Chin. Ann. Math.
28B(3), 2007, 353–362
DOI: 10.1007/s11401-005-0427-y

Chinese Annals of Mathematics, Series B

Isomorphisms Between Quasi-Banach Algebras**

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Abstract In this paper, the author proves the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras. This is used to investigate isomorphisms between quasi-Banach algebras.

Keywords Cauchy functional equation, Jensen functional equation, Quasi-Banach algebra, Hyers-Ulam-Rassias stability, Homomorphism, p-Banach algebra
2000 MR Subject Classification 39B72, 46B03, 47J

1 Introduction and Preliminaries

The stability problem of functional equations originated from a question in [37] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta \quad \text{for all } x, y \in G_1,$$

then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon \quad \text{for all } x \in G_1?$$

If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D. H. Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon \quad \text{for all } x, y \in X \text{ and some } \varepsilon \ge 0.$$

Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon \quad \text{for all } x \in X.$$

Manuscript received October 9, 2005. Revised August 25, 2006. Published online April 30, 2007.
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**Project supported by the Korea Research Foundation (No. KRF-2005-041-C00027).

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. T. M. Rassias [27] introduced the following inequality: Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p) \quad \text{for all } x, y \in X.$$

T. M. Rassias [27] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p \quad \text{for all } x \in X.$$

The above inequality has provided a lot of influence in the development of what is now known as Hyers-Ulam-Rassias stability of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [9] generalized the Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 3, 5–9, 11–26, 28–32, 35]).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.1 (See [4, 34]) *Let X be a real linear space. A quasi-norm is a real-valued function on X satisfying the following:*

(1) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $x \in X$.

(3) *There is a constant* $K \geq 1$ *such that* $\|x + y\| \leq K(\|x\| + \|y\|)$ *for all* $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \quad \textit{for all } x, y \in X.$$

In this case, a quasi-Banach space is called a p-Banach space.

Given a p-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on X. By the Aoki-Rolewicz theorem in [34] (see also [4]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

Definition 1.2 (See [1]) *Let $(A, \|\cdot\|)$ be a quasi-normed space. The quasi-normed space $(A, \|\cdot\|)$ is called a quasi-normed algebra if A is an algebra and there is a constant $K > 0$ such that $\|xy\| \leq K\|x\| \cdot \|y\|$ for all $x, y \in A$.*

A quasi-Banach algebra is a complete quasi-normed algebra.

If the quasi-norm $\|\cdot\|$ is a p-norm then the quasi-Banach algebra is called a p-Banach algebra.

In Section 2, we prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation and the Jensen functional equation.

In Section 3, we investigate isomorphisms between quasi-Banach algebras.

2 Stability of Homomorphisms in Quasi-Banach Algebras

Throughout this section, assume that A is a quasi-normed algebra with quasi-norm $\|\cdot\|_A$ and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$. Let K be the modulus of concavity of $\|\cdot\|_B$.

We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Cauchy functional equation.

Theorem 2.1 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a mapping such that*

$$\|f(x+y) - f(x) - f(y)\|_B \le \theta(\|x\|_A^r + \|y\|_A^r), \tag{2.1}$$

$$\|f(xy) - f(x)f(y)\|_B \le \theta(\|x\|_A^r + \|y\|_A^r) \tag{2.2}$$

for all $x, y \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{2\theta}{(2^{pr} - 2^p)^{\frac{1}{p}}}\|x\|_A^r \quad \textit{for all } x \in A. \tag{2.3}$$

Proof Letting $y = x$ in (2.1), we get

$$\|f(2x) - 2f(x)\|_B \le 2\theta\|x\|_A^r \quad \text{for all } x \in A. \tag{2.4}$$

So

$$\left\|f(x) - 2f\Big(\frac{x}{2}\Big)\right\|_B \le \frac{2\theta}{2^r}\|x\|_A^r \quad \text{for all } x \in A.$$

Since B is a p-Banach algebra,

$$\left\|2^l f\Big(\frac{x}{2^l}\Big) - 2^m f\Big(\frac{x}{2^m}\Big)\right\|_B^p \le \sum_{j=l}^{m-1}\left\|2^j f\Big(\frac{x}{2^j}\Big) - 2^{j+1} f\Big(\frac{x}{2^{j+1}}\Big)\right\|_B^p \le \frac{2^p\theta^p}{2^{pr}}\sum_{j=l}^{m-1}\frac{2^{pj}}{2^{prj}}\|x\|_A^{pr} \tag{2.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.5) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) \quad \text{for all } x \in A.$$

It follows from (2.1) that

$$\begin{aligned}\|H(x+y)-H(x)-H(y)\|_B &= \lim_{n\to\infty} 2^n\Big\|f\Big(\frac{x+y}{2^n}\Big)-f\Big(\frac{x}{2^n}\Big)-f\Big(\frac{y}{2^n}\Big)\Big\|_B \\ &\le \lim_{n\to\infty}\frac{2^n\theta}{2^{nr}}(\|x\|_A^r+\|y\|_A^r)=0 \quad \text{for all } x,y\in A.\end{aligned}$$

So

$$H(x+y)=H(x)+H(y) \quad \text{for all } x,y\in A.$$

Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.5), we get (2.3).

By the same reasoning as in the proof of Theorem of [27], the mapping $H : A\to B$ is $\mathbb{R}$-linear.

It follows from (2.2) that

$$\begin{aligned}\|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty} 4^n\Big\|f\Big(\frac{xy}{2^n\cdot 2^n}\Big)-f\Big(\frac{x}{2^n}\Big)f\Big(\frac{y}{2^n}\Big)\Big\|_B \\ &\le \lim_{n\to\infty}\frac{4^n\theta}{2^{nr}}(\|x\|_A^r+\|y\|_A^r)=0 \quad \text{for all } x,y\in A.\end{aligned}$$

So

$$H(xy)=H(x)H(y) \quad \text{for all } x,y\in A.$$

Now, let $T : A\to B$ be another Cauchy additive mapping satisfying (2.3). Then we have

$$\begin{aligned}\|H(x)-T(x)\|_B &= 2^n\Big\|H\Big(\frac{x}{2^n}\Big)-T\Big(\frac{x}{2^n}\Big)\Big\|_B \\ &\le 2^nK\Big(\Big\|H\Big(\frac{x}{2^n}\Big)-f\Big(\frac{x}{2^n}\Big)\Big\|_B+\Big\|T\Big(\frac{x}{2^n}\Big)-f\Big(\frac{x}{2^n}\Big)\Big\|_B\Big) \\ &\le \frac{2^{n+2}K\theta}{(2^{pr}-2^p)^{\frac{1}{p}}2^{nr}}\|x\|_A^r,\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x\in A$. So we can conclude that $H(x)=T(x)$ for all $x\in A$. This proves the uniqueness of H. Thus the mapping $H : A\to B$ is a unique homomorphism satisfying (2.3).

Theorem 2.2 *Let $r<1$ and θ be positive real numbers, and let $f : A\to B$ be a mapping satisfying* (2.1) *and* (2.2). *If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, then there exists a unique homomorphism $H : A\to B$ such that*

$$\|f(x)-H(x)\|_B \le \frac{2\theta}{(2^p-2^{pr})^{\frac{1}{p}}}\|x\|_A^r \quad \textit{for all } x\in A. \tag{2.6}$$

Proof It follows from (2.4) that

$$\Big\|f(x)-\frac{1}{2}f(2x)\Big\|_B \le \theta\|x\|_A^r \quad \text{for all } x\in A.$$

Since B is a p-Banach algebra,

$$\Big\|\frac{1}{2^l}f(2^lx)-\frac{1}{2^m}f(2^mx)\Big\|_B^p \le \sum_{j=l}^{m-1}\Big\|\frac{1}{2^j}f(2^jx)-\frac{1}{2^{j+1}}f(2^{j+1}x)\Big\|_B^p \le \theta^p\sum_{j=l}^{m-1}\frac{2^{prj}}{2^{pj}}\|x\|_A^{pr} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.7) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \quad \text{for all } x \in A.$$

The rest of the proof is similar to the proof of Theorem 2.1.

We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Jensen functional equation.

Theorem 2.3 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a mapping with $f(0) = 0$ satisfying* (2.2) *such that*

$$\left\|2f\Big(\frac{x+y}{2}\Big) - f(x) - f(y)\right\|_B \le \theta(\|x\|_A^r + \|y\|_A^r) \quad \textit{for all } x, y \in A. \tag{2.8}$$

If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{K(3+3^r)\theta}{(3^p - 3^{pr})^{\frac{1}{p}}} \|x\|_A^r \quad \textit{for all } x \in A. \tag{2.9}$$

Proof Letting $y = -x$ in (2.8), we get

$$\| - f(x) - f(-x)\|_B \le 2\theta\|x\|_A^r \quad \text{for all } x \in A.$$

Letting $y = 3x$ and replacing x by $-x$ in (2.8), we get

$$\|2f(x) - f(-x) - f(3x)\|_B \le (3^r + 1)\theta\|x\|_A^r \quad \text{for all } x \in A.$$

Thus

$$\|3f(x) - f(3x)\|_B \le K(3^r + 3)\theta\|x\|_A^r \quad \text{for all } x \in A. \tag{2.10}$$

So

$$\left\|f(x) - \frac{1}{3} f(3x)\right\|_B \le \frac{K(3^r + 3)\theta}{3} \|x\|_A^r \quad \text{for all } x \in A.$$

Since B is a p-Banach algebra,

$$\begin{aligned}\left\|\frac{1}{3^l} f(3^l x) - \frac{1}{3^m} f(3^m x)\right\|_B^p &\le \sum_{j=l}^{m-1} \left\|\frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x)\right\|_B^p \\ &\le \frac{K^p(3^r+3)^p\theta^p}{3^p} \sum_{j=l}^{m-1} \frac{3^{prj}}{3^{pj}} \|x\|_A^{pr}\end{aligned} \tag{2.11}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.11) that the sequence $\{\frac{1}{3^n} f(3^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{3^n} f(3^n x)\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \quad \text{for all } x \in A.$$

By (2.8),

$$\begin{aligned}\Big\|2H\Big(\frac{x+y}{2}\Big)-H(x)-H(y)\Big\|_B &= \lim_{n\to\infty}\frac{1}{3^n}\Big\|2f\Big(3^n\cdot\frac{x+y}{2}\Big)-f(3^nx)-f(3^ny)\Big\|_B\\ &\le \lim_{n\to\infty}\frac{3^{rn}}{3^n}\theta(\|x\|_A^r+\|y\|_A^r)=0 \quad \text{for all } x,y\in A.\end{aligned}$$

So

$$2H\Big(\frac{x+y}{2}\Big)=H(x)+H(y) \quad \text{for all } x,y\in A.$$

Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.11), we get (2.9).

It follows from (2.2) that

$$\begin{aligned}\|H(xy)-H(x)H(y)\|_B &= \lim_{n\to\infty}\frac{1}{9^n}\|f(9^nxy)-f(3^nx)f(3^ny)\|_B\\ &\le \lim_{n\to\infty}\frac{3^{nr}\theta}{9^n}(\|x\|_A^r+\|y\|_A^r)=0 \quad \text{for all } x,y\in A.\end{aligned}$$

So

$$H(xy)=H(x)H(y) \quad \text{for all } x,y\in A.$$

Now, let $T:A\to B$ be another Jensen additive mapping satisfying (2.9). Then we have

$$\begin{aligned}\|H(x)-T(x)\|_B^p &= \frac{1}{3^{pn}}\|H(3^nx)-T(3^nx)\|_B^p\\ &\le \frac{1}{3^{pn}}(\|H(3^nx)-f(3^nx)\|_B^p+\|T(3^nx)-f(3^nx)\|_B^p)\\ &\le 2\cdot\frac{3^{prn}}{3^{pn}}\cdot\frac{K^p(3+3^r)^p\theta^p}{3^p-3^{pr}}\|x\|_A^{pr},\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x\in A$. So we can conclude that $H(x)=T(x)$ for all $x\in A$. This proves the uniqueness of H.

The rest of the proof is similar to the proof of Theorem 2.1.

Theorem 2.4 *Let $r>2$ and θ be positive real numbers, and let $f:A\to B$ be a mapping with $f(0)=0$ satisfying (2.2) and (2.8). If $f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, then there exists a unique homomorphism $H:A\to B$ such that*

$$\|f(x)-H(x)\|_B\le\frac{K(3^r+3)\theta}{(3^{pr}-3^p)^{\frac{1}{p}}}\|x\|_A^r \quad \textit{for all } x\in A. \tag{2.12}$$

Proof It follows from (2.10) that

$$\Big\|f(x)-3f\Big(\frac{x}{3}\Big)\Big\|_B\le\frac{K(3^r+3)\theta}{3^r}\|x\|_A^r \quad \text{for all } x\in A.$$

Since B is a p-Banach algebra,

$$\begin{aligned}\Big\|3^lf\Big(\frac{x}{3^l}\Big)-3^mf\Big(\frac{x}{3^m}\Big)\Big\|_B^p &\le \sum_{j=l}^{m-1}\Big\|3^jf\Big(\frac{x}{3^j}\Big)-3^{j+1}f\Big(\frac{x}{3^{j+1}}\Big)\Big\|_B^p\\ &\le \frac{K^p(3^r+3)^p\theta^p}{3^{pr}}\sum_{j=l}^{m-1}\frac{3^{pj}}{3^{prj}}\|x\|_A^{pr}\end{aligned} \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It follows from (2.13) that the sequence $\{3^n f(\frac{x}{3^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{3^n f(\frac{x}{3^n})\}$ converges. So one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 3^n f\Big(\frac{x}{3^n}\Big) \quad \text{for all } x \in A.$$

The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3.

3 Isomorphisms Between Quasi-Banach Algebras

Throughout this section, assume that A is a quasi-Banach algebra with quasi-norm $\|\cdot\|_A$ and unit e and that B is a p-Banach algebra with p-norm $\|\cdot\|_B$ and unit e'. Let K be the modulus of concavity of $\|\cdot\|_B$.

We investigate isomorphisms between quasi-Banach algebras, associated to the Cauchy functional equation.

Theorem 3.1 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *such that*

$$f(xy) = f(x)f(y) \quad \textit{for all } x, y \in A. \tag{3.1}$$

If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and

$$\lim_{n\to\infty} 2^n f\Big(\frac{e}{2^n}\Big) = e',$$

then the mapping $f : A \to B$ is an isomorphism.

Proof Since

$$f(xy) - f(x)f(y) = 0 \quad \text{for all } x, y \in A,$$

the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.1, there exists a homomorphism $H : A \to B$ satisfying (2.3). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} 2^n f\Big(\frac{x}{2^n}\Big) \quad \text{for all } x \in A.$$

It follows from (3.1) that

$$\begin{aligned} H(x) = H(ex) &= \lim_{n\to\infty} 2^n f\Big(\frac{ex}{2^n}\Big) = \lim_{n\to\infty} 2^n f\Big(\frac{e}{2^n}\cdot x\Big) = \lim_{n\to\infty} 2^n f\Big(\frac{e}{2^n}\Big) f(x) \\ &= e' f(x) = f(x) \quad \text{for all } x \in A. \end{aligned}$$

So the bijective mapping $f : A \to B$ is an isomorphism.

Theorem 3.2 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping satisfying* (2.1) *and* (3.1). *If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and*

$$\lim_{n\to\infty} \frac{1}{2^n} f(2^n e) = e',$$

then the mapping $f : A \to B$ is an isomorphism.

Proof Since

$$f(xy) - f(x)f(y) = 0 \quad \text{for all } x, y \in A,$$

the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.2, there exists a homomorphism $H : A \to B$ satisfying (2.6). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \quad \text{for all } x \in A.$$

The rest of the proof is similar to the proof of Theorem 3.1.

We investigate isomorphisms between quasi-Banach algebras, associated to the Jensen functional equation.

Theorem 3.3 *Let $r < 1$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping with $f(0) = 0$ satisfying* (2.8) *and* (3.1)*. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and*

$$\lim_{n\to\infty} \frac{1}{3^n} f(3^n e) = e',$$

then the mapping $f : A \to B$ is an isomorphism.

Proof Since

$$f(xy) - f(x)f(y) = 0 \quad \text{for all } x, y \in A,$$

the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.3, there exists a homomorphism $H : A \to B$ satisfying (2.9). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) \quad \text{for all } x \in A.$$

The rest of the proof is similar to the proof of Theorem 3.1.

Theorem 3.4 *Let $r > 2$ and θ be positive real numbers, and let $f : A \to B$ be a bijective mapping with $f(0) = 0$ satisfying* (2.8) *and* (3.1)*. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and*

$$\lim_{n\to\infty} 3^n f\left(\frac{e}{3^n}\right) = e',$$

then the mapping $f : A \to B$ is an isomorphism.

Proof Since

$$f(xy) - f(x)f(y) = 0 \quad \text{for all } x, y \in A,$$

the mapping $f : A \to B$ satisfies (2.2). By Theorem 2.4, there exists a homomorphism $H : A \to B$ satisfying (2.12). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right) \quad \text{for all } x \in A.$$

The rest of the proof is similar to the proof of Theorem 3.1.

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