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Analytic Extension of Functions from Analytic Hilbert Spaces**

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Abstract Let M be an invariant subspace of H_v^2. It is shown that for each $f \in M^\perp$, f can be analytically extended across $\partial\mathbb{B}_d \backslash \sigma(S_{z_1}, \cdots, S_{z_d})$.

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Let $d \geq 1$ and

$$\mathbb{B}_d = \Big\{ z \in \mathbb{C}^d : |z|^2 = \sum_{i=1}^d |z_i|^2 < 1 \Big\}.$$

In this note, we mainly consider the reproducing kernel space H_v^2 $(v > 0)$ over $\mathbb{B}_d$ with reproducing kernel

$$K_\lambda^v(z) = \frac{1}{(1 - \langle z, \lambda \rangle)^v} = \sum_{n=0}^\infty a_n^{(v)} \langle z, \lambda \rangle^n,$$

where

$$\langle z, \lambda \rangle = \sum_{i=1}^d z_i \overline{\lambda}_i \quad \text{and} \quad a_0^{(v)} = 1, \quad a_n^{(v)} = \frac{v(v+1)\cdots(v+n-1)}{n!} \quad \text{for } n \geq 1.$$

When $v = 1$, the Hilbert space H_v^2 is the Symmetric Fock space, which was deeply studied by Arveson [2]. When $v = d$, the space H_v^2 is the usual Hardy space $H^2(\mathbb{B}_d)$, and when $v = d+1$, it is the usual Bergman space $L_a^2(\mathbb{B}_d)$ on the unit ball. We refer the reader to [4] for details.

Let M be an invariant subspace of H_v^2, that is, $pM \subset M$ for any polynomial p. Let $S_p = P_{M^\perp} M_p|_{M^\perp}$ be the compression operator on $M^\perp$ for a polynomial p, where $P_{M^\perp}$ is the orthogonal projection onto $M^\perp$. Those operators carry key information about the invariant subspace (see [1, 2, 9, 10, 12, 17]). We denote the Taylor spectrum (see [6, 16]) of the tuple $\{S_{z_1}, \cdots, S_{z_d}\}$ by $\sigma(S_{z_1}, \cdots, S_{z_d})$, and write $\{S_1, \cdots, S_d\}$ for $\{S_{z_1}, \cdots, S_{z_d}\}$.

The following notations are standard. For any ordered d-tuple of nonnegative integers $\alpha = (\alpha_1, \cdots, \alpha_d)$, $z = (z_1 \cdots, z_d) \in \mathbb{C}^d$, write

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \quad \alpha! = \alpha_1! \cdots \alpha_d!, \quad z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}, \quad S^\alpha = S_1^{\alpha_1} \cdots S_d^{\alpha_d}.$$

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Theorem 1 *Let M be an invariant subspace of H_v^2. Then for any $f \in M^{\perp}$, f can be analytically extended across $\partial\mathbb{B}_d \backslash \sigma(S_1, \cdots, S_d)$.*

Proof First, we will show that $\sigma(S_1, \cdots, S_d) \subseteq \mathbb{B}_d$. For given μ, $|\mu| > 1$ and the function $f_\mu(z) = \langle z, \mu\rangle - |\mu|^2$, from the proof of Theorem 4.5 in [11], the multiplication operator M_{f_μ} acting H_v^2 is invertible with the inverse M_{1/f_μ}. A simple analysis shows that S_{f_μ} acting on $M^{\perp}$ is invertible with the inverse S_{1/f_μ} and hence $0 \notin \sigma(S_{f_\mu})$. Therefore, by the spectral mapping theorem (see [16, Theorem 4.8]), $\mu \notin \sigma(S_1, \cdots, S_d)$. It follows that

$$\sigma(S_1, \cdots, S_d) \subseteq \mathbb{B}_d.$$

Letting I be the identity operator on $M^{\perp}$, by [16, Theorem 4.8], we have

$$\sigma\Big(I - \sum_{i=1}^{d} \lambda_i S_i^*\Big) = \Big\{1 - \sum_{i=1}^{d} \lambda_i \overline{z}_i : z \in \sigma(S_1, \cdots, S_d)\Big\}.$$

For $\lambda \in \overline{\mathbb{B}}_d$, $z \in \overline{\mathbb{B}}_d$, we have

$$\mathrm{Re}\Big(1 - \sum_{i=1}^{d} \lambda_i \overline{z}_i\Big) \geq 1 - |\lambda||z| \geq 0,$$

and hence we get

$$\mathrm{Re}\Big(1 - \sum_{i=1}^{d} \lambda_i \overline{z}_i\Big) > 0, \quad \text{if } \lambda \in \mathbb{B}_d \text{ or } \lambda \neq z.$$

This implies that for any point $\lambda \in \mathbb{B}_d \cup (\partial\mathbb{B}_d \backslash \sigma(S_1, \cdots, S_d))$,

$$\sigma\Big(I - \sum_{i=1}^{d} \lambda_i S_i^*\Big) \subseteq \{w \in \mathbb{C} : \mathrm{Re}\, w > 0\}.$$

Choose an open set U of $\mathbb{C}^d$ satisfying $\mathbb{B}_d \cup (\partial\mathbb{B}_d \backslash \sigma(S_1, \cdots, S_d)) \subseteq U$, and

$$\mathrm{Re}\Big(1 - \sum_{i=1}^{d} \lambda_i \overline{z}_i\Big) > 0 \quad \text{for each } \lambda \in U,\ z \in \sigma(S_1, \cdots, S_d).$$

This means that for any $\lambda \in U$,

$$\sigma\Big(I - \sum_{i=1}^{d} \lambda_i S_i^*\Big) \subseteq \{\mathrm{Re}\, w > 0 : w \in \mathbb{C}\}.$$

Since w^{-v} is analytic on the domain $\{\mathrm{Re}\, w > 0 : w \in \mathbb{C}\}$, we see that the operator value function

$$\lambda \to \Big(I - \sum_{i=1}^{d} \lambda_i S_i^*\Big)^{-v}$$

is analytic on U.

Furthermore, given any $f \in M^\perp$ and $\lambda \in \mathbb{B}_d$, we have

$$\begin{aligned}\left\langle \Big(I - \sum_{i=1}^d \lambda_i S_i^*\Big)^{-v} f, 1\right\rangle &= \sum_{n=0}^\infty a_n^{(v)} \sum_{|\alpha|=n} \frac{n!}{\alpha!}\lambda^\alpha \langle S^{*\alpha} f, 1\rangle = \sum_{n=0}^\infty a_n^{(v)} \sum_{|\alpha|=n} \frac{n!}{\alpha!}\lambda^\alpha \langle M_z^{*\alpha} f, 1\rangle \\ &= \sum_{n=0}^\infty a_n^{(v)} \sum_{|\alpha|=n} \frac{n!}{\alpha!}\lambda^\alpha \langle f, z^\alpha\rangle = \langle f, K_\lambda\rangle = f(\lambda).\end{aligned}$$

Since

$$\Big(I - \sum_{i=1}^d \lambda_i S_i^*\Big)^{-v}$$

is analytic in U, it follows that the analytic function

$$f(\lambda) = \left\langle \Big(I - \sum_{i=1}^d \lambda_i S_{z_i}^*\Big)^{-v} f, 1\right\rangle$$

has an analytic continuation across $\partial\mathbb{B}_d \backslash \sigma(S_1, \cdots, S_d)$. This leads to the desired result.

Using an easy argument, we obtain the following corollary from Theorem 1.

Corollary 1 *Let M be an invariant subspace of H_v^2. Then*

$$\bigcup_{f \in M^\perp} \{\lambda \in \partial\mathbb{B}_d : f \textit{ is not analytic at } \lambda\} \subseteq \sigma(S_1, \cdots, S_d).$$

In the classical Hardy space $H^2(\mathbb{D})$, by the elegant Beurling theorem, any invariant subspace M is generated by an inner function η. Arveson [1] showed that, in this case, $\sigma(S) = Z(\eta)$ and $\sigma_e(S) = Z_\partial(\eta)$, where

$$\begin{aligned}Z(\eta) &= \{\lambda \in \mathbb{D} : \eta(\lambda) = 0\} \cup \{\lambda \in \mathbb{T} : \eta \text{ is not analytic at } \lambda\}, \\ Z_\partial(\eta) &= Z(\eta) \cap \mathbb{T}.\end{aligned}$$

In higher dimensions, one see a similar result from the above corollary. When the dimension $d > 1$, any inner functions η on $\mathbb{B}_d$ is not analytic at any point in $\partial\mathbb{B}_d$ (see [14]). Hence, the function $P_{M^\perp} K_\lambda = (1 - \overline{\eta(\lambda)}\eta) K_\lambda$ is not analytic at any point in $\partial\mathbb{B}_d$. If $M = [\eta]$ is an invariant subspace of $H^2(\mathbb{B}_d)$ generated by η, by the above corollary, a simple argument shows that

$$\sigma(S_1, \cdots, S_d) = \{\lambda \in \mathbb{B}_d : \eta(\lambda) = 0\} \cup \partial\mathbb{B}_d.$$

Another important type of invariant subspaces is that generated by polynomials (see [1, 4, 7, 8, 17]). For a polynomial p, let $[p]$ denote the invariant subspace of H_v^2 generated by p. Then $S_p = 0$ since for any $f_1, f_2 \in [p]^\perp$,

$$\langle S_p f_1, f_2\rangle = \langle p f_1, f_2\rangle = 0.$$

From $S_p = p(S_1, \cdots, S_d)$, applying [16, Theorem 4.8] shows

$$\sigma(S_1, \cdots, S_d) \subseteq Z(p),$$

where $Z(p) = \{\lambda \in \mathbb{C}^d : \ p(\lambda) = 0\}$. Combing this fact with Theorem 1, we have

Corollary 2 *Let p be a polynomial. Then for each function $f \in H_v^2$ and $f \perp [p]$, f can be analytically extended across $\partial\mathbb{B}_d \backslash Z(p)$.*

In the case $v = d$, H_v^2 is the usual Hardy space. It follows from Corollary 2 that $f \in H^2(\mathbb{B}_d)$ has an analytic continuation across $\partial\mathbb{B}_d \backslash Z(p)$ if $f \perp [p]$. The following example shows that there exists a function $f \in [p]^\perp$ such that f can not be analytically extended across each point in $\partial\mathbb{B}_d \backslash Z(p)$.

Example 1 Consider the invariant subspace $[z-w]$ of the Hardy space $H^2(\mathbb{B}_2)$. It is easy to check $[z-w]^\perp = \overline{\text{span}}\{(z+w)^n;\ n = 0, 1, \cdots, \}$. Let $B(z)$ be a Blaschke product satisfying that the closure of its zero set contains the unit circle. Then $B(\frac{z+w}{\sqrt{2}})$ is in $[z-w]^\perp$, and it can not be analytically extended across each point in $\{(\frac{\sqrt{2}}{2}\, e^{i\theta}, \frac{\sqrt{2}}{2}\, e^{i\theta});\ 0 \leq \theta \leq 2\pi\}$.

We will show by the next example that in some cases, for any fixed point in $\partial\mathbb{B}_d \cap \sigma(S_1, \cdots, S_d)$, there is a function in $M^\perp$ such that it can not be analytically extended across this point.

Example 2 Fix $\lambda_0 \in \partial\mathbb{B}_d$, and let M be the invariant subspace of H_v^2 defined by

$$M = \{f \in H_v^2 : f(\xi\lambda_0) = 0,\ \text{where } \xi \in \mathbb{C},\ |\xi| < 1\}.$$

Using the same argument as in Corollary 2, one can verify that

$$\sigma(S_1, \cdots, S_d) \subseteq \{\xi\lambda_0 : \xi \in \mathbb{C},\ |\xi| \leq 1\}.$$

Applying Theorem 1 shows that f is analytic in $\overline{\mathbb{B}}_d \backslash \{\xi\lambda_0 : \xi \in \mathbb{C},\ |\xi| = 1\}$ for any $f \in M^\perp$.

It is not difficult to check that the set $\text{span}\{K_{\xi\lambda_0} : |\xi| < 1\}$ is dense in $M^\perp$. From [11, Example 2], we have $\langle z, \lambda_0\rangle^n \perp \langle z, \lambda_0\rangle^m$ if $m \neq n$. It follows that if $\xi \to 0$,

$$\frac{K_{\xi\lambda_0}(z) - K_0(z)}{\bar{\xi}} = \frac{\frac{1}{(1-\langle z, \xi\lambda_0\rangle)^v} - 1}{\bar{\xi}} = \sum_{n=1}^{\infty} a_n^{(v)} \bar{\xi}^{n-1} \langle z, \lambda_0\rangle^n \to a_1^{(v)} \langle z, \lambda_0\rangle$$

in the norm of H_v^2. Hence $\langle z, \lambda_0\rangle \in M^\perp$. Using the same argument, we have $\langle z, \lambda_0\rangle^n \in M^\perp$ for any $n > 0$. Since

$$K_{\xi\lambda_0}(z) = \sum_{n=0}^{\infty} a_n^{(v)} \bar{\xi}^n \langle z, \lambda_0\rangle^n \in \overline{\text{span}}\{1, \langle z, \lambda_0\rangle, \langle z, \lambda_0\rangle^2, \cdots\} \quad \text{for any } |\xi| < 1,$$

this means

$$M^\perp = \overline{\text{span}}\{1, \langle z, \lambda_0\rangle, \langle z, \lambda_0\rangle^2, \cdots\}.$$

Below, we will show that there is $G_{\xi_0} \in M^\perp$ such that G_{ξ_0} can not be analytically extended across the point $\xi_0\lambda_0$ for any $\xi_0 \in \mathbb{C}$, $|\xi_0| = 1$. First, we calculate the norm of $\langle z, \lambda_0\rangle^n$. Noticing

$$\|K_{\xi\lambda_0}\|^2 = K_{\xi\lambda_0}(\xi\lambda_0) = \frac{1}{(1-|\xi|^2)^v} = \sum_{n=0}^{\infty} a_n^{(v)} |\xi|^{2n},$$

$$\|K_{\xi\lambda_0}\|^2 = \Big\| \sum_{n=0}^{\infty} a_n^{(v)} \xi^n \langle z, \lambda_0\rangle^n \Big\|^2 = \sum_{n=0}^{\infty} |\xi|^{2n} \|a_n^{(v)} \langle z, \lambda_0\rangle^n\|^2,$$

and comparing the coefficients of $|\xi|^{2n}$, we get

$$\|\langle z, \lambda_0\rangle^n\|^2 = \frac{1}{a_n^{(v)}}.$$

Set

$$g(w) = \sum_{n=2}^{\infty} \frac{w^n}{n \ln n}.$$

Then it is analytic in the unit disk $\{|w| < 1\}$, but it can not be analytically extended across the point $w = 1$. Let

$$G_{\xi_0}(z) = g(\langle z, \xi_0\lambda_0\rangle) = \sum_{n=2}^{\infty} \frac{\langle z, \xi_0\lambda_0\rangle^n}{n \ln n}.$$

Since

$$a_n^{(v)} = \frac{v(v+1)\cdots(v+n-1)}{n!} > \frac{v}{n} \quad \text{for } n > 1,$$

we have

$$\Big\|\sum_{n=2}^{\infty} \frac{\langle z, \xi_0\lambda_0\rangle^n}{n \ln n}\Big\|^2 = \sum_{n=2}^{\infty} \frac{1}{a_n^{(v)} n^2 \ln^2 n} < \frac{1}{v}\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} < \infty.$$

This implies that $G_{\xi_0} \in M_{\lambda_0}^{\perp}$ and it can not be extended across the point $\xi_0\lambda_0$.

In the case $d = 1, v = 2$, H_v^2 is the classical Bergman space $L_a^2(\mathbb{D})$. As in the situation of the classical Hardy space over the unit disk, Bergman inner functions play an important role in the study of invariant subspaces of the Bergman space $L_a^2(\mathbb{D})$ (see [5, 13, 3] and references therein). A function Φ in $L_a^2(\mathbb{D})$ is called a Bergman inner function if

$$\int_{\mathbb{D}} (|\Phi(z)|^2 - 1) z^n d\, A = 0$$

for all nonnegative integers n, where dA is the normalized area measure. There are a number of references concerning analytic extension of Bergman inner functions. We refer the reader to [5, 13, 15] and the references therein for detailed results on this problem. Notice that if Φ is a Bergman inner function, then $\Phi \perp [z\Phi]$. Set

$$Z(\Phi) = \Big\{\lambda \in \mathbb{C} : |\lambda| = 1,\ \liminf_{|z|<1,\, z\to\lambda} \Phi(z) = 0\Big\} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1,\ \Phi(\lambda) = 0\}.$$

Then by Hedenmalm [12, Theorem 1.3], the operator S (acting on $[z\Phi]^{\perp}$) has spectrum

$$\sigma(S) = Z(\Phi) \cup \{0\}.$$

As an application of Theorem 1, we come to a well-known result that Φ can be analytically extended across $\partial\mathbb{D}\backslash Z(\Phi)$ (see the above mentioned references).

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