h^{(n)}_{i,j-1} = d^{(n)}_{ij} := -\displaystyle\sum_{l=0}^{m+1} c_{il} h^{(n+1)}_{lj}, & 0 \le i \le m,\ 1 \le j \le n, \\ -h^{(n)}_{in} = d^{(n)}_{i,n+1} := -\displaystyle\sum_{l=0}^{m+1} c_{il} h^{(n+1)}_{l,n+1}, & 0 \le i \le m, \end{cases} \tag{3.10}$$

where $0 \le n \le m - 1$. Now, we see that all the coefficients $h^{(n)}_{ij}$ are recursively defined from $n = m$ to $n = 0$ by the linear systems in (3.9) and (3.10). Note that all of these linear systems are of the same type as the one in (2.1), and thus it follows from the existence of solutions and Theorem 2.1 that there is a unique solution to the polynomial problem (1.1) and the solution can be defined by (1.2), with (3.3) and (3.4). The proof is finished.

The solutions in the cases of $m = 0$ and $m = 1$ are as follows. If $m = 0$, we have

$$h_0 = -c_{01}, \quad h_1 = c_{00};$$

and if $m = 1$, we have

$$\begin{cases} h_0 = -(c_{01} + c_{12})y - c_{02}x - c_{01}c_{12} + c_{02}c_{11}, \\ h_1 = -c_{12}x + c_{00}y + c_{00}c_{12} - c_{02}c_{10}, \\ h_2 = (c_{00} + c_{11})x + c_{10}y - c_{00}c_{11} + c_{01}c_{10}. \end{cases}$$

Let us now show some polynomial identities of c_{ij} from the existence of solutions of (1.1). First, based on Theorem 2.1, there are the conditions, defined by (2.2), for the existence of solutions of the linear systems in (3.10), although the linear system (3.9) always has a unique solution. Using the definitions of $d^{(n)}_{i+k,k}$ in (3.10), we see that those conditions given by (2.2) are altogether equivalent to

$$\sum_{k=0}^{n+1} \sum_{l=0}^{m+1} c_{i+k,l} h^{(n+1)}_{lk} = 0, \quad 0 \le n \le m-1, \; 0 \le i \le m-n-1. \tag{3.11}$$

Second, collecting the constant terms in (1.1) leads to

$$\sum_{l=0}^{m+1} c_{il} h^{(0)}_{l0} = 0, \quad 0 \le i \le m. \tag{3.12}$$

Noting that all the coefficients $h^{(n)}_{lk}$ are homogeneous polynomials in c_{ij} of degree $m - n + 1$, we can sum up the results in (3.11) and (3.12) to yield the following theorem.

Theorem 3.2 *Let* $h^{(n)}_{ij}$ *be recursively defined by* (3.3) *and* (3.4)*. Then we have the following homogeneous polynomial identities of* c_{ij}*:*

$$\sum_{k=0}^{n} \sum_{l=0}^{m+1} c_{i+k,l} h^{(n)}_{lk} = 0, \quad 0 \le n \le m, \; 0 \le i \le m-n. \tag{3.13}$$

Actually, we can prove the identities in (3.13) directly from (3.7). First, when $0 \le n \le m-1$, the coefficients of $x^k y^{n+1-k}$ with $0 \le k \le n+1$ in each pair of $\widetilde{h}_i = \widetilde{r}_i$ and $h_i = x^i y^{m+1-i} - \widetilde{r}_i$ are the same, and thus collecting the coefficients of $x^k y^{n+1-k}$ of the identities in (3.7) leads to

$$\begin{cases} \displaystyle\sum_{l=0}^{m+1} c_{il} h^{(n+1)}_{l0} + h^{(n)}_{i+1,0} = 0, \\ \displaystyle\sum_{l=0}^{m+1} c_{i+k,l} h^{(n+1)}_{lk} - h^{(n)}_{i+k,k-1} + h^{(n)}_{i+k+1,k} = 0, \quad 1 \le k \le n, \\ \displaystyle\sum_{l=0}^{m+1} c_{i+n+1,l} h^{(n+1)}_{l,n+1} - h^{(n)}_{i+n+1,n} = 0, \end{cases}$$

where $0 \le i \le m-n-1$. Adding up those equalities, we obtain (3.11), which gives the identities with $n > 0$ in (3.13). On the other hand, computing the constant terms of the identities in (3.7) yields (3.12), which gives the identities with $n = 0$ in (3.13).

Theorem 3.2 provides a set of interesting identities of c_{ij}. In particular, when $n = m$ and $n = 1$, we obtain from (3.13) that

$$\sum_{k=0}^{m}\sum_{l=0}^{m+1} c_{kl} h_{lk}^{(m)} = 0$$

and

$$\sum_{l=0}^{m+1} c_{il} h_{l0}^{(1)} + \sum_{l=0}^{m+1} c_{i+1,l} h_{l1}^{(1)} = 0, \quad 0 \le i \le m-1,$$

respectively. The first equation gives the following identity

$$\sum_{l=1}^{m+1}\sum_{k=0}^{l-1}\sum_{p=0}^{k} c_{kl}c_{l-k-1+p,p} - \sum_{l=0}^{m}\sum_{k=l}^{m}\sum_{p=k+1}^{m+1} c_{kl}c_{l-k-1+p,p} = 0,$$

which can also be verified directly. However, the identities defined by both the second equation and the equation (3.12) are far away from obvious.

Theorem 3.1 also immediately implies a number of interesting conclusions for the structure of the manifold $\mathfrak{J}_m$.

Theorem 3.3 (a) *The set of coefficients of polynomials h_i that satisfy the algebraic equation* (1.1) *is a polynomial image of $\mathbb{F}^{(m+1)(m+2)}$ in $\mathbb{F}^{\frac{m(m+1)(m+2)}{2}}$, and the induced polynomial map is a bijection. Hence* (*cf.* [4, *p.* 196]) *the set of coefficients of polynomials h_i* (*thus the set* $\mathfrak{J}_m$) *is an irreducible algebraic variety in $\mathbb{F}^{\frac{m(m+1)(m+2)}{2}}$, and the dimension of this variety is* $(m+1)(m+2)$.

(b) *If the h_i's satisfy* (1.1)*, then the coefficients of $h_i^{(m)}$ and the constants c_{jk} are linear functions of each other. Hence the determining constants c_{jk} are uniquely determined by the ideal $J \in \mathfrak{J}_m$.*

Conversely, given arbitrary polynomials $\overline{h}_i^{(m)} \in \mathbb{F}_m[x,y]$, there exists a unique ideal $J = \langle x^i y^{m+1-i} - h_i : i = 0, \cdots, m+1\rangle \in \mathfrak{J}_m$ such that the leading polynomials $h_i^{(m)}$ of h_i is $\overline{h}_i^{(m)}$. This is in stark contrast with the one-dimensional case where every polynomial $h \in \mathbb{F}_{\le m}[x]$ defines an ideal $J = \langle x^{m+1} - h\rangle \in \mathfrak{J}_m$, even if the leading coefficient of h is zero.

(c) *If $J = \langle x^i y^{m+1-i} - h_i : i = 0, \cdots, m+1\rangle \in \mathfrak{J}_m$ is a polynomial ideal defined by the matrix C from Theorem* 3.1*, then the columns of the matrix $A + C$ define generators for the first syzygy module,* Syz(J) (*cf.* [5, *p.* 1980])*, of the ideal J, where the matrix A is given by* (3.1)*. Since these rows are linearly independent over the ring $\mathbb{F}[x,y]$, we conclude that the set of rows of the matrix $A + C$ is the module basis for* Syz(J).

In particular, if $J \in \mathfrak{J}_m$, then Syz(J) *is a free submodule of the module* $(\mathbb{F}[x,y])^{m+1}$.

4 Concluding Remarks

Theorem 3.1 shows the correspondence between the matrices C, the ideals $J \in \mathfrak{J}_m$ and the polynomials h_i defined by the equation (3.2). It would be interesting to classify the ideals $J \in \mathfrak{J}_m$ by considering properties of the matrices C. Let $J(C)$ stand for the ideal generated by the matrix C via the polynomials $h_i(C)$:

$$J(C) = \langle x^i y^{m+1-i} - h_i(C) : i = 0, \cdots, m+1\rangle.$$

Now, for instance, it is clear from (3.2) that $C = 0$ implies that $J(C)$ is a monomial ideal generated by $\{x^i y^{m+1-i} : i = 0, \cdots, m+1\}$. If the structure of the matrix C resembles that of the matrix A, i.e., if

$$C = \begin{bmatrix} -a & b & & & 0 \\ & -a & b & & 0 \\ & & \ddots & \ddots & \\ 0 & & & -a & b \end{bmatrix}_{(m+1)\times(m+2)},$$

then

$$J(C) = \langle (x-a)^i (y-b)^{m+1-i} : i = 0, \cdots, m+1 \rangle.$$

More generally, if we choose

$$C = \begin{bmatrix} -a_0 & b_0 & & & 0 \\ & -a_1 & b_1 & & \\ & & \ddots & \ddots & \\ 0 & & & -a_m & b_m \end{bmatrix}_{(m+1)\times(m+2)},$$

where a_i and b_i are arbitrary constants in $\mathbb{F}$, then the collection of polynomials that solves (1.1) reads

$$h_i(C) = x^i y^{m+1-i} - (-1)^i \prod_{j=0}^{i-1} (x - a_j) \prod_{j=i}^{m} (y - b_j), \quad 0 \le i \le m+1.$$

The special choices of the constants $c_{ij} \in \mathbb{F}$ in (3.5) can also provide three-term recurrence relations for the involved polynomials as in the theory of orthogonal polynomials in one variable. Like the above examples, the resulting polynomials in two variables may possess some specific properties as expected.

Here are a few open questions:

(1) For what C, the ideal $J(C)$ is regular? Prime? Primary?

(2) What relationship does exist between the structure of C and the cardinality of the variety

$$V(C) := \{(x, y) \in \mathbb{F}^2 : f(x, y) = 0 \text{ for all } f \in J(C)\}?$$

If $\#V(C) = 1$, then $J^\perp(C)$, the space of all functional annihilating J, is the space of polynomials of dimension m, which is invariant with respect to differentiation. Hence

(3) How can one describe all D-invariant m-dimensional subspaces of $\mathbb{F}[x, y]$?

On the other hand, the solution presented in Theorem 3.1 can also be used to solve the Cauchy problem of partial differential equations. For example, the collection of polynomials in (3.8) provides a polynomial solution: $u_i = \widetilde{r}_i$, $0 \le i \le m+1$, to the following linear system of partial differential equations:

$$q \partial_x^p \partial_y^{q-1} u_{i+1} - p \partial_x^{p-1} \partial_y^q u_i = -\sum_{j=0}^{m+1} c_{ij} \partial_x^p \partial_y^q u_j, \quad 0 \le i, p, q \le m,$$

with the initial conditions

$$\sum_{j=0}^{m+1} c_{ij} u_j|_{x=0,y=0} = 0.$$

It is not clear to us yet how to solve this Cauchy type problem generally.

We also remark that all results of this paper could be extended to the case of multivariate polynomials like multivariate polynomial interpolations (see [2]). The technique using determinant identities, like the determinant identities in (3.6) in the proof of the existence of solutions to the polynomial problem (1.1) can also be applied to present solutions to other polynomial problems.

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