Chin. Ann. Math.
28B(3), 2007, 299–310
DOI: 10.1007/s11401-006-0041-7

Chinese Annals of Mathematics, Series B

Conformal CMC-Surfaces in Lorentzian Space Forms**

Changxiong NIE* Xiang MA* Changping WANG*

Abstract Let $\mathbb{Q}^3$ be the common conformal compactification space of the Lorentzian space forms $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$. We study the conformal geometry of space-like surfaces in $\mathbb{Q}^3$. It is shown that any conformal CMC-surface in $\mathbb{Q}^3$ must be conformally equivalent to a constant mean curvature surface in $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$. We also show that if $x : M \to \mathbb{Q}^3$ is a space-like Willmore surface whose conformal metric g has constant curvature K, then either $K = -1$ and x is conformally equivalent to a minimal surface in $\mathbb{R}^3_1$, or $K = 0$ and x is conformally equivalent to the surface $\mathbb{H}^1(\frac{1}{\sqrt{2}}) \times \mathbb{H}^1(\frac{1}{\sqrt{2}})$ in $\mathbb{H}^3_1$.

Keywords Conformal geometry, Willmore surfaces, Lorentzian space
2000 MR Subject Classification 53A30, 53B30

1 Introduction

In the study of the classical conformal geometry of $\mathbb{R}^n$, it is preferable to consider in its conformal compactification $\mathbb{S}^n$, on which the Moebius group acts transitively. At the same time, it unifies the study of the three standard Riemannian space forms, making the Euclidean geometry, the spherical geometry and the hyperbolic geometry as its sub-geometries.

Interestingly, in Lorentzian geometry we have similar constructions. Let $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ be the three-dimensional Lorentzian space forms of curvature 0, 1 and -1, respectively (cf. [10]), where

$$\mathbb{R}^3_1 = (\mathbb{R}^3, (\ ,\ )), \quad (u, v) = u_1v_1 + u_2v_2 - u_3v_3, \tag{1.1}$$
$$\mathbb{S}^3_1 = \{u \in \mathbb{R}^4_1 \mid u_1^2 + u_2^2 + u_3^2 - u_4^2 = 1\}, \tag{1.2}$$
$$\mathbb{H}^3_1 = \{u \in \mathbb{R}^4_1 \mid u_1^2 + u_2^2 - u_3^2 - u_4^2 = -1\}, \tag{1.3}$$

together with the induced metric. Let $\mathbb{Q}^3$ be the quadric in $\mathbb{R}P^4$ defined by

$$\mathbb{Q}^3 = \{[x] \in \mathbb{R}P^4 \mid x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2 = 0\}. \tag{1.4}$$

As well-known $\mathbb{Q}^3$ is a compact manifold homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1/\{\pm 1\}$. There is a standard conformal structure of Lorentzian type on $\mathbb{Q}^3$ with the conformal group $O(3,2)/\{\pm 1\}$ (cf. [5]).

Manuscript received February 13, 2006. Published online April 30, 2007.
*Lab of Mathematics and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing 100871, China. E-mail: niecx740218@163.com maxiang@math.pku.edu.cn cpwang@math.pku.edu.cn
**Project supported by the National Natural Science Foundation of China (No. 10125105) and the Research Fund for the Doctoral Program of Higher Education.

And there are conformal embeddings of $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ into $\mathbb{Q}^3$ separately defined by

$$\sigma: \mathbb{R}^3_1 \to \mathbb{Q}^3, \quad \sigma(u) = \Big[\Big(\frac{1}{2}(1-(u,u)), u, \frac{1}{2}(1+(u,u))\Big)\Big], \tag{1.5}$$

$$\tau: \mathbb{S}^3_1 \to \mathbb{Q}^3, \quad \tau(u) = [(u,1)], \tag{1.6}$$

$$\varrho: \mathbb{H}^3_1 \to \mathbb{Q}^3, \quad \varrho(u) = [(1,u)]. \tag{1.7}$$

In this paper we study the geometry of surfaces in $\mathbb{Q}^3$ under the conformal group $O(3,2)/\{\pm 1\}$. By using conformal embeddings σ, τ or ϱ, respectively, we can regard surfaces in $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ as surfaces in $\mathbb{Q}^3$.

Let $x: M \to \mathbb{Q}^3$ be an immersed space-like surface without umbilical point. Similar to the conformal geometry of surfaces in $\mathbb{S}^3$, here one can also define a conformal Gauss map $\xi: M \to \mathbb{R}^5_2$ of the surface x (cf. [1, 4]). With this construction at hand, the surface theory can be developed as usual: the conformal Gauss map is interpreted as the mean curvature sphere (see Section 4); it induces an invariant conformal metric $g = \langle d\xi, d\xi\rangle$; the critical surfaces with respect to the volume functional of g are called Willmore surfaces in $\mathbb{Q}^3$.

Let $\pi: \mathbb{E} \to M$ be the canonical line bundle over M defined by $\pi^{-1}(p) = \mathbb{R}y$, where $[y] = x(p)$. For any non-zero local section $y: U \to \mathbb{E}$, there exists a unique $y^*: U \to \mathbb{R}^5_2$, such that $y^* \perp \{\xi, d\xi\}$, $\langle y^*, y^*\rangle = 0$ and $\langle y^*, y\rangle = 1$. We write $g = e^{2\omega}|dz|^2$ for some complex coordinate z. Then $s = e^{-2\omega}\langle \xi_{z\bar{z}}, y^*\rangle y$ is independent of the choice of y and z, and thus it is a global section of $\mathbb{E}$. A surface in $\mathbb{Q}^3$ is called conformal CMC if s is a non-zero parallel section with respect to the standard connection on $\mathbb{E}$. Willmore surfaces in $\mathbb{Q}^3$ are exactly the surfaces with $s \equiv 0$.

Our main results are as follows:

Theorem 1.1 *Let $x: M \to \mathbb{Q}^3$ be a space-like conformal CMC-surface in $\mathbb{Q}^3$. Then x is conformally equivalent to a constant mean curvature surface in $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$.*

Theorem 1.2 *Let $x: M \to \mathbb{Q}^3$ be a space-like umbilic free Willmore surface whose conformal metric g has constant curvature K. Then either $K = -1$ and x is conformally equivalent to a minimal surface in $\mathbb{R}^3_1$, or $K = 0$ and x is conformally equivalent to the minimal surface $\mathbb{H}^1(\frac{1}{\sqrt{2}}) \times \mathbb{H}^1(\frac{1}{\sqrt{2}})$ in $\mathbb{H}^3_1$.*

We mention that there have been a lot of interesting results concerning Willmore surfaces in $\mathbb{S}^3$ (cf. [2, 4, 7, 8, 11, 12]). These are then generalized to Lorentzian conformal geometry in [1, 6]. Also note that similar results on conformal CMC-surfaces and Willmore surfaces of constant Moebius curvature in $\mathbb{S}^3$ have been obtained (see [9]).

This paper is organized as follows. In Section 2, we introduce the conformal space $\mathbb{Q}^3$ as the common compactification space of the three dimensional Lorentzian space forms. The conformal surfaces theory in $\mathbb{Q}^3$ is established in Section 3. Then we investigate conformal CMC-surfaces and prove Theorem 1.1 in Section 4. Finally, in Section 5, we study Willmore surfaces in $\mathbb{Q}^3$ and prove Theorem 1.2.

2 The Conformal Space $\mathbb{Q}^3$

Let $\mathbb{R}^5_2$ be the space $\mathbb{R}^5$, equipped with the inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4 - x_5y_5. \tag{2.1}$$

Let $\mathbb{Q}^3$ be the quadric in the real projective space $\mathbb{R}P^4$ given by (1.4). We define

$$\mathbb{S}^2 \times \mathbb{S}^1 = \{x \in \mathbb{R}^5_2 \mid x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 = 1\}. \tag{2.2}$$

Then the standard projection $\pi : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{Q}^3$ is a 2-1 covering. For any local section $Z : U \to \mathbb{S}^2 \times \mathbb{S}^1 \subset \mathbb{R}^5_2$ of π defined on an open set $U \subset \mathbb{Q}^3$, we can define a metric

$$\langle dZ, dZ\rangle = dZ_1 \otimes dZ_1 + dZ_2 \otimes dZ_2 + dZ_3 \otimes dZ_3 - dZ_4 \otimes dZ_4 - dZ_5 \otimes dZ_5,$$

which is independent of the choice of Z. It determines a global Lorentzian metric h on $\mathbb{Q}^3$. Moreover, if we define on $\mathbb{S}^2 \times \mathbb{S}^1$ the standard Lorentzian metric $g(S^2) \oplus -g(S^1)$, then the projection $\pi : \mathbb{S}^2 \times \mathbb{S}^1 \to (\mathbb{Q}^3, h)$ is a 2:1 isometric covering. It is known from a theorem of Cahen and Kerbrat (cf. [5]) that the conformal group of $(\mathbb{Q}^3, [h])$ is exactly the orthogonal group $O(3,2)/\{\pm 1\}$, which keeps the inner product (2.1) in $\mathbb{R}^5_2$ invariant and acts on $\mathbb{Q}^3$ by

$$T([x]) = [xT], \quad T \in O(3,2). \tag{2.3}$$

Let $\mathbb{R}^3_1$ be the Lorentzian space equipped with the inner product (1.1). Let $\sigma : \mathbb{R}^3_1 \to \mathbb{Q}^3$ be the conformal embedding of $\mathbb{R}^3_1$ in $\mathbb{Q}^3$ defined by (1.5). We introduce the light-cone at infinity as

$$C_\infty = \{[(-a, u, a)] \in \mathbb{R}P^4 \mid (u, u) = 0, a \in \mathbb{R}\}. \tag{2.4}$$

Then we have

$$\mathbb{Q}^3 = \sigma(\mathbb{R}^3_1) \cup C_\infty. \tag{2.5}$$

Thus $\mathbb{Q}^3$ is a compactification of $\mathbb{R}^3_1$ by attaching the light-cone C_∞ to $\mathbb{R}^3_1$ at infinity.

Let $\gamma = \{u \in \mathbb{R}^3_1 \mid (u-p, u-p) = \varepsilon r^2\}$ be the two-sheet hyperboloid (with $\varepsilon = -1$) or the one-sheet hyperboloid (with $\varepsilon = 1$) in $\mathbb{R}^3_1$ centered at p with radius r. Then γ defines an inversion γ in $\mathbb{R}^3_1$ by

$$\gamma : \mathbb{R}^3_1 \to \mathbb{R}^3_1 \cup C_\infty, \quad \gamma(u) = p + \frac{\varepsilon r^2 (u-p)}{(u-p, u-p)}, \quad u \in \mathbb{R}^3_1.$$

It is a conformal transformation in $\mathbb{R}^3_1$ which fixes every point on γ and takes the light cone centered at p to the cone at infinity. If we define a vector $\gamma \in \mathbb{R}^5_2$ by

$$\gamma = \frac{1}{r}\Big(\frac{1}{2}(1 - (p,p) + \varepsilon r^2), p, \frac{1}{2}(1 + (p,p) - \varepsilon r^2)\Big), \tag{2.6}$$

then $\langle \gamma, \gamma\rangle = \varepsilon$, and γ defines a reflection $T_\gamma \in O(3,2)$:

$$T_\gamma : \mathbb{R}^5_2 \to \mathbb{R}^5_2, \quad T_\gamma(x) = x - 2\varepsilon\langle x, \gamma\rangle\gamma. \tag{2.7}$$

It is straightforward to check the following commuting diagram

$$\sigma \circ \gamma(u) = T_\gamma \circ \sigma(u), \quad u \in \mathbb{R}^3_1. \tag{2.8}$$

Thus all inversions in $\mathbb{R}^3_1 \cup C_\infty$ generate the conformal group $O(3,2)$ in $(\mathbb{Q}^3, [h])$.

Let $\mathbb{R}^4_1$ be the Lorentzian space with inner product

$$(u,v)=u_1v_1+u_2v_2+u_3v_3-u_4v_4. \tag{2.9}$$

Let $\mathbb{S}^3_1=\{x\in\mathbb{R}^4_1\mid(x,x)=1\}$ be the de Sitter space in $\mathbb{R}^4_1$. Let $\tau:\mathbb{S}^3_1\to\mathbb{Q}^3$ be the conformal embedding defined by (1.6). Denote

$$\mathbb{S}^2_\infty=\{[(u,0)]\mid u\in\mathbb{R}^4_1\backslash\{0\},u_1^2+u_2^2+u_3^2=u_4^2\}. \tag{2.10}$$

Then we have

$$\mathbb{Q}^3=\tau(\mathbb{S}^3_1)\cup\mathbb{S}^2_\infty. \tag{2.11}$$

Hence $\mathbb{Q}^3$ is viewed here as a conformal compactification of $\mathbb{S}^3_1$ by attaching a sphere $\mathbb{S}^2_\infty$ at infinity of $\mathbb{R}^4_1$.

Let $\mathbb{R}^4_2$ be the space $\mathbb{R}^4$, equipped with the inner product

$$(u,v)=u_1v_1+u_2v_2-u_3v_3-u_4v_4. \tag{2.12}$$

There is a hyperboloid $\mathbb{H}^3_1=\{x\in\mathbb{R}^4_2\mid(x,x)=-1\}$ in $\mathbb{R}^4_2$, which could also be embedded into $\mathbb{Q}^3$ via $\varrho:\mathbb{H}^3_1\to\mathbb{Q}^3$ (1.7). Set

$$\mathbb{T}^2_\infty=\{[(0,u)]\mid u\in\mathbb{R}^4_2\backslash\{0\},u_1^2+u_2^2=u_3^2+u_4^2\}. \tag{2.13}$$

It follows that

$$\mathbb{Q}^3=\varrho(\mathbb{H}^3_1)\cup\mathbb{T}^2_\infty. \tag{2.14}$$

So $\mathbb{Q}^3$ is also a conformal compactification of $\mathbb{H}^3_1$ by attaching a torus $\mathbb{T}^2_\infty$ at infinity of $\mathbb{R}^4_2$.

We note that $\mathbb{R}^3_1$, $\mathbb{S}^3_1$ and $\mathbb{H}^3_1$ are the Lorentzian space forms of curvature 0, 1 and -1, respectively. Thus $\mathbb{Q}^3$ is the common conformal compactification of the Lorentzian space forms.

3 Conformal Geometry of Surfaces in $\mathbb{Q}^3$

In this section we study the geometry of surfaces in $\mathbb{Q}^3$ under the conformal group $O(3,2)$. Two surfaces in $\mathbb{Q}^3$ (or in $\mathbb{R}^3_1,\mathbb{S}^3_1,\mathbb{H}^3_1\subset\mathbb{Q}^3$) are said to be conformally equivalent if there exists $T\in O(3,2)$ taking one surface to the other.

Consider a space-like surface $x:M\to\mathbb{Q}^3$. That means x^*h, the pull-back of the standard Lorentzian metric on $\mathbb{Q}^3$, is positive difinite over M. Let $y:U\to\mathbb{R}^5_2\backslash\{0\}$ be a local lift of x with $x=[y]$ defined on an open set U of M. By the definition of h we can find a positive function λ on U such that $x^*h=\lambda^2\langle dy,dy\rangle$. Since x is space-like, we can write

$$\langle dy,dy\rangle=\frac{1}{2}e^{2\rho}(dz\otimes d\overline{z}+d\overline{z}\otimes dz) \tag{3.1}$$

for a local complex coordinate $\{z\}$. From (3.1) and the fact that $\langle y,y\rangle=0$, we get

$$\langle y,y_z\rangle=\langle y,y_{\overline{z}}\rangle=0,\quad \langle y_z,y_z\rangle=0,\quad \langle y_z,y_{\overline{z}}\rangle=\frac{1}{2}e^{2\rho}=-\langle y,y_{z\overline{z}}\rangle. \tag{3.2}$$

It follows that

$$\mathbb{V} = \mathrm{span}\{y, y_{z\overline{z}}\} \oplus \{\overline{\alpha} y_z + \alpha y_{\overline{z}} \mid \alpha \in \mathbb{C}\} \tag{3.3}$$

is a 4-dimensional non-degenerate subspace in $\mathbb{R}^5_2$ of signature $(+,-,+,+)$. It is easy to verify that $\mathbb{V}$ is independent to the choice of local life y and complex coordinate z. So there is a well-defined map $\xi : M \to \mathbb{R}^5_2$ (up to signs) such that

$$\xi \perp \mathbb{V}, \quad \mathbb{R}^5_2 = \mathbb{V} \oplus \mathbb{R}\xi, \quad \langle \xi, \xi \rangle = -1. \tag{3.4}$$

Denote

$$\mathbb{H}^4_1 = \{X \in \mathbb{R}^5_2 \mid \langle X, X \rangle = -1\} \subset \mathbb{R}^5_2. \tag{3.5}$$

We call $\xi : M \to \mathbb{H}^4_1 \subset \mathbb{R}^5_2$ the conformal Gauss map of x.

Denote $\langle y_{zz}, \xi \rangle = \Omega$. It is essentially the Hopf differential of the surface x, yet is dependent on the choice of y and z. Nevertheless, the vanishing of Ω at a point p is a well-defined property, which is also invariant under the action of the conformal group $O(3,2)$. Such points are exactly the umbilical points of x. By (3.3) and (3.4) we have

$$\begin{aligned} &\langle \xi_z, \xi \rangle = 0, && \langle \xi_z, y \rangle = -\langle \xi, y_z \rangle = 0, \\ &\langle \xi_z, y_z \rangle = -\Omega, && \langle \xi_z, y_{\overline{z}} \rangle = -\langle \xi, y_{z\overline{z}} \rangle = 0. \end{aligned}$$

Since $\{y, y_{z\overline{z}}, y_z, y_{\overline{z}}, \xi\}$ is a local frame in $\mathbb{R}^5_2$, we get

$$\xi_z = \beta y - 2e^{-2\rho}\Omega y_{\overline{z}} \tag{3.6}$$

for some function β. It follows that

$$\langle \xi_z, \xi_z \rangle = 0, \quad \langle \xi_z, \xi_{\overline{z}} \rangle = 2e^{-2\rho}|\Omega|^2. \tag{3.7}$$

So $\xi : M \to \mathbb{H}^4_1$ is a conformal immersion at any non-umbilical point of x, and hence shares the same complex coordinate z. This justifies the name of the conformal Gauss map.

From now on, suppose $x : M \to \mathbb{Q}^3$ is a umbilic free surface. Then $\xi : M \to \mathbb{H}^4_1$ induces an invariant metric $g = \langle d\xi, d\xi \rangle$, which is called the conformal metric of x. Write

$$g = \langle d\xi, d\xi \rangle = \frac{1}{2} e^{2\omega}(dz \otimes d\overline{z} + d\overline{z} \otimes dz). \tag{3.8}$$

From (3.7) we get

$$e^{2\omega} = 4e^{-2\rho}|\Omega|^2. \tag{3.9}$$

Since

$$\mathbb{W} = \{\overline{\alpha}\xi_z + \alpha\xi_{\overline{z}} \mid \alpha \in \mathbb{C}\} \oplus \mathbb{R}\xi \tag{3.10}$$

is a 3-dimensional subspace in $\mathbb{R}^5_2$ with signature $(+,+,-)$, we know that $\mathbb{W}^{\perp}$ is a 2-dimensional subspace in $\mathbb{R}^5_2$ with a non-degenerate metric of signature $(+,-)$. By (3.4) and (3.6) we have $y \in \mathbb{W}^{\perp}$. So there exists a unique $y^* \in \mathbb{W}^{\perp}$ such that $\mathbb{W}^{\perp} = \mathrm{span}\{y, y^*\}$ and

$$\langle y, y \rangle = \langle y^*, y^* \rangle = 0, \quad \langle y^*, y \rangle = 1. \tag{3.11}$$

Hence $\{y, y^*, \xi_z, \xi_{\overline{z}}, \xi\}$ is a conformally invariant moving frame in $\mathbb{R}^5_2$, which depends on the choice of $\{y, z\}$. Define

$$\varphi = \langle y_z, y^* \rangle, \quad \mathcal{H} = e^{-2\omega}\langle \xi_{z\overline{z}}, y^* \rangle, \quad \Omega = \langle \xi_{zz}, y \rangle, \quad \Omega^* = \langle \xi_{zz}, y^* \rangle. \tag{3.12}$$

Taking derivatives of the frame $\{y, y^*, \xi_z, \xi_{\overline{z}}, \xi\}$ yields

$$y_z = \varphi y - 2e^{-2\omega}\Omega\xi_{\overline{z}}, \tag{3.13}$$
$$y^*_z = -\varphi y^* - 2\mathcal{H}\xi_z - 2e^{-2\omega}\Omega^*\xi_{\overline{z}}, \tag{3.14}$$
$$\xi_{zz} = \Omega^* y + \Omega y^* + 2\omega_z\xi_z, \tag{3.15}$$
$$\xi_{z\overline{z}} = e^{2\omega}\mathcal{H}y + \frac{1}{2}e^{2\omega}\xi. \tag{3.16}$$

Using the identities $y_{z\overline{z}} = y_{\overline{z}z}$, $y^*_{z\overline{z}} = y^*_{\overline{z}z}$ and $\xi_{zz\overline{z}} = \xi_{z\overline{z}z}$, we get the integrability conditions as follows:

$$\varphi_{\overline{z}} - \overline{\varphi}_z = 2e^{-2\omega}(\Omega\overline{\Omega}^* - \overline{\Omega}\Omega^*), \tag{3.17}$$
$$\Omega_{\overline{z}} = \overline{\varphi}\Omega, \tag{3.18}$$
$$\Omega^*_{\overline{z}} + \overline{\varphi}\Omega^* = e^{2\omega}(\mathcal{H}_z + \varphi\mathcal{H}), \tag{3.19}$$
$$4e^{-4\omega}(\Omega\overline{\Omega}^* + \overline{\Omega}\Omega^*) = -(1+K). \tag{3.20}$$

Here $K = -4e^{-2\omega}\omega_{z\overline{z}}$ is the Gauss curvature of the conformal metric g.

Now Let us make it clear how these quantities depends on the choice of $\{y, z\}$. Let $\{\widetilde{y}, \tau\}$ be another pair of local lift and complex coordinate defined on V. Then on $U \cap V$ holds

$$\widetilde{y} = \lambda y, \quad \tau = \tau(z)$$

for some non-zero real function λ and a holomorphic function $\tau(z)$, which implies that

$$\widetilde{y}^* = \lambda^{-1}y^*, \quad e^{2\widetilde{\omega}} = e^{2\omega}\left|\frac{dz}{d\tau}\right|^2.$$

It follows that

$$\widetilde{\varphi} = (\varphi + \lambda^{-1}\lambda_z)\frac{dz}{d\tau}, \quad \widetilde{\mathcal{H}} = \lambda^{-1}\mathcal{H}, \quad \widetilde{\Omega} = \lambda\Omega\Big(\frac{dz}{d\tau}\Big)^2, \quad \widetilde{\Omega}^* = \lambda^{-1}\Omega^*\Big(\frac{dz}{d\tau}\Big)^2. \tag{3.21}$$

Thus we know that the complex function F and the complex 4-form Φ given by

$$F = e^{-4\omega}\Omega\overline{\Omega}^*, \quad \Phi = \Omega\Omega^* dz^4 \tag{3.22}$$

are independent of the choice of $\{y, z\}$, which are globally defined conformal invariants on M. From (3.21) we get

$$\widetilde{\mathcal{H}}_\tau + \widetilde{\varphi}\widetilde{\mathcal{H}} = \lambda^{-1}(\mathcal{H}_z + \varphi\mathcal{H})\frac{dz}{d\tau}. \tag{3.23}$$

Thus the equation $\mathcal{H}_z + \varphi\mathcal{H} = 0$ is conformally invariant.

Next consider the canonical line bundle $\pi : \mathbb{E} \to M$ with $\pi^{-1}(p) = \mathbb{R}y$, where $[y] = x(p)$. We define a standard connection ∇ on $\mathbb{E}$ by

$$\nabla y = (\varphi dz + \overline{\varphi}d\overline{z})y. \tag{3.24}$$

Using (3.21) we can easily verify that ∇ is independent of the choice of $\{y, z\}$, and hence is globally defined on $\mathbb{E}$. Note that $s = e^{-2\omega}\langle \xi_{z\bar{z}}, y^*\rangle y = \mathcal{H}y$ is a global section of $\mathbb{E}$, and

$$\nabla s = ((\mathcal{H}_z + \varphi\mathcal{H})dz + (\mathcal{H}_{\bar{z}} + \overline{\varphi}\mathcal{H})d\bar{z})y.$$

We call a surface in $\mathbb{Q}^3$ a conformal CMC (constant mean curvature) surface if s is a nonzero parallel section of $(\mathbb{E}, \nabla)$, or equivalently, if

$$\mathcal{H}_z + \varphi\mathcal{H} = 0, \quad \mathcal{H} \neq 0. \tag{3.25}$$

We observe that according to (3.16), $s = \mathcal{H}y$ is nothing else but the mean curvature vector of the immersion $\xi : M \to \mathbb{H}^4_1$. This is part of the reason why conformal CMC surfaces are so called. On the other hand, the condition that $\mathcal{H} = 0$ characterizes $\xi : M \to \mathbb{H}^4_1$ as a harmonic map, which amounts to say that $x : M \to \mathbb{Q}^3$ is a Willmore surface (cf. [1]). Such surfaces are exactly the critical surfaces with respect to the induced area

$$W(M) = \frac{i}{2}\int_M e^{2\omega} dz \wedge d\bar{z}. \tag{3.26}$$

This conformally invariant functional is viewed as the generalization of the usual Willmore functional of a surface in $\mathbb{S}^3$ (cf. [1, 4]).

4 Conformal CMC-Surfaces in Lorentzian Space Forms

In this section, we study the relationship between the geometry of surfaces in the Lorentzian space forms and the conformal geometry of surfaces in $\mathbb{Q}^3$.

Let $f : M \to \mathbb{R}^3_1$ be a space-like surface in $\mathbb{R}^3_1$. Let z be a complex coordinate of f, such that

$$(df, df) = \frac{1}{2}e^{2\rho}(dz \otimes d\bar{z} + d\bar{z} \otimes dz), \tag{4.1}$$

where $(\ ,\ )$ is the Lorentzian inner product in $\mathbb{R}^3_1$ given in (1.1). Let n be the unit normal of f in $\mathbb{R}^3_1$ with $(n, n) = -1$. The structure equations of $f : M \to \mathbb{R}^3_1$ is given by

$$f_{zz} = 2\rho_z f_z - \Omega n, \quad f_{z\bar{z}} = -\frac{1}{2}e^{2\rho}Hn, \quad n_z = -Hf_z - 2\Omega e^{-2\rho} f_{\bar{z}}, \tag{4.2}$$

where $\Omega dz^2 = (f_{zz}, n)dz^2$ is the Hopf-form of f and $H = \frac{k_1+k_2}{2}$ is the mean curvature of f. We have the relation $\|\Omega\|^2 = e^{-4\rho}|\Omega|^2 = \frac{(k_1-k_2)^2}{16}$, where k_1 and k_2 are principal curvatures of f.

Let $\sigma : \mathbb{R}^3_1 \to \mathbb{Q}^3$ be the conformal embedding defined by (1.5). Then

$$x = \sigma \circ f = \Big[\Big(\frac{1}{2}(1 - (f, f)), f, \frac{1}{2}(1 + (f, f))\Big)\Big] : M \to \mathbb{Q}^3 \tag{4.3}$$

is a surface in $\mathbb{Q}^3$. There is a standard lift of x given by

$$y = \Big(\frac{1}{2}(1 - (f, f)), f, \frac{1}{2}(1 + (f, f))\Big), \tag{4.4}$$

which satisfies $\langle dy, dy\rangle = (df, df)$. It follows from (4.2) that under a given coordinate z,

$$\begin{aligned}
y_z &= (-(f_z, f), f_z, (f_z, f)),\\
y_{zz} &= 2\rho_z(-(f_z, f), f_z, (f_z, f)) - \Omega(-(n, f), n, (n, f)),\\
y_{z\bar{z}} &= -\frac{1}{2}e^{2\rho}H(-(n, f), n, (n, f)) - \frac{1}{2}e^{2\rho}(1, \mathbf{0}, -1).
\end{aligned}$$

Since the conformal Gauss map $\xi \perp \{y, y_z, y_{\overline{z}}, y_{z\overline{z}}\}$ and $\langle \xi, \xi \rangle = -1$, we get

$$\xi = H\Big(\frac{1}{2}(1-(f,f)), f, \frac{1}{2}(1+(f,f))\Big) + (-(n,f), n, (n,f)), \tag{4.5}$$

which implies that

$$\langle y_{zz}, \xi \rangle = (f_{zz}, n) = \Omega.$$

Thus $p \in M$ is an umbilical point of $x = \sigma \circ f$ if and only if p is an umbilical point for $f : M \to \mathbb{R}^3_1$, i.e., $k_1(p) = k_2(p)$.

Now let $\mathbf{c}$ be the light-like vector in $\mathbb{R}^5_2$ given by

$$\mathbf{c} = (1, \mathbf{0}, -1), \quad \mathbf{0} \in \mathbb{R}^3_1, \quad \langle \mathbf{c}, \mathbf{c} \rangle = 0. \tag{4.6}$$

By (4.4) and (4.5) we get

$$\langle y, \mathbf{c} \rangle = 1, \quad \langle \xi, \mathbf{c} \rangle = H. \tag{4.7}$$

It follows from (3.16), (4.7) and (3.9) that

$$\mathcal{H} = e^{-2\omega} H_{z\overline{z}} - \frac{1}{2} H = \frac{1}{16\|\Omega\|^2}(\Delta H - 8\|\Omega\|^2 H), \tag{4.8}$$

where Δ is the Laplacian operator of the metric (df, df). We note that $\mathcal{H} = 0$ is exactly the Euler-Lagrange equation for Willmore functional of surfaces in $\mathbb{R}^3_1$. Thus $\mathcal{H} = 0$ if and only if $f : M \to \mathbb{R}^3_1$ is a Willmore surface in $\mathbb{R}^3_1$.

If $f : M \to \mathbb{R}^3_1$ is a constant mean curvature surface with $H \neq 0$, then we know from (4.8) that $\mathcal{H} = -\frac{1}{2} H \neq 0$ is also a constant. From (3.13) and (4.7) we get $\varphi = 2e^{-2\omega}\Omega H_{\overline{z}} = 0$. Thus $x = \sigma \circ f : M \to \mathbb{Q}^3$ satisfies $\mathcal{H}_z + \varphi \mathcal{H} = 0$. Thus any CMC-surface in $\mathbb{R}^3_1$ is a conformal CMC-surface in $\mathbb{Q}^3$ (under the conformal embedding $\sigma : \mathbb{R}^3_1 \to \mathbb{Q}^3$).

When the ambient space is $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$, we have similar conclusions. For a space-like surface $f : M \to \mathbb{S}^3_1 \subset \mathbb{R}^4_1$, let n be the normal vector of f with $(n, n) = -1$, and H be its mean curvature. Using the embedding $\tau : \mathbb{S}^3_1 \to \mathbb{Q}^3$ defined by (1.6), we get a surface $x = [(f, 1)] : M \to \mathbb{Q}^3$ with the lift $y = (f, 1)$, whose conformal Gauss map is computed out as

$$\xi = H(f, 1) + (n, 0). \tag{4.9}$$

For the time-like vector $\mathbf{c} \in \mathbb{R}^5_2$ given by

$$\mathbf{c} = (\mathbf{0}, -1), \quad \mathbf{0} \in \mathbb{R}^4_1, \tag{4.10}$$

we have

$$\langle \mathbf{c}, \mathbf{c} \rangle = -1, \quad \langle y, \mathbf{c} \rangle = 1, \quad \langle \xi, \mathbf{c} \rangle = H. \tag{4.11}$$

In case of a space-like surface $f : M \to \mathbb{H}^3_1$ we can do almost the same, with

$$\xi = H(1, f) + (0, n), \quad \mathbf{c} = (1, \mathbf{0}), \quad \mathbf{0} \in \mathbb{R}^4_2, \tag{4.12}$$

$$\langle \mathbf{c}, \mathbf{c} \rangle = 1, \quad \langle y, \mathbf{c} \rangle = 1, \quad \langle \xi, \mathbf{c} \rangle = H. \tag{4.13}$$

It is straightforward to verify that any CMC-surface in $\mathbb{S}^3_1$ or $\mathbb{H}^3_1$ is a conformal CMC-surface in $\mathbb{Q}^3$ (under the conformal embedding τ or ϱ, respectively).

Now we come to the proof of Theorem 1.1. Let $x : M \to \mathbb{Q}^3$ be a conformal CMC-surface. Then by (3.25) we have $\mathcal{H}_z + \varphi\mathcal{H} = 0$ and $\mathcal{H} \neq 0$. Since

$$\widetilde{y} = \mathcal{H}y = e^{-2\omega}\langle \xi_{z\overline{z}}, y^*\rangle y$$

is independent of the choice of $\{y, z\}$, it defines a global lift of x. Moreover,

$$\widetilde{\varphi} = \langle \widetilde{y}_z, \widetilde{y}^*\rangle = \left\langle (\mathcal{H}y)_z, \frac{1}{\mathcal{H}}y^*\right\rangle = \frac{1}{\mathcal{H}}(\mathcal{H}_z + \varphi\mathcal{H}) = 0.$$

Since $\widetilde{\mathcal{H}}\widetilde{y} = \mathcal{H}y$, we get $\widetilde{\mathcal{H}} = 1$. Thus, by taking new $y = \widetilde{y}$ as the lift of x if necessary, we may assume that $\varphi = 0$ and $\mathcal{H} = 1$. It follows from (3.18) and (3.19) that both Ωdz^2 and $\Omega^* dz^2$ are holomorphic 2-forms on M. By (3.17) we get $\Omega^*/\Omega = \overline{\Omega^*/\Omega}$, which implies that $\Omega^* = \mu\Omega$ for some real constant μ. From (3.13) and (3.14) we get

$$y^* - \mu y + 2\xi = \mathbf{c} \tag{4.14}$$

for some constant vector $\mathbf{c} \in \mathbb{R}^5_2$. It follows that

$$\langle y, \mathbf{c}\rangle = 1, \quad \langle \xi, \mathbf{c}\rangle = -2, \quad \langle \mathbf{c}, \mathbf{c}\rangle = -2\mu - 4. \tag{4.15}$$

First we consider the case that $\mu = -2$. Then $\mathbf{c}$ is a light-like vector in $\mathbb{R}^5_2$. Taking a conformal transformation $T \in O(3,2)$, if necessary, we may assume that $\mathbf{c} = (1, \mathbf{0}, -1) \in \mathbb{R}^5_2$. Since $\langle y, \mathbf{c}\rangle = 1$, we can write y by (4.4) for some $f : M \to \mathbb{R}^3_1$. From the fact that $\langle dy, dy\rangle = (df, df)$, we know that f is an immersion, and the conformal Gauss map ξ is given by (4.5). It follows from (4.7) and (4.15) that $H = -2$. Thus $f : M \to \mathbb{R}^3_1$ is a constant mean curvature surface in $\mathbb{R}^3_1$.

Next we consider the case that $\mu = -2 + \frac{1}{2}r^2$ for some $r > 0$. Then $\mathbf{c}$ is a time-like vector in $\mathbb{R}^5_2$. Taking a conformal transformation $T \in O(3,2)$, if necessary, we may assume that $\mathbf{c} = r(\mathbf{0}, -1) \in \mathbb{R}^5_2$. Since $\langle y, \mathbf{c}\rangle = 1$, we can write $y = r^{-1}(f, 1)$ for some $f : M \to \mathbb{S}^3_1$. From the fact that $\langle dy, dy\rangle = r^{-2}(df, df)$, we know that f is an immersion, and the conformal Gauss map ξ is given by (4.9). It follows from (4.11) and (4.15) that $H = -2r^{-1}$. Thus $f : M \to \mathbb{S}^3_1$ is a constant mean curvature surface in $\mathbb{S}^3_1$.

Finally we consider the case that $\mu = -2 - \frac{1}{2}r^2$ for some $r > 0$. Then $\mathbf{c}$ is a space-like vector in $\mathbb{R}^5_2$. By using a conformal transformation $T \in O(3,2)$, if necessary, we may assume that $\mathbf{c} = r(1, \mathbf{0}) \in \mathbb{R}^5_2$. Since $\langle y, \mathbf{c}\rangle = 1$, we can write $y = r^{-1}(1, f)$ for some $f : M \to \mathbb{H}^3_1$. From the fact that $\langle dy, dy\rangle = r^{-2}(df, df)$, we know that f is an immersion, and the conformal Gauss map ξ is given by (4.12). It follows from (4.13) that $H = -2r^{-1}$. Thus $f : M \to \mathbb{H}^3_1$ is a constant mean curvature surface in $\mathbb{H}^3_1$.

Thus we complete the proof of Theorem 1.1.

5 Willmore Surfaces of Constant Curvature in $\mathbb{Q}^3$

In this section, we study Willmore surfaces in $\mathbb{Q}^3$ and prove Theorem 1.2.

Let $x : M \to \mathbb{Q}^3$ be a Willmore surface of constant curvature K. Then $\mathcal{H} = 0$. From (3.18) and (3.19) we get

$$(\Omega\Omega^*)_{\overline{z}} = \Omega_{\overline{z}}\Omega^* + \Omega\Omega^*_{\overline{z}} = 0. \tag{5.1}$$

Thus

$$\Omega\Omega^* dz^4 = \langle \xi_{zz}, y\rangle\langle \xi_{zz}, y^*\rangle dz^4 \tag{5.2}$$

is a globally defined holomorphic 4-form on M. We know that either $\Omega^* \equiv 0$ or $\Omega^* \neq 0$ on M (except at some isolated points).

First we consider the case that $\Omega^* \equiv 0$. By (3.17) we get $\varphi_{\overline{z}} = \overline{\varphi}_z$, which implies that $d(\varphi dz + \overline{\varphi} d\overline{z}) = 0$. Thus we can find $\lambda > 0$ such that $\varphi dz + \overline{\varphi} d\overline{z} = -d\log\lambda$. It follows from (3.24) that $\nabla(\lambda y) = 0$. Taking $\widetilde{y} = \lambda y$, we have $\widetilde{\varphi} = 0$. By changing y to $\widetilde{y}$, if necessary, we may assume that $\varphi = 0$. It follows from (3.14) that y^* is a constant light-like vector in $\mathbb{R}^5_2$. By making a conformal transformation $T \in O(3,2)$, if necessary, we may assume that $y^* = \mathbf{c} = (1, \mathbf{0}, -1)$. Since $\langle y, \mathbf{c}\rangle = 1$, we can write y as (4.4) for some surface $f : M \to \mathbb{R}^3_1$. It follows from (4.7) that

$$H = \langle \xi, \mathbf{c}\rangle = \langle \xi, y^*\rangle = 0.$$

Thus $f : M \to \mathbb{R}^3_1$ is a minimal surface in $\mathbb{R}^3_1$. Since $\Omega^* = 0$, we get from (3.20) that $K = -1$.

Now we consider the case that $\Omega^* \neq 0$. Then $F = e^{-4\omega}\Omega\overline{\Omega}^* \neq 0$ is a globally defined complex function on M. We write

$$F = |F|e^{i\psi} = |F|(\cos\psi + i\sin\psi). \tag{5.3}$$

Let Δ be the Laplacian operator of g. Using (3.17)–(3.19) we get

$$\Delta \log F = 4e^{-2\omega}(\log F)_{z\overline{z}} = 4K - 16i|F|\sin\psi. \tag{5.4}$$

It follows from (5.4) and (3.20) that

$$\Delta \log |F| = 4K, \tag{5.5}$$

$$\Delta\psi = -16|F|\sin\psi, \tag{5.6}$$

$$8|F|\cos\psi = -(1+K). \tag{5.7}$$

First we show that ψ must be a constant. Assume, on the contrary, that ψ is not a constant, then we can find an open set U of M such that $\cos\psi \neq 0$ and $\psi_z \neq 0$ on U. In the following, we consider $\psi : U \to \mathbb{R}$. From (5.7) we have

$$|F| = -\frac{1}{8}\frac{(1+K)}{\cos\psi}. \tag{5.8}$$

It follows from (5.6) and (5.8) that

$$\psi_{z\overline{z}} = -4e^{2\omega}|F|\sin\psi = \frac{1}{2}(1+K)e^{2\omega}\tan\psi. \tag{5.9}$$

By (5.5) and (5.8) we get

$$e^{2\omega}K = (\log|F|)_{z\overline{z}} = -(\log|\cos\psi|)_{z\overline{z}} = \psi_{z\overline{z}}\tan\psi + \frac{\psi_z\psi_{\overline{z}}}{\cos^2\psi}. \tag{5.10}$$

From (5.9) and (5.10) we get

$$0 < e^{-2\omega}\psi_z\psi_{\overline{z}} = K\cos^2\psi - \frac{1+K}{2}\sin^2\psi = \lambda(\psi). \tag{5.11}$$

Using (5.9) and (5.11) we can eliminate ω and obtain

$$\psi_{z\overline{z}} = \frac{1}{2}(1+K)\frac{\tan\psi}{\lambda(\psi)}\psi_z\psi_{\overline{z}}. \tag{5.12}$$

Let $\eta = \frac{\psi_z}{\sqrt{\lambda}}$. Then by (5.12) we get

$$\eta_{\overline{z}} = \frac{1}{2\lambda(\psi)}((1+K)\tan\psi + (1+3K)\sin\psi\cos\psi)\psi_{\overline{z}}\eta. \tag{5.13}$$

Let $\mu = \mu(\psi)$ be a solution of the ODE

$$\frac{d\mu}{d\psi} = \frac{1}{2\lambda(\psi)}((1+K)\tan\psi + (1+3K)\sin\psi\cos\psi). \tag{5.14}$$

Then we get from (5.13) that

$$\eta_{\overline{z}} = \frac{d\mu}{d\psi}\psi_{\overline{z}}\eta = \mu_{\overline{z}}\eta, \quad (e^{-\mu}\eta)_{\overline{z}} = 0. \tag{5.15}$$

It follow that $\eta = e^{\mu}\eta_0$ for some non-zero holomorphic function η_0. By (5.11) we have $e^{2\omega} = |\eta|^2$. Thus we get from (5.15) that

$$2\omega_{z\overline{z}} = (\log|\eta|^2)_{z\overline{z}} = (2\mu + \log|\eta_0|^2)_{z\overline{z}} = 2\mu_{\overline{z}z} = 2\frac{d}{dz}\Big(\frac{\eta_{\overline{z}}}{\eta}\Big).$$

Thus we get from (5.13) that

$$2\omega_{z\overline{z}} = \frac{d}{dz}\Big(\frac{1}{\lambda(\psi)}((1+K)\tan\psi + (1+3K)\sin\psi\cos\psi)\psi_{\overline{z}}\Big). \tag{5.16}$$

It follows from (5.16), (5.11) and (5.12) that

$$\begin{aligned} K = -4e^{-2\omega}\omega_{z\overline{z}} = &-(1+K)((1+K)\tan^2\psi + (1+3K)\sin^2\psi)\frac{1}{\lambda(\psi)} \\ &- 2(1+3K)((1+K)\sin^2\psi + (1+3K)\sin^2\psi\cos^2\psi)\frac{1}{\lambda(\psi)} \\ &- 2\Big(\frac{1+K}{\cos^2\psi} + (1+3K)\cos 2\psi\Big), \end{aligned} \tag{5.17}$$

where $\lambda(\psi)$ is defined by (5.11). Let $t = \tan\frac{\psi}{2}$. Then we have

$$\cos\psi = \frac{1-t^2}{1+t^2}, \quad \sin\psi = \frac{2t}{1+t^2}, \quad \tan\psi = \frac{2t}{1-t^2}. \tag{5.18}$$

From (5.17) and (5.18) we know that $t = \tan\frac{\psi}{2}$ satisfies a non-trivial polynomial with constant coefficients. Thus t and ψ is a constant, which contradicts our assumption that $\psi_z \neq 0$. We conclude that ψ must be a constant.

Now we consider the case that ψ is a constant. By (5.6) we get $\sin\psi = 0$ and $\cos\psi = \pm 1$. Thus (5.7) implies that $8|F| = \pm(1+K)$ is a constant. By (5.5) we get $K = 0$. Thus $F = |F|\cos\psi = -\frac{1}{8}$. Since

$$F = e^{-4\omega}\Omega\overline{\Omega}^* = e^{-4\omega}|\Omega|^2\overline{\Omega^*/\Omega} = -\frac{1}{8},$$

we know that $\Omega^*/\Omega = \overline{\Omega^*/\Omega}$ is a negative function. If we change y to λy, then Ω^*/Ω will change to $\lambda^{-2}\Omega^*/\Omega$. Thus, by changing y to $\widetilde{y} = \lambda y$ for some suitable λ, if necessary, we may assume that $\Omega^*/\Omega = -1$. Since $\Omega^*\Omega$ is holomorphic and $\Omega^* = -\Omega$, we know that both Ω and Ω^* are holomorphic. It follows from (3.18) that $\varphi = 0$. Since $K = 0$, we can choose z such that $g = |dz|^2$ and $\omega = 0$. These properties are preserved under the coordinate change $z \to e^{i\theta}z$ for constant θ. Since $F = \Omega\overline{\Omega}^* = -\frac{1}{8}$, we get $|\Omega|^2 = |\Omega^*|^2 = \frac{1}{8}$. Since $\Omega^* = -\Omega$ is holomorphic, we know that both Ω and Ω^* are constant. If we change z to $e^{i\theta}z$ for some constant θ, then Ω will change to $e^{-2i\theta}\Omega$. Thus we may assume that Ω is a positive real number, which implies that $\Omega^* = -\Omega = -\frac{\sqrt{2}}{4}$. It follows from (3.13) and (3.14) that $y + y^* = \mathbf{c}$ for some constant vector $\mathbf{c} \in \mathbb{R}^5_2$. Since $\langle \mathbf{c}, \mathbf{c} \rangle = 2$, by making a conformal transformation $T \in O(3,2)$, if necessary, we may assume that $\mathbf{c} = \sqrt{2}(1, \mathbf{0})$. Since $\langle \mathbf{c}, y \rangle = 1$, We can write $y = \frac{1}{\sqrt{2}}(1, f)$ for some surface $f : M \to \mathbb{H}^3_1$. Since $\langle \mathbf{c}, \xi \rangle = 0$, we get from (4.13) $H = 0$ and $\xi = (0, n)$. We write $z = u + iv$. From (3,13) we get $y_z = -\frac{1}{\sqrt{2}}\xi_{\overline{z}}$. It follows that

$$f_u = -n_u, \quad f_v = n_v. \tag{5.19}$$

Thus the principal curvatures of f are given by $k_1 = 1$ and $k_2 = -1$. Thus $f : M \to \mathbb{H}^3_1$ is an open part of the surface

$$\mathbb{H}^1\left(\frac{1}{\sqrt{2}}\right) \times \mathbb{H}^1\left(\frac{1}{\sqrt{2}}\right) = \left\{ x \in \mathbb{R}^4_2 \,\middle|\, x_1^2 - x_3^2 = x_2^2 - x_4^2 = -\frac{1}{2} \right\} \subset \mathbb{H}^3_1.$$

Thus we complete the proof of Theorem 1.2.

References

[1] Alias, L. J. and Palmer, B., Conformal geometry of surfaces in Lorentzian space forms, *Geometriae Dedicata*, **60**, 1996, 301–315.

[2] Babich, M. and Bobenko, A., Willmore tori with umbilic lines and minimal surfaces in hyperbolic space, *Duke Math. J.*, **72**, 1993, 151–185.

[3] Blaschke, W., Vorlesungen über Differentialgeometrie, Vol. 3, Springer, Berlin, 1929.

[4] Bryant, R. L., A duality theorem for Willmore surfaces, *J. Diff. Geom.*, **20**, 1984, 23–53.

[5] Cahen, M. and Kerbrat, Y., Domaines symetriques des quadriques projectives, *J. Math. Pures et Appl.*, **62**, 1983, 327–348.

[6] Deng, Y. J. and Wang, C. P., Willmore surfaces in Lorentzian space, *Sci. China Ser. A*, **35**, 2005, 1361–1372.

[7] Hertrich-Jeromin, U., Introduction to Möbius Differential Geometry, London Math. Soc. Lecture Note Series, Vol. 300, Cambridge University Press, Cambridge, 2003.

[8] Hertrich-Jeromin, U. and Pinkall, U., Ein Beweis der Willmoreschen Vermutung für Kanaltori, *J. Reihe Angew. Math.*, **430**, 1992, 21–34.

[9] Ma, X. and Wang, C. P., Willmore surfaces of constant Moebius curvature, *Annals of Global Analysis and Geometry,* to appear.

[10] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Pure and Applied Mathematics, **103**, Academic Press, New York, 1983.

[11] Pinkall, U., Hopf tori in S^3, *Invent. Math.*, **81**, 1985, 379–386.

[12] Pinkall, U. and Sterling, I., Willmore surfaces, *Math. Intell.*, **9**, 1987, 38–43.