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On the Collision Local Time of Fractional Brownian Motions***

Yiming JIANG* Yongjin WANG**

Abstract In this paper, the existence and smoothness of the collision local time are proved for two independent fractional Brownian motions, through L^2 convergence and Chaos expansion. Furthermore, the regularity of the collision local time process is studied.

Keywords Collision local time, Fractional Brownian motion, Chaos expansion, Hölder continuity
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1 Introduction

Let $B^{H_i} = \{B_t^{H_i}, t \geq 0\}$ $(i \in \{1, 2\})$ be two independent fractional Brownian motions with respective Hurst parameters on the same probability space $\{\Omega, \mathcal{F}, P\}$. For each $i = 1, 2$, B^{H_i} is a centered Gaussian process with covariance

$$R_{H_i}(t, s) := E(B_t^{H_i} B_s^{H_i}) = \frac{1}{2}(t^{2H_i} + s^{2H_i} - |t - s|^{2H_i}) \quad \text{for } s, t \geq 0. \tag{1.1}$$

Here we are interested in the so-called collision local time, which might be formally expressed as follows, i.e., for $T > 0$,

$$I(H_1, H_2, T) = \int_0^T \delta(B_t^{H_1} - B_t^{H_2}) dt, \tag{1.2}$$

where $\delta(\cdot)$ denotes the Dirac delta function on $\mathbb{R}$, i.e., $\int_{\mathbb{R}} \delta(x) f(x) dx = f(0)$.

First of all, we would mention some works, which motivated our present consideration. In [1], Rosen proved the existence of normalized intersection local time of fractional Brownian motion in the plane. On the other hand, since the Chaos expansion was first brought into the study of local times by Nualart and Vive [2], it has been extensively employed to study the self-intersection of fractional Brownian motion by many authors (see e.g. [3–6]). In this paper, we are going to develop this technique and establish the existence of the collision local time $I(H_1, H_2, T)$ for two independent fractional Brownian motions. Furthermore, the smoothness and the regularity of $I(H_1, H_2, T)$ will be demonstrated under some restrictive conditions.

The paper is organized as follows. In Section 2, we will introduce the Chaos expansion. The main results on the collision local time of two independent fractional Brownian motions will be

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*Corresponding author. LPMC, School of Mathematical Sciences, Nankai University, Tianjin 300071, China.
E-mail: ymjiangnk@yahoo.com.cn
**LPMC, School of Mathematical Sciences, Nankai University, Tianjin 300071, China.
E-mail: yjwang@nankai.edu.cn
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given in Section 3 as well as their proofs. In Section 4, we will further discuss the regularity of the collision local time process $\{I(H_1, H_2, t), t \in [0,T]\}$.

2 Preliminaries

In this section, we introduce the Chaos expansion, which is an orthogonal decomposition of $L^2(\Omega, P)$. The readers may refer to [7, 8] and the references therein for more details. Let $X := \{X_t, t \in [0,T]\}$ be a Gaussian process defined on the probability space $(\Omega,\mathcal{F},P)$. If $p_n(x)$ is a polynomial of degree n in x, then we call $p_n(X_t)$ a polynomial function of X with $t \in [0,T]$. Let P_n be the completion with respect to the $L^2(\Omega, P)$ norm of the set $\{p_m(X_t) : 0 \le m \le n, t \in [0,T]\}$. Clearly P_n is a subspace of $L^2(\Omega, P)$. If let $\mathcal{C}_n$ denote the orthogonal complement of P_{n-1} in P_n, then $L^2(\Omega, P)$ is actually the direct sum of $\mathcal{C}_n$, i.e.,

$$L^2(\Omega, P) = \bigoplus_{n=0}^{\infty} \mathcal{C}_n. \tag{2.1}$$

This means that for each element $F \in L^2(\Omega, P)$, there exists $F_n \in \mathcal{C}_n$ with $n \in \mathbf{Z}_+$ such that

$$F = \sum_{n=0}^{\infty} F_n. \tag{2.2}$$

The decomposition (2.2) and F_n are called the Chaos expansion and n-th Chaos of F, respectively. By the orthogonality,

$$E(|F|^2) = \sum_{n=0}^{\infty} E(|F_n|^2). \tag{2.3}$$

Next, for $F \in L^2(\Omega, P)$, define an operator $\Gamma(u)$ for every $u \in [0,1]$ by

$$\Gamma(u)F := \sum_{n=0}^{\infty} u^n F_n. \tag{2.4}$$

Set $\Theta(u) := \Gamma(\sqrt{u}\,)F$. Then $\Theta(1) = F$. Define $\Phi_\Theta(u) := \frac{d}{du}(\|\Theta(u)\|^2)$, where $\|F\|^2 := E(|F|^2)$ for $F \in L^2(\Omega, P)$. Then

$$\Phi_\Theta(u) = \sum_{n=0}^{\infty} n u^{n-1} E(|F_n|^2).$$

Note that $\|\Theta(u)\|^2 = E(|\Theta(u)|^2) = \sum\limits_{n=0}^{\infty} E(u^n |F_n|^2)$.

Definition 2.1 *Let* $\mathbf{U} = \Big\{F \in L^2(\Omega, P) : F = \sum\limits_{n=0}^{\infty} F_n$ *and* $\sum\limits_{n=0}^{\infty} nE(|F_n|^2) < \infty\Big\}$. *Then* $\mathbf{U}$ *is called Meyer-Watanabe test function space (see* [9]*).*

Obviously, we have the following proposition from the definition.

Proposition 2.1 *Let* $F \in L^2(\Omega, P)$. *Then* $F \in \mathbf{U}$ *if and only if* $\Phi_\Theta(1) < \infty$.

Let $H_n(x),\ x \in \mathbb{R}$ be the Hermite polynomials of degree n,

$$H_n(x) = \frac{(-1)^n}{n!} \exp\Big(\frac{x^2}{2}\Big) \frac{\partial^n}{\partial x^n} \exp\Big(-\frac{x^2}{2}\Big).$$

Then

$$\exp\Big(tx-\frac{t^2}{2}\Big)=\sum_{n=0}^{\infty}t^nH_n(x)\quad\text{for all } t\in[0,T],\ x\in\mathbb{R}. \tag{2.5}$$

This implies that

$$\exp\Big(iu\xi(B_t^{H_1}-B_t^{H_2})+\frac{1}{2}u^2\xi^2\mathrm{Var}(B_t^{H_1}-B_t^{H_2})\Big)=\sum_{n=0}^{\infty}(iu)^n\sigma^n(t,\xi)H_n\Big(\frac{\xi(B_t^{H_1}-B_t^{H_2})}{\sigma(t,\xi)}\Big), \tag{2.6}$$

where $\sigma(t,\xi)=\sqrt{\mathrm{Var}(B_t^{H_1}-B_t^{H_2})\xi^2}$ for $\xi\in\mathbb{R}$. Because of the orthogonality of $\{H_n(x), x\in\mathbb{R}\}_{n\in\mathbf{Z}_+}$, we know from (2.2) that $(iu)^n\sigma^n(t,\xi)H_n\Big(\frac{\xi(B_t^{H_1}-B_t^{H_2})}{\sigma(t,\xi)}\Big)$ is the n-th Chaos of $\exp\big(iu\xi(B_t^{H_1}-B_t^{H_2})+\frac{1}{2}u^2\xi^2\mathrm{Var}(B_t^{H_1}-B_t^{H_2})\big)$.

3 Main Results and Proofs

Let $B^{H_i}=\{B_t^{H_i},t\geq 0\}$ $(i\in\{1,2\})$ be two independent fractional Brownian motions, and we use the following approximation to establish the existence of the so-called collision local time.

Theorem 3.1 (Existence of the Collision Local Time) *Define*

$$I_\epsilon(H_1,H_2,T):=\int_0^T p_\epsilon(B_t^{H_1}-B_t^{H_2})dt,$$

where $p_\epsilon(\cdot)$ is the density function of normal distribution $N(0,\epsilon)$. If $H_i\in(0,1)$, for each $i=1,2$, then $I_\epsilon(H_1,H_2,T)$ converges in $L^2(\Omega,P)$ sense, as $\epsilon\to 0$. Moreover, denote the limit by $I(H_1,H_2,T)$. Then $I(H_1,H_2,T)$ is an element in $L^2(\Omega,P)$.

Proof We proceed the proof of the theorem in two steps.

Step 1 Show that for each $\epsilon>0$, $I_\epsilon(H_1,H_2,T)\in L^2(\Omega,P)$.

Actually, since

$$\begin{aligned} I_\epsilon(H_1,H_2,T)&=\int_0^T p_\epsilon(B_t^{H_1}-B_t^{H_2})dt\\ &=\frac{1}{2\pi}\int_0^T\int_{\mathbb{R}}\exp(i\xi(B_t^{H_1}-B_t^{H_2}))\times\exp\Big(-\frac{\epsilon|\xi|^2}{2}\Big)d\xi dt, \end{aligned} \tag{3.1}$$

we have

$$\begin{aligned} E(|I_\epsilon(H_1,H_2,T)|^2)&=E\Big(\frac{1}{4\pi^2}\int_0^T\int_0^T\int_{\mathbb{R}^2}\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2}))\\ &\qquad\times\exp\Big(-\frac{\epsilon(|\xi|^2+|\eta|^2)}{2}\Big)d\xi d\eta ds dt\Big)\\ &=\frac{1}{4\pi^2}\int_0^T\int_0^T\int_{\mathbb{R}^2}E(\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2})))\\ &\qquad\times\exp\Big(-\frac{\epsilon(|\xi|^2+|\eta|^2)}{2}\Big)d\xi d\eta ds dt. \end{aligned} \tag{3.2}$$

However, noting that

$$\begin{aligned}&E(\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2})))\\&=\exp\Big(-\frac{\mathrm{Var}(\xi(B_t^{H_1}-B_t^{H_2})+\eta(B_s^{H_1}-B_s^{H_2}))}{2}\Big)=\exp\Big(-\frac{1}{2}M\Big),\end{aligned}$$

where

$$M:=\mathrm{Var}(\xi(B_t^{H_1}-B_t^{H_2})+\eta(B_s^{H_1}-B_s^{H_2})),$$

Then use the property of the local nondeterminism of the fractional Brownian motions (see [10, 11]). Let $s\le t$ and there is a constant $k>0$ so that

$$\begin{aligned}M&=\mathrm{Var}(\xi[(B_t^{H_1}-B_t^{H_2})-(B_s^{H_1}-B_s^{H_2})]+(\xi+\eta)(B_s^{H_1}-B_s^{H_2}))\\&\ge k(\xi^2((t-s)^{2H_1}+(t-s)^{2H_2})+(\xi+\eta)^2(s^{2H_1}+s^{2H_2})).\end{aligned}$$

So we have

$$\begin{aligned}&\int_0^T\int_0^T\int_{\mathbf{R}^2}E(\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2})))d\xi d\eta dsdt\\&\le 2\int_0^T\int_0^t\int_{\mathbf{R}^2}\exp\Big(-\frac{k}{2}(\xi^2((t-s)^{2H_1}+(t-s)^{2H_2})+(\xi+\eta)^2(s^{2H_1}+s^{2H_2}))\Big)d\xi d\eta dsdt\\&=\frac{4\pi}{k}\int_0^T\int_0^t(((t-s)^{2H_1}+(t-s)^{2H_2})(s^{2H_1}+s^{2H_2}))^{-\frac{1}{2}}dsdt\\&\le\frac{4\pi}{k}\int_0^T\int_0^t(t-s)^{-H_i}s^{-H_i}dsdt\\&<\infty,\end{aligned}\tag{3.3}$$

if $H_i\in(0,1)$ for $i=1,2$. This implies that for all $\epsilon>0$, $E(|I_\epsilon(H_1,H_2,T)|^2)<\infty$, if $H_i\in(0,1)$, $i=1,2$.

Step 2 Show that $\{I_\epsilon(H_1,H_2,T),\epsilon>0\}$ is a Cauchy sequence in $L^2(\Omega,P)$.

Note that

$$\begin{aligned}&E((I_\epsilon(H_1,H_2,T)-I_\delta(H_1,H_2,T))^2)\\&=E\Big(\frac{1}{4\pi^2}\int_0^T\int_0^T\int_{\mathbb{R}^2}\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2}))\\&\quad\times(e^{-\epsilon|\xi|^2/2}-e^{-\delta|\xi|^2/2})(e^{-\epsilon|\eta|^2/2}-e^{-\delta|\eta|^2/2})d\xi d\eta dsdt\Big)\\&=\frac{1}{4\pi^2}\int_0^T\int_0^T\int_{\mathbb{R}^2}E(\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2})))\\&\quad\times\Big(1-\exp\Big(-\frac{|\epsilon-\delta||\xi|^2}{2}\Big)\Big)\Big(1-\exp\Big(-\frac{|\epsilon-\delta||\eta|^2}{2}\Big)\Big)\\&\quad\times\exp\Big(-\frac{\min\{\epsilon,\delta\}(|\xi|^2+|\eta|^2)}{2}\Big)d\xi d\eta dsdt\\&\le\frac{1}{4\pi^2}\sup_{\xi\in\mathbb{R}}\Big(1-\exp\Big(-\frac{|\epsilon-\delta||\xi|^2}{2}\Big)\Big)^2\\&\quad\times\int_0^T\int_0^T\int_{\mathbb{R}^2}E(\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2})))d\xi d\eta dsdt.\end{aligned}\tag{3.4}$$

It follows from (3.3) that

$$\int_0^T\int_0^T\int_{\mathbb{R}^2} E(\exp(i\xi(B_t^{H_1}-B_t^{H_2})+i\eta(B_s^{H_1}-B_s^{H_2})))d\xi d\eta ds dt<\infty.$$

So

$$E((I_\epsilon(H_1,H_2,T)-I_\delta(H_1,H_2,T))^2)\to 0, \quad \text{as } \epsilon\to 0 \text{ and } \delta\to 0. \tag{3.5}$$

Hence $\{I_\epsilon(H_1,H_2,T),\epsilon>0\}$ is a Cauchy sequence in $L^2(\Omega,P)$, i.e., $\lim\limits_{\epsilon\to 0} I_\epsilon(H_1,H_2,T)$ exists. Thus $I(H_1,H_2,T):=\lim\limits_{\epsilon\to 0} I_\epsilon(H_1,H_2,T)\in L^2(\Omega,P)$, since $L^2(\Omega,P)$ is a Banach space. This completes the proof of the theorem.

We have established the existence of the collision local time. Next we shall prove the smoothness of the collision local time under some restrictive Hurst parameters.

Theorem 3.2 (Smoothness of the Collision Local Time) *Let $I(H_1,H_2,T)$ be the collision local time of two independent fractional Brownian motions with Hurst parameters H_1 and H_2. If $\min\{H_1,H_2\}<\frac{1}{3}$ and $\max\{H_1,H_2\}<\frac{1}{2}$, then $I(H_1,H_2,T)\in\mathbf{U}$.*

In order to prove Theorem 3.2, we need the following lemma. And the following notations are needed. Let

$$\begin{aligned} a_t &:= \mathrm{Var}(B_t^{H_1}-B_t^{H_2}) = t^{2H_1}+t^{2H_2},\\ \rho_{s,t} &:= E((B_t^{H_1}-B_t^{H_2})(B_s^{H_1}-B_s^{H_2}))\\ &= \frac{1}{2}(t^{2H_1}+s^{2H_1}-|t-s|^{2H_1}+t^{2H_2}+s^{2H_2}-|t-s|^{2H_2}). \end{aligned}$$

Lemma 3.1 *Let*

$$d_H(u;s,t)=a_s a_t-u^2\rho_{s,t}^2 \quad \textit{for } (u,s,t)\in[0,1]\times[0,T]^2.$$

If $\int_0^T\int_0^T \rho_{s,t}^2(d_H(1;s,t))^{-\frac{3}{2}}dsdt<+\infty$, then

$$I(H_1,H_2,T)\in\mathbf{U}.$$

Proof Recall $I_\epsilon(H_1,H_2,T)$ defined in Theorem 3.1. $I_\epsilon(H_1,H_2,T)\in L^2(\Omega,P)$ for every $\epsilon>0$.

To seek the Chaos expansion of $I_\epsilon(H_1,H_2,T)$, we consider

$$\begin{aligned} \exp(i\xi(B_t^{H_1}-B_t^{H_2})) &= \exp\Big(-\frac{1}{2}\xi^2\mathrm{Var}(B_t^{H_1}-B_t^{H_2})\Big)\\ &\quad\times\exp\Big(i\xi(B_t^{H_1}-B_t^{H_2})+\frac{1}{2}\xi^2\mathrm{Var}(B_t^{H_1}-B_t^{H_2})\Big). \end{aligned} \tag{3.6}$$

For any random variable X, define

$$\Upsilon_u(X):=\exp\Big(uX-\frac{1}{2}u^2\mathrm{Var}(X)\Big), \quad \text{as } u\in[0,1]. \tag{3.7}$$

Then by (2.6),

$$\Upsilon_1(i\xi(B_t^{H_1}-B_t^{H_2}))=\sum_{n=0}^{\infty} i^n\sigma^n(t,\xi)H_n\Big(\frac{\xi(B_t^{H_1}-B_t^{H_2})}{\sigma(t,\xi)}\Big). \tag{3.8}$$

Let

$$\psi_u(t,\xi) = \exp\Big(-\frac{1}{2}\mathrm{Var}(B_t^{H_1} - B_t^{H_2})\xi^2\Big)\Upsilon_u(i\xi(B_t^{H_1} - B_t^{H_2})). \tag{3.9}$$

It follows from (2.4) and (3.1) that

$$\Gamma(u)(I_\epsilon(H_1,H_2,T)) = \frac{1}{2\pi}\int_0^T\int_{\mathbb{R}}\psi_u(t,\xi)\exp\Big(-\frac{\epsilon|\xi|^2}{2}\Big)d\xi dt. \tag{3.10}$$

Set

$$K_\epsilon(u,T,X) := E(|\Gamma(\sqrt{u})I_\epsilon(H_1,H_2,T)|^2). \tag{3.11}$$

Then from (3.10), we have

$$K_\epsilon(u,T,X) = \frac{1}{(2\pi)^2}\int_0^T\int_0^T\int_{\mathbb{R}^2}E(\psi_{\sqrt{u}}(t,\xi)\psi_{\sqrt{u}}(s,\eta))\exp\Big(-\frac{\epsilon(|\xi|^2+|\eta|^2)}{2}\Big)d\xi d\eta ds dt. \tag{3.12}$$

Note that for any non-degenerate two-dimensional centered Gaussian random vector (X,Y),

$$E(\Upsilon_u(X)\Upsilon_v(Y)) = \exp(uv\mathrm{Cov}(X,Y)). \tag{3.13}$$

Then for any $0 \le s < t \le T$ fixed, set

$$\begin{cases} X = B_t^{H_1} - B_t^{H_2}, \\ Y = B_s^{H_1} - B_s^{H_2}. \end{cases}$$

It is easy to get that the (X,Y) defined above satisfies (3.13). Since

$$E(\xi(B_t^{H_1} - B_t^{H_2})\eta(B_s^{H_1} - B_s^{H_2})) = \xi\eta E((B_t^{H_1} - B_t^{H_2})(B_s^{H_1} - B_s^{H_2})),$$

we have

$$\begin{aligned} &E(\Upsilon_{\sqrt{u}}(i\xi(B_t^{H_1} - B_t^{H_2}))\Upsilon_{\sqrt{u}}(i\eta(B_s^{H_1} - B_s^{H_2}))) \\ &= \exp(-u\xi\eta E((B_t^{H_1} - B_t^{H_2})(B_s^{H_1} - B_s^{H_2}))). \end{aligned} \tag{3.14}$$

Then by (3.9) and (3.14),

$$\begin{aligned} E(\psi_{\sqrt{u}}(t,\xi)\psi_{\sqrt{u}}(s,\eta)) &= \exp\Big(-\frac{1}{2}\mathrm{Var}(B_t^{H_1} - B_t^{H_2})\xi^2 - u\xi\eta E((B_t^{H_1} - B_t^{H_2})(B_s^{H_1} - B_s^{H_2})) \\ &\quad - \frac{1}{2}\mathrm{Var}(B_s^{H_1} - B_s^{H_2})\eta^2\Big) \\ &> 0. \end{aligned} \tag{3.15}$$

So

$$\begin{aligned} \int_{\mathbb{R}^2}E(\psi_{\sqrt{u}}(t,\xi)\psi_{\sqrt{u}}(s,\eta))\exp\Big(-\frac{\epsilon(|\xi|^2+|\eta|^2)}{2}\Big)d\xi d\eta &\le \int_{\mathbb{R}^2}E(\psi_{\sqrt{u}}(t,\xi)\psi_{\sqrt{u}}(s,\eta))d\xi d\eta \\ &\le 2\pi(d_H(u;s,t))^{-\frac{1}{2}}, \end{aligned} \tag{3.16}$$

where $d_H(u;s,t)$ is defined in Lemma 3.1. From (3.12) we obtain

$$K_\epsilon(u,T,X) \le \frac{1}{2\pi}\int_0^T\int_0^T(d_H(u;s,t))^{-\frac{1}{2}}ds dt.$$

Define

$$d_H^\epsilon(u;s,t) := (a_t+\epsilon)(a_s+\epsilon) - u^2\rho_{s,t}.$$

Then it follows from (3.9) and (3.12) that

$$K_\epsilon(u,T,X) = \frac{1}{2\pi}\int_0^T\int_0^T (d_H^\epsilon(u;s,t))^{-\frac{1}{2}}dsdt.$$

Hence

$$\frac{\partial}{\partial u}K_\epsilon(u,T,X) = -c\int_0^T\int_0^T (d_H^\epsilon(u;s,t))^{-\frac{3}{2}}\frac{\partial}{\partial u}(d_H^\epsilon(u;s,t))dsdt, \tag{3.17}$$

where $c > 0$ is a constant. It is easy to verify that

$$\frac{\partial}{\partial u}(d_H^\epsilon(u;s,t)) = -2u\rho_{s,t}^2. \tag{3.18}$$

Therefore

$$\begin{aligned}(d_H^\epsilon(u;s,t))^{-\frac{3}{2}}\frac{\partial}{\partial u}(d_H^\epsilon(u;s,t)) &= -2u\rho_{s,t}^2((a_t+\epsilon)(a_s+\epsilon)-u^2\rho_{s,t}^2)^{-\frac{3}{2}}\\ &\geq -2u\rho_{s,t}^2(a_ta_s-u^2\rho_{s,t}^2)^{-\frac{3}{2}}\\ &= -2u\rho_{s,t}^2(d_H(u;s,t))^{-\frac{3}{2}}.\end{aligned}$$

Thus by (3.17),

$$\frac{\partial}{\partial u}K_\epsilon(u,T,X) \leq 2c\int_0^T\int_0^T u\rho_{s,t}^2(d_H(u;s,t))^{-\frac{3}{2}}dsdt.$$

Recall $\Phi_\Theta(u) = \frac{d}{du}(\|\Theta(u)\|^2)$ and (3.11). We have

$$\Phi_{K_\epsilon}(u) \leq 2c\int_0^T\int_0^T u\rho_{s,t}^2(d_H(u;s,t))^{-\frac{3}{2}}dsdt.$$

Then it follows from Proposition 2.1 that $I(H_1,H_2,T)\in\mathbf{U}$ if and only if $\Phi_{K_\epsilon}(1)<\infty$. Thus we complete the proof of Lemma 3.1.

Now, we are ready to prove Theorem 3.2.

Proof of Theorem 3.2 By Lemma 3.1, it suffices to show that

$$\int_0^T\int_0^T (d_H(1;s,t))^{-\frac{3}{2}}\rho_{s,t}^2dsdt < \infty.$$

Recall that

$$\begin{aligned}a_t &= \mathrm{Var}(B_t^{H_1}-B_t^{H_2}) = t^{2H_1}+t^{2H_2},\\ a_s &= \mathrm{Var}(B_s^{H_1}-B_s^{H_2}) = s^{2H_1}+s^{2H_2},\\ \rho_{s,t} &= E((B_t^{H_1}-B_t^{H_2})(B_s^{H_1}-B_s^{H_2}))\\ &= \frac{1}{2}(t^{2H_1}+s^{2H_1}-|t-s|^{2H_1}+t^{2H_2}+s^{2H_2}-|t-s|^{2H_2}).\end{aligned}$$

Let $s \le t$ and $s = xt$, where $x \in [0,1]$, and set

$$\begin{aligned} a_{x,t} &:= x^{2H_1}t^{2H_1} + x^{2H_2}t^{2H_2}, \\ \widetilde{\rho}_{x,t} &:= \frac{1}{2}(t^{2H_1} + x^{2H_1}t^{2H_1} - |1-x|^{2H_1}t^{2H_1} + t^{2H_2} + x^{2H_2}t^{2H_2} - |1-x|^{2H_2}t^{2H_2}). \end{aligned}$$

Clearly, $a_{x,t} = a_s$ and $\widetilde{\rho}_{x,t} = \rho_{s,t}$. Then

$$\begin{aligned} &a_t a_{x,t} - \widetilde{\rho}^2_{x,t} \\ =& \frac{1}{4}\{t^{4H_1}[4x^{2H_1} + 2(1+x^{2H_1})(1-x)^{2H_1} - (1+x^{2H_1})^2 - (1-x)^{4H_1}] \\ &+ t^{4H_2}[4x^{2H_2} + 2(1+x^{2H_2})(1-x)^{2H_2} - (1+x^{2H_2})^2 - (1-x)^{4H_2}] \\ &+ t^{2H_1+2H_2}[4x^{2H_1} + 4x^{2H_2} - 2((1+x^{2H_1}) - (1-x)^{2H_1})((1+x^{2H_2}) - (1-x)^{2H_2})]\} \\ =& \frac{1}{4}[t^{4H_1}f(x,H_1) + t^{4H_2}f(x,H_2) + t^{2H_1+2H_2}g(x,H_1,H_2)], \end{aligned}$$

where

$$\begin{aligned} f(x,H) &= 4x^{2H} + 2(1+x^{2H})(1-x)^{2H} - (1+x^{2H})^2 - (1-x)^{4H}, \\ g(x,H_1,H_2) &= 4x^{2H_1} + 4x^{2H_2} - 2[(1+x^{2H_1}) - (1-x)^{2H_1}][(1+x^{2H_2}) - (1-x)^{2H_2}]. \end{aligned}$$

Note that

$$x^{2H} + (1-x)^{2H} \ge 1, \quad \text{as } H < \frac{1}{2}. \tag{3.19}$$

Since $\max\{H_1, H_2\} < \frac{1}{2}$, we have

$$\begin{aligned} f(x,H_i) &= 2x^{2H_i} + 2(1-x)^{2H_i} + 2x^{2H_i}(1-x)^{2H_i} - 1 - x^{4H_i} - (1-x)^{4H_i} \\ &\ge x^{2H_i} + (1-x)^{2H_i} - 1 + 2x^{2H_i}(1-x)^{2H_i} \\ &\ge 2x^{2H_i}(1-x)^{2H_i}, \quad i = 1,2, \\ g(x,H_1,H_2) &= 2[x^{2H_1} + x^{2H_2} + (1-x)^{2H_1} + (1-x)^{2H_2} + x^{2H_1}(1-x)^{2H_2} \\ &\quad + x^{2H_2}(1-x)^{2H_1} - (1-x)^{2H_1+2H_2} - x^{2H_1+2H_2} - 1] \\ &\ge 2[x^{2H_1}(1-x)^{2H_2} + x^{2H_2}(1-x)^{2H_1}]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} a_t a_{x,t} - \widetilde{\rho}^2_{x,t} &\ge \frac{1}{2}[t^{4H_1}x^{2H_1}(1-x)^{2H_1} + t^{4H_2}x^{2H_2}(1-x)^{2H_2} \\ &\quad + t^{2H_1+2H_2}(x^{2H_1}(1-x)^{2H_2} + x^{2H_2}(1-x)^{2H_1})] \\ &= \frac{1}{2}(x^{2H_1}t^{2H_1} + x^{2H_2}t^{2H_2})[(1-x)^{2H_1}t^{2H_1} + (1-x)^{2H_2}t^{2H_2}]. \end{aligned} \tag{3.20}$$

On the other hand,

$$\begin{aligned} \widetilde{\rho}^2_{x,t} &= \frac{1}{4}[t^{4H_1}(1+x^{2H_1} - (1-x)^{2H_1})^2 + t^{4H_2}(1+x^{2H_2} - (1-x)^{2H_2})^2 \\ &\quad + 2t^{2H_1+2H_2}(1+x^{2H_1} - (1-x)^{2H_1})(1+x^{2H_2} - (1-x)^{2H_2})] \\ &= \frac{1}{4}[t^{4H_1}h(x,H_1) + t^{4H_2}h(x,H_2) + 2t^{2H_1+2H_2}k(x,H_1,H_2)], \end{aligned}$$

where

$$h(x,H) = (1 + x^{2H} - (1-x)^{2H})^2,$$
$$k(x,H_1,H_2) = [1 + x^{2H_1} - (1-x)^{2H_1}][1 + x^{2H_2} - (1-x)^{2H_2}].$$

Clearly, by (3.19), we have

$$h(x,H_i) \le 4x^{4H_i}, \quad i = 1,2 \quad \text{and} \quad k(x,H_1,H_2) \le 4x^{2H_1+2H_2}.$$

Therefore

$$\widetilde{\rho}^2_{x,t} \le t^{4H_1}x^{4H_1} + t^{4H_1}x^{4H_1} + 2t^{2H_1+2H_2}x^{2H_1+2H_2} = (t^{2H_1}x^{2H_1} + t^{2H_2}x^{2H_2})^2. \tag{3.21}$$

Recall the definition of $d_H(u;s,t)$ in Lemma 3.1. By (3.20) and (3.21) we have

$$\begin{aligned}
&\int_0^T\int_0^T (d_H(1;s,t))^{-\frac{3}{2}}\rho^2_{s,t}dsdt \\
&= \int_0^T\int_0^1 (a_t a_{x,t} - \widetilde{\rho}^2_{x,t})^{-\frac{3}{2}}\widetilde{\rho}^2_{x,t}tdxdt \\
&\le \int_0^T\int_0^1 \Big\{\frac{1}{2}(x^{2H_1}t^{2H_1} + x^{2H_2}t^{2H_2})[(1-x)^{2H_1}t^{2H_1} + (1-x)^{2H_2}t^{2H_2}]\Big\}^{-\frac{3}{2}} \\
&\quad \times (t^{2H_1}x^{2H_1} + t^{2H_2}x^{2H_2})^2 tdxdt \\
&= 2^{\frac{3}{2}}\int_0^T\int_0^1 \sqrt{x^{2H_1}t^{2H_1} + x^{2H_2}t^{2H_2}}\,((1-x)^{2H_1}t^{2H_1} + (1-x)^{2H_2}t^{2H_2})^{-\frac{3}{2}}tdxdt \\
&< 2^{\frac{3}{2}}\int_0^T\int_0^1 \sqrt{x^{2H_1}t^{2H_1} + x^{2H_2}t^{2H_2}}\,((1-x)^{-3\min\{H_1,H_2\}}t^{1-3\min\{H_1,H_2\}})dxdt \\
&< \infty,
\end{aligned}$$

under the condition $\min\{H_1,H_2\} < \frac{1}{3}$. Thus the proof of the theorem is completed.

4 Regularity of the Collision Local Time Process

In this section, we will discuss the regularity of process $\{I(H_1,H_2,t), t \in [0,T]\}$. The following theorem gives the Hölder continuity of process $\{I(H_1,H_2,t), t \in [0,T]\}$.

Theorem 4.1 (Regularity of the Collision Local Time Process) *The collision local time process* $\{I(H_1,H_2,t), t \in [0,T]\}$ *is Hölder continuous with order* $1 - \max\{H_1,H_2\}$.

Proof Let

$$L_t = I(H_1,H_2,t) = \int_0^t \delta(B^{H_1}_s - B^{H_2}_s)ds \quad \text{and} \quad L^\epsilon_t = I_\epsilon(H_1,H_2,t) = \int_0^t p_\epsilon(B^{H_1}_s - B^{H_2}_s)ds.$$

Suppose $0 \le s < t \le T$. Then

$$\begin{aligned}
E(|L^\epsilon_t - L^\epsilon_s|) &= E\Big(\int_s^t p_\epsilon(B^{H_1}_u - B^{H_2}_u)du\Big) \\
&= \frac{1}{2\pi}\int_s^t\int_{\mathbb{R}} E(\exp(i\xi(B^{H_1}_u - B^{H_2}_u)))\exp(-\epsilon|\xi|^2)d\xi du \\
&\le \frac{1}{2\pi}\int_s^t\int_{\mathbb{R}} \exp\Big(-\frac{1}{2}\xi^2(u^{2H_1} + u^{2H_2})\Big)d\xi du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_s^t \int_{\mathbb{R}} \sqrt{\frac{2}{u^{2H_1}+u^{2H_2}}}\ \exp(-x^2)dxdu \\
&= \frac{1}{\sqrt{2\pi}}\int_s^t \sqrt{\frac{1}{u^{2H_1}+u^{2H_2}}}\,du \\
&\le \frac{1}{\sqrt{2\pi}}\int_s^t u^{-\max\{H_1,H_2\}}du \\
&= \frac{t^{1-\max\{H_1,H_2\}}-s^{1-\max\{H_1,H_2\}}}{\sqrt{2\pi}\,(1-\max\{H_1,H_2\})} \\
&\le K|t-s|^{1-\max\{H_1,H_2\}},
\end{aligned}
$$

where $K=\frac{1}{\sqrt{2\pi}\,(1-\max\{H_1,H_2\})}>0$. By Theorem 3.1,

$$
|L_t-L_s| = \lim_{\epsilon\to 0}|L_t^\epsilon - L_s^\epsilon|, \quad P\text{-a.s.}
$$

Thus by Fatou's lemma, we obtain

$$
E(|L_t-L_s|) = E\Big(\lim_{\epsilon\to 0}|L_t^\epsilon - L_s^\epsilon|\Big) \le \liminf_{\epsilon\to 0} E(|L_t^\epsilon - L_s^\epsilon|) \le K|t-s|^{1-\max\{H_1,H_2\}}.
$$

Thus we complete the proof of the theorem.

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