

Proof of Murphy-Cohen Conjecture on One-Dimensional Hard Ball Systems**

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Abstract We prove the Murphy and Cohen's conjecture that the maximum number of collisions of $n + 1$ elastic particles moving freely on a line is $\frac{n(n+1)}{2}$ if no interior particle has mass less than the arithmetic mean of the masses of its immediate neighbors. In fact, we prove the stronger result that, for the same conclusion, the condition that no interior particle has mass less than the geometric mean, rather than the arithmetic mean, of the masses of its immediate neighbors suffices.

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1 Introduction

We consider a system of $n + 1$ hard balls (rods) moving freely on a straight line, where the only interaction is the elastic collision between two adjacent balls. In an elastic collision of two adjacent hard balls, their velocities are redistributed according to the laws of conservation of energy and momentum. It is well known that such a dynamical system is isomorphic to a billiard inside a polyhedral angle. It is also assumed that multiple collisions, i.e., collisions essential between three or more hard balls, do not occur, corresponding to the fact that if the billiard ball hits a corner, its further motion is not defined (with some exceptions).

Estimates of the number of collisions in hard ball systems, more generally in semi-dispersing billiards, have been studied for a long time because of their importance for proper generalization of the Boltzmann equation. Sinai [8] proved the existence of a uniform estimate of the number of collisions of billiard trajectories in a polyhedral angle. Gal'perin [3] obtained an explicit estimate for a system of elastic particles (balls of zero radii) on a line. And Sevryuk [7] gave a uniform estimate for billiards in a polyhedral angle in terms of a geometrical characteristic of the angle. The most general results on this problem were obtained by Burago, Ferleger and Kononenko [2]. In particular, they provided an explicit estimate depending only on the masses of balls for generalized hard ball systems on simply connected Riemannian spaces of non-positive sectional curvature. However, the maximum number of collisions that a hard ball system may undergo was known only for systems of three identical balls in Euclidean space

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of dimension at least 2, besides being known for one-dimensional systems of three balls with different masses which are almost trivial under the billiard approach. Due to Foch, an example of initial conditions which led three identical hard balls to four collisions was known (cf. [5, Appendix B]). Thurston and Sandri [9] discovered a system of three identical balls suffering four collisions as well. Then Sandri et al. [6] conjectured that four is the maximum number of collisions in 1964. A rigorous proof was published until 1993 by Murphy and Cohen [4].

One of the features of a hard ball system in one dimension is that the balls always remain the same order on the line. Since we are only interested in upper bounds of the number of collisions, the information on length or distance of the system can be completely ignored for our purpose. It allows us to reduce directly the original system to an action of a reflection group, generated by n orthogonal reflections, acting on a sphere in n -dimensional Euclidean space $\mathbb{E}^n$, and then to a numbers game. This natural method reveals the geometric meaning of the constructions of “relative mass” and “relative velocity” appeared in [5]. The numbers game is intimately related to Coxeter groups (see, for example, [1, Chapter 4]) although the main interests there are different from ours.

We number the hard balls $0, 1, \dots, n$ in the order of increasing coordinates and write m_i for the mass of ball i . In [5], Murphy and Cohen showed that for some initial conditions at least $\frac{n(n+1)}{2}$ collisions occur and conjectured that if $m_i \geq \frac{m_{i-1}+m_{i+1}}{2}$, $i = 1, \dots, n-1$, then the maximum number of collisions is $\frac{n(n+1)}{2}$. The purpose of this paper is to prove the following theorem.

Theorem 1.1 *If*

$$m_i \geq \sqrt{m_{i-1}m_{i+1}}, \quad i = 1, \dots, n-1,$$

then the maximum number of collisions is $\frac{n(n+1)}{2}$.

Remark 1.1 Since $\sqrt{ab} \leq \frac{a+b}{2}$ for $a, b > 0$, and equality holds if and only if $a = b$, Theorem 1.1 proves the conjecture of Murphy and Cohen with a weaker assumption. To see why and how the number $\frac{n(n+1)}{2}$ arises, consider the simplest case of equal mass: $m_0 = m_1 = \dots = m_n$. Suppose that v_{i-1} and v_i are the velocities of ball $i-1$ and ball i respectively ($1 \leq i \leq n$) before an elastic collision between them. Then the post-collision velocities v'_{i-1} and v'_i are, in general, given by

$$v'_{i-1} = \frac{(m_{i-1} - m_i)v_{i-1} + 2m_iv_i}{m_{i-1} + m_i}, \quad v'_i = \frac{2m_{i-1}v_{i-1} + (m_i - m_{i-1})v_i}{m_{i-1} + m_i}.$$

In the case of equal mass, the two collision balls simply exchange their velocities. The consequences will become apparent if we observe changes of the inversion number of the sequence of velocities $(v_0, v_1, \dots, v_n)$, which remains constant between collisions. The inversion number of a sequence of numbers $\mathbf{q} = (q_0, q_1, \dots, q_n)$ is defined as the number of its inversions, that is,

$$\text{inv}(\mathbf{q}) = \text{card} \{(i, j) \mid i < j, q_i > q_j\}.$$

It is obvious that

$$0 \leq \text{inv}(\mathbf{q}) \leq \frac{n(n+1)}{2}$$

and $\text{inv}(\mathbf{q}) = 0$, iff $\mathbf{q}$ is an increasing sequence. When $v_{i-1} > v_i$, a collision between ball $i-1$ and i exchanges the values of the two velocities in the sequence of velocities so that its inversion number decreases 1. The collisions then sort the sequence by binary exchanges until the sequence is in increasing order, after which there can be no more collision. Therefore, the total number of collisions equals the inversion number of the sequence of the initial velocities. In the proof of Theorem 1.1, we will construct a sequence (depending on the time) with the similar property: its inversion number remains constant between collisions and decreases at least 1 in any collision.

2 Proof of Theorem 1.1

Let

$$\mathbf{1}_i = (\delta_{i0}, \delta_{i1}, \dots, \delta_{in})^T \in \mathbb{E}^{n+1}, \quad i = 0, 1, \dots, n,$$

where δ_{ij} is the Kronecker delta. Write v_i for the velocity of ball i and set

$$\mathbf{m} = \sum_{j=0}^n \sqrt{m_j} \mathbf{1}_j, \quad \mathbf{v} = \sum_{j=0}^n \sqrt{m_j} v_j \mathbf{1}_j \in \mathbb{E}^{n+1}.$$

Then the momentum and energy of the system read $(\mathbf{m}, \mathbf{v})$ and $\frac{1}{2}\|\mathbf{v}\|^2$ respectively, where $(\cdot, \cdot)$ is the standard scalar product on $\mathbb{E}^{n+1}$ and $\|\cdot\|$ is the norm determined by the scalar product. For $i = 1, \dots, n$, let

$$\alpha_i = \frac{\frac{1_i}{\sqrt{m_i}} - \frac{1_{i-1}}{\sqrt{m_{i-1}}}}{\left\| \frac{1_i}{\sqrt{m_i}} - \frac{1_{i-1}}{\sqrt{m_{i-1}}} \right\|}$$

and σ_i be the orthogonal reflection with respect to the hyperplane passing through the origin with α_i as a unit normal, that is,

$$\sigma_i : \beta \mapsto \beta - 2(\alpha_i, \beta)\alpha_i.$$

It is readily seen that

$$\begin{aligned} (\alpha_i, \mathbf{m}) &= 0, \quad (\alpha_i, \alpha_i) = 1, \quad i = 1, \dots, n, \\ (\alpha_i, \alpha_j) &= 0, \quad |i - j| > 1, \\ (\alpha_i, \alpha_{i+1}) &= -\frac{1}{m_i} \cdot \frac{1}{\sqrt{\frac{1}{m_i} + \frac{1}{m_{i-1}}}} \cdot \frac{1}{\sqrt{\frac{1}{m_{i+1}} + \frac{1}{m_i}}}, \quad i = 1, \dots, n-1, \end{aligned}$$

and $\mathbf{m}, \alpha_1, \dots, \alpha_n$ form a basis of $\mathbb{E}^{n+1}$ as a vector space. A collision between ball $i-1$ and ball i is now realized geometrically by the reflection σ_i , according to momentum conservation $(\mathbf{m}, \mathbf{v}) = (\mathbf{m}, \sigma_i(\mathbf{v}))$ and energy conservation $\|\mathbf{v}\|^2 = \|\sigma_i(\mathbf{v})\|^2$. A necessary condition for the collision really to take place is $v_{i-1} > v_i$, equivalently, $(\alpha_i, \mathbf{v}) < 0$, since

$$(\alpha_i, \mathbf{v}) = -(\alpha_i, \sigma_i(\mathbf{v})) = \frac{v_i - v_{i-1}}{\sqrt{\frac{1}{m_i} + \frac{1}{m_{i-1}}}}.$$

If

$$|i - j| > 1,$$

then

$$(\alpha_i, \alpha_j) = 0,$$

i.e., σ_i commutes with σ_j . It reflects the fact that several binary collisions may take place simultaneously.

We are now in a position to play a numbers game. Let $\mathbf{p} = (p_1, \dots, p_n)$ be thought of as a position in the game. A position $\mathbf{p}$ is called nonnegative if p_i is nonnegative for all $i = 1, \dots, n$. Choose the weights

$$k_{ij} = -2(\alpha_i, \alpha_j), \quad 1 \leq i, j \leq n.$$

Thus

$$\begin{aligned} k_{ii} &= -2, \quad k_{ij} = k_{ji}, \quad 1 \leq i, j \leq n, \\ k_{ij} &= 0, \quad |i - j| > 1, \\ k_{i,i+1} &= \frac{1}{m_i} \sqrt{\frac{2}{\frac{1}{m_i} + \frac{1}{m_{i-1}}} \cdot \frac{2}{\frac{1}{m_{i+1}} + \frac{1}{m_i}}}, \quad i = 1, \dots, n-1. \end{aligned}$$

Moves in the game are defined as follows. A firing of i changes a position $\mathbf{p}$ by adding $p_i k_{ij}$ to the j -th component of $\mathbf{p}$ for all j . More explicitly, a firing of i changes $\mathbf{p}$ in the following way: switch the sign of the i -th component, add $p_i k_{ij}$ to each adjacent component p_j , and leave all other components unchanged. Such a move is called negative if $p_i < 0$. A negative game is one that is played with negative moves from a given starting position. The negative game terminates when it arrives a nonnegative position.

A history of the original hard ball system generates an orbit of the action of the reflection group generated by $\sigma_1, \dots, \sigma_n$, which records the elastic collision sequence. And the orbit corresponds to a negative play sequence of the numbers game with the weights k_{ij} by setting

$$\mathbf{p} = (p_1, \dots, p_n) = ((\alpha_1, \mathbf{v}), \dots, (\alpha_n, \mathbf{v})).$$

We will show that the negative game defined as above must always terminate in $\frac{n(n+1)}{2}$ steps no matter what the starting position is and how it is played.

Let $\mathbf{p} = (p_1, \dots, p_n)$ be a position in the numbers game. To avoid analysis case by case, from now on let

$$k_{i0} = k_{i,n+1} = p_0 = p_{n+1} = p_{n+2} = \dots = 0$$

and the same symbol $\mathbf{p}$ denote the augmented position $(0, p_1, \dots, p_n, 0, 0, \dots)$. (The values of $p_0, p_{n+1}, p_{n+2}, \dots$ do not change in the whole game.) Define

$$\begin{aligned} q_i &= \sum_{j=0}^i p_j, \quad i = 0, 1, 2, \dots, \\ \mathbf{q} &= (q_0, q_1, \dots, q_n). \end{aligned}$$

We will call $\mathbf{q}$ the potential associated to the position $\mathbf{p}$. Then a position is nonnegative if and only if its potential is an increasing sequence. Suppose now we fire i , $1 \leq i \leq n$. The augmented position after the firing is

$$\mathbf{p}' = (p_0, \dots, p_{i-2}, p_{i-1} + p_i k_{i,i-1}, -p_i, p_{i+1} + p_i k_{i,i+1}, p_{i+2}, p_{i+3}, \dots),$$

and hence the potential associated to it becomes $\mathbf{q}' = (q'_0, q'_1, \dots, q'_n)$ where

$$q'_j = \begin{cases} q_j, & j \leq i-2, \\ q_i - p_i(1 - k_{i,i-1}), & j = i-1, \\ q_{i-1} - p_i(1 - k_{i,i-1}), & j = i, \\ q_j - p_i(1 - k_{i,i-1} + 1 - k_{i,i+1}), & j \geq i+1. \end{cases}$$

Using the elementary inequality $\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab}$, for $a, b > 0$, we have

$$k_{i,i+1} \leq \frac{1}{m_i} \sqrt{\sqrt{m_i m_{i-1}} \cdot \sqrt{m_{i+1} m_i}} = \sqrt{\frac{\sqrt{m_{i-1} m_{i+1}}}{m_i}}, \quad i = 1, \dots, n-1.$$

If

$$m_i \geq \sqrt{m_{i-1} m_{i+1}},$$

then

$$k_{i,i+1} \leq 1,$$

i.e.,

$$(\alpha_i, \alpha_{i+1}) \geq -\frac{1}{2}, \quad i = 1, \dots, n-1.$$

It follows that, when $p_i < 0$, equivalently, $q_{i-1} > q_i$, the sequence

$$-p_i(0, \dots, 0, 1 - k_{i,i-1}, 1 - k_{i,i-1}, 1 - k_{i,i-1} + 1 - k_{i,i+1}, \dots)$$

is increasing. Therefore, the inversion number of the potential after firing i ($1 \leq i \leq n$)

$$\text{inv}(\mathbf{q}') \leq \text{inv}(q_0, \dots, q_{i-2}, q_i, q_{i-1}, q_{i+1}, \dots, q_n) = \text{inv}(\mathbf{q}) - 1.$$

The proof is completed since

$$0 \leq \text{inv}(\mathbf{q}) \leq \frac{n(n+1)}{2}.$$

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