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# Homogenization of Reynolds Equation by Two-Scale Convergence

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**Abstract** To increase the hydrodynamic performance in different machine elements, as e.g. journal bearings and thrust bearings, during lubrication it is important to understand the influence of surface roughness. In this connection one encounters homogenization of the incompressible Reynolds equation, where the roughness of the lubricated surface is assumed to be periodic. This problem has recently been studied in more engineering-oriented papers by using the formal method of multiple scale expansion. In this paper, we rigorously prove both homogenization and corrector results by using two-scale convergence, which may be regarded as a justification of the formal multiple scale expansion method described above. Moreover, some numerical illustrations and results are presented.

**Keywords** Homogenization, Two-scale convergence, Reynolds equation, Lubrication
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## 1 Introduction

An important problem in the theory of lubrication for thin films is to describe the flow behavior between two surfaces in relative motion. In this connection one encounter the incompressible Reynolds equation. If $\mu$ is the viscosity of the lubricant and the relative motion only takes place in the $x_1$ direction at the speed $V$, then the equation is

$$\mathrm{div}(h^3(x)\nabla p(x)) = \Lambda\frac{\partial h(x)}{\partial x_1} \quad \text{on } \Omega \subset \mathbb{R}^2,$$

where $p$ is the pressure, $h$ the film thickness and $\Lambda = 6\mu V$.

In this paper, we focus on the effects of a periodical surface roughness. The film thickness is assumed to be described by

$$h_\varepsilon(x) = h_0(x) + h_1\Big(\frac{x}{\varepsilon}\Big), \quad \varepsilon > 0,$$

where $h_0$ is the global film thickness and $h_1$ is a periodic function which represents the roughness contribution. Thus $\varepsilon$ is a parameter which describes the roughness wavelength. This together with the Reynolds equation leads to the differential equation

$$\mathrm{div}(h_\varepsilon^3(x)\nabla p_\varepsilon(x)) = \Lambda\frac{\partial h_\varepsilon(x)}{\partial x_1} \quad \text{on } \Omega \subset \mathbb{R}^2. \tag{1.1}$$

For small values of $\varepsilon$ the coefficient $h_\varepsilon^3$ includes rapidly oscillating functions. Therefore it is natural to apply some type of asymptotic analysis in the study of this equation. The mathematical theory which has been developed for this purpose is known as homogenization. We will

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see that the solutions $p_\varepsilon$ converges (in a sense which will be specified below) to the solution of a homogenized problem. We remark that the homogenization of (1.1) is not that of the classical problem (see e.g. [1, 6] or [9]) due to the fact that the right hand side does not converge in $W^{-1,2}(\Omega)$.

In the more engineering-oriented papers [3, 8] the formal method of multiple scale expansion was applied to study the homogenization of (1.1). Indeed it was assumed that $p_\varepsilon$ has the form

$$p_\varepsilon(x) = p_0(x) + \varepsilon p_1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2 p_2\Big(x, \frac{x}{\varepsilon}\Big) + \cdots .$$

The result of this analysis is that $p_0$ is the solution of a homogenized problem of the type

$$\operatorname{div}(B(x)\nabla p_0(x)) = \operatorname{div}(c(x)) \quad \text{on } \Omega \subset \mathbb{R}^2, \tag{1.2}$$

where $B$ and $c$ do not involve any rapid oscillations.

In 1989, Nguetseng introduced a new method for analyzing homogenization problems (see [9]). This method was later developed by Allaire in [1] and called two-scale convergence. The method of two scale convergence is now frequently used among mathematicians in the study of different homogenization problems. In this paper, we use this two-scale convergence technique to rigorously homogenize (1.1). This together with a so-called corrector result, which we also prove, may be regarded as justification of the formal multiple scale expansion method described above.

We want to mention that the compressible Reynolds equation has been studied by two scale convergence in [7] and the incompressible Reynolds equation has been analyzed by the theory of $G$-convergence in [2, 5]. The solution of the homogenized equation (1.2) is fairly complex since it for each $x \in \Omega$ involves the solutions of three periodic problems. The numerical aspects in this connection are considered in [4]. The stochastic Reynolds equation for hydrodynamic lubrication with random homogeneous roughness of the lubricated surface has been studied in [12] by using series expansions. We remark that the homogenization of (1.1) may also be done by moving the right hand side to the left hand side and then applying Tartar's method of oscillating test functions. However, the focus of this work is two-scale convergence.

## 2 Preliminaries and Notation

Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$ and $Y$ the unit cube. The set of infinitely differentiable $Y$-periodic functions is denoted by $C^\infty_{\text{per}}(Y)$. Let us recall some basic facts concerning two-scale convergence. For details the reader is referred to [1, 9, 10] or [11].

**Definition 2.1** *Let $(p_\varepsilon)$ be a bounded sequence in $L^2(\Omega)$ and $p_0 \in L^2(\Omega \times Y)$. Then we say that $(p_\varepsilon)$ two-scale converges weakly to $p_0$ (we write $p_\varepsilon \overset{2}{\rightharpoonup} p_0$) if*

$$\int_\Omega p_\varepsilon(x)\phi\Big(x, \frac{x}{\varepsilon}\Big)\,dx \to \int_\Omega\int_Y p_0(x,y)\phi(x,y)\,dydx, \quad \text{as } \varepsilon \to 0 \tag{2.1}$$

*for every test function $\phi$ of the form $\phi(x,y) = \psi(x)\sigma(y)$, where $\psi \in C_0^\infty(\Omega)$ and $\sigma \in C^\infty_{\text{per}}(Y)$ (infinitely differentiable $Y$-periodic functions).*

Weak two-scale convergence implies weak convergence in $L^2(\Omega)$. This is seen by choosing test functions $\phi$ in (2.1), which are independent of $y$. One of the crucial results concerning two-scale convergence is the following compactness theorem.

**Theorem 2.1** *If $(p_\varepsilon)$ is a bounded sequence in $L^2(\Omega)$, then there exists a subsequence which two-scale converges weakly.*

In addition to weak two-scale convergence we also define strong two-scale convergence.

**Definition 2.2** *Let $(p_\varepsilon)$ be a bounded sequence in $L^2(\Omega)$ and $p_0 \in L^2(\Omega \times Y)$. Then we say that $(p_\varepsilon)$ two-scale converges strongly to $p_0$ (we write $p_\varepsilon \xrightarrow{2} p_0$) if for any bounded sequence $(v_\varepsilon)$ in $L^2(\Omega)$ which two-scale converges weakly to $v \in L^2(\Omega \times Y)$, we have that*

$$\int_\Omega p_\varepsilon(x) v_\varepsilon(x)\, dx \to \int_\Omega \int_Y p_0(x,y) v(x,y)\, dydx.$$

**Theorem 2.2** *Weak two-scale convergence of the sequence $(p_\varepsilon)$ in $L^2(\Omega)$ to $p_0 \in L^2(\Omega \times Y)$ together with*

$$\lim_{\varepsilon \to 0} \int_\Omega |p_\varepsilon|^2 \, dx = \int_\Omega \int_Y |p_0|^2 \, dydx$$

*is equivalent to strong two-scale convergence of $(p_\varepsilon)$ to $p_0$.*

In applications of weak two-scale convergence it is often important to enlarge the class of test functions $\phi$ for which the convergence (2.1) holds true. Let $\mathcal{A}$ be the class of $Y$-periodic extensions of functions $\phi$ in $L^2(\Omega \times Y)$ for which it holds that $\phi(x, \frac{x}{\varepsilon}) \xrightarrow{2} \phi$. The functions in $\mathcal{A}$ will be referred to as admissible test functions. A corollary of Theorem 2.2 is

**Corollary 2.1** *If a sequence $(p_\varepsilon)$ in $L^2(\Omega)$ two-scale converges weakly to $p_0 \in L^2(\Omega \times Y)$, then*

$$\int_\Omega p_\varepsilon(x) \phi\Big(x, \frac{x}{\varepsilon}\Big)\, dx \to \int_\Omega \int_Y p_0(x,y)\phi(x,y)\, dydx, \quad \forall \phi \in \mathcal{A}.$$

One important subset of $\mathcal{A}$ is $L^2_{\text{per}}(Y; C(\overline{\Omega}))$, more precisely the class of functions $\phi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ which satisfies:

(1) The function $x \to \phi(x,y)$ is continuous for $\mu$-almost every $y$.

(2) The function $y \to \phi(x,y)$ is $\mu$-measurable and $Y$-periodic for every $x \in \Omega$.

(3) The function $y \to \sup\limits_{x \in \Omega} |\phi(x,y)|$ is in $L^2_{\text{per}}(Y)$.

An other subset of $\mathcal{A}$ is the set of functions $\phi$ on the form $\phi(x,y) = \phi_1(x)\phi_2(y)$, $\phi_1 \in L^{2s}(\Omega)$, $\phi_2 \in L^{2t}_{\text{per}}(Y)$ with $1 \leq s, t \leq \infty$ and such that $\frac{1}{s} + \frac{1}{t} = 1$.

Denote by $W^{1,2}_{\text{per}}(Y)$ the completion of the set of functions in $C^\infty_{\text{per}}(Y)$ which have mean value zero with respect to the usual norm on $W^{1,2}(Y)$. The next fundamental result concerns weak two-scale convergence in Sobolev spaces.

**Theorem 2.3** *If $(p_\varepsilon)$ is a sequence in $W^{1,2}_0(\Omega)$ such that $p_\varepsilon(x) \overset{2}{\rightharpoonup} p_0(x,y)$ and $\nabla p_\varepsilon(x) \overset{2}{\rightharpoonup} z(x,y)$. Then the weak two-scale limit $p_0$ is independent of $y$ and belongs to $W^{1,2}_0(\Omega)$, i.e., $p_0(x,y) = p_0(x) \in W^{1,2}_0(\Omega)$. Moreover, $z(x,y) = \nabla p_0(x) + \nabla_y p_1(x,y)$, where $p_1 \in L^2(\Omega; W^{1,2}_{\text{per}}(Y))$.*

## 3 Homogenization of Reynolds Equation

Let $\Omega$ be an open bounded subset of $\mathbb{R}^2$ and $Y$ the unit cube. Let $h : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be of the form $h(x,y) = h_0(x) + h_1(y)$, where $h_0 \in C(\overline{\Omega})$, $h_1 \in L^\infty(\mathbb{R}^N)$ and $h_1$ is a $Y$-periodic function. We also assume that there exists a constant $\alpha > 0$ such that $h(x,y) \geq \alpha$. Define

$$h_\varepsilon(x) = h\Big(x, \frac{x}{\varepsilon}\Big) = h_0(x) + h_1\Big(\frac{x}{\varepsilon}\Big).$$

Consider Reynolds equation: Find $p_\varepsilon \in W_0^{1,2}(\Omega)$ such that

$$\operatorname{div}(h_\varepsilon^3 \nabla p_\varepsilon) = \Lambda \frac{\partial h_\varepsilon}{\partial x_1} \quad \text{on } \Omega. \tag{3.1}$$

By definition $p_\varepsilon$ is a solution of (3.1) if the following integral identity holds:

$$\int_\Omega h_\varepsilon^3 \nabla p_\varepsilon \cdot \nabla \phi \, dx = \Lambda \int_\Omega h_\varepsilon \frac{\partial \phi}{\partial x_1} \, dx, \quad \forall \phi \in W_0^{1,2}(\Omega). \tag{3.2}$$

Consider the three periodic problems: Find $v_1, v_2, v_3 \in L^2(\Omega; W_{\mathrm{per}}^{1,2}(Y))$ such that

$$\int_Y h^3 [e_1 + \nabla_y v_1] \cdot \nabla w \, dy = 0, \quad \forall\, w \in C_{\mathrm{per}}^\infty(Y), \tag{3.3}$$

$$\int_Y h^3 [e_2 + \nabla_y v_2] \cdot \nabla w \, dy = 0, \quad \forall\, w \in C_{\mathrm{per}}^\infty(Y), \tag{3.4}$$

$$\int_Y h^3 \nabla_y v_3 \cdot \nabla w \, dy = \Lambda \int_Y h \frac{\partial w}{\partial y_1} \, dy, \quad \forall\, w \in C_{\mathrm{per}}^\infty(Y), \tag{3.5}$$

where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$. We remark that the periodic problems (3.3)–(3.5) have a unique solution. Define the matrix function $B(x) = (b_{ij}(x))$ in terms of $v_1$ and $v_2$

$$\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \int_Y h^3 \left[ e_1 + \nabla_y v_1 \right] \, dy, \quad \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = \int_Y h^3 \left[ e_2 + \nabla_y v_2 \right] \, dy. \tag{3.6}$$

Moreover, let the vector function $c(x) = (c_1(x), c_2(x))$ be defined via $v_3$

$$c_1 = \int_Y \left( \Lambda h - h^3 \frac{\partial v_3}{\partial y_1} \right) dy \quad \text{and} \quad c_2 = \int_Y -h^3 \frac{\partial v_3}{\partial y_2} \, dy. \tag{3.7}$$

The homogenization problem corresponding to (3.1) considers the asymptotic behavior of $p_\varepsilon$ as $\varepsilon \to 0$. We have the following homogenization result.

**Theorem 3.1** *The sequence of solutions $p_\varepsilon$ of* (3.1) *converges weakly in $W_0^{1,2}(\Omega)$ to the solution $p_0 \in W_0^{1,2}(\Omega)$ of the homogenized equation*

$$\int_\Omega B(x) \nabla p_0 \cdot \nabla \phi \, dx = \int_\Omega c(x) \cdot \nabla \phi \, dx, \quad \phi \in C_0^\infty(\Omega), \tag{3.8}$$

*where $B$ and $c$ are defined as in* (3.6) *and* (3.7). *Moreover, $\nabla p_\varepsilon \overset{2}{\rightharpoonup} \nabla p_0(x) + \nabla_y p_1(x, y)$, where $p_1 \in L^2(\Omega; W_{\mathrm{per}}^{1,2}(Y))$ and may be expressed in the solutions of the periodic problems* (3.3)–(3.5)*:*

$$p_1(x, y) = v_1(x, y) \frac{\partial p_0}{\partial x_1} + v_2(x, y) \frac{\partial p_0}{\partial x_2} + v_3(x, y). \tag{3.9}$$

**Proof** Choose $\phi = p_\varepsilon$ as a test function in (3.2)

$$\int_\Omega h_\varepsilon^3 \left| \nabla p_\varepsilon \right|^2 \, dx = \Lambda \int_\Omega h_\varepsilon \frac{\partial p_\varepsilon}{\partial x_1} \, dx \le \Lambda \int_\Omega h_\varepsilon \left| \nabla p_\varepsilon \right| \, dx.$$

By the assumptions on $h_0$ and $h_1$ it follows that there exists a constant $c$ such that

$$\int_\Omega \left| \nabla p_\varepsilon \right|^2 \, dx \le c.$$

This together with the Poincare inequality implies that the sequences $(p_\varepsilon)$ and $(\nabla p_\varepsilon)$ are bounded in $L^2(\Omega)$ and $L^2(\Omega)^2$, respectively. By Theorem 2.1 there exists a subsequence (still denoted by $\varepsilon$) such that $p_\varepsilon(x) \overset{2}{\rightharpoonup} p_0(x,y)$ and $\nabla p_\varepsilon(x) \overset{2}{\rightharpoonup} z(x,y)$. By Theorem 2.3 it follows that $p_0(x,y) = p_0(x) \in W_0^{1,2}(\Omega)$ and $z(x,y) = \nabla p_0(x) + \nabla_y p_1(x,y)$, where $p_1 \in L^2(\Omega; W^{1,2}_{\text{per}}(Y))$. Moreover, Corollary 2.1 implies that

$$h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon(x) \overset{2}{\rightharpoonup} h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]. \tag{3.10}$$

Since weak two-scale convergence implies weak convergence in $L^2(\Omega)$ we have that

$$h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon \to \int_Y h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]\, dy \quad \text{weakly in } L^2(\Omega).$$

We can now pass to the limit in (3.2) and obtain

$$\int_\Omega \int_Y h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]\, dy \cdot \nabla \phi \, dx = \Lambda \int_\Omega \int_Y h(x,y) \frac{\partial \phi}{\partial x_1}\, dy dx \tag{3.11}$$

for every $\phi \in C_0^\infty(\Omega)$.

Let $w_\varepsilon(x) \overset{\text{def}}{=} \varepsilon \varphi(x) w(\frac{x}{\varepsilon})$, where $\varphi \in C_0^\infty(\Omega)$ and $w \in C^\infty_{\text{per}}(Y)$. Then $w_\varepsilon \in C_0^\infty(\Omega)$ and can thus be used as a test function in (3.2) which gives

$$\begin{aligned} &\int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon \cdot \varphi \nabla w\Big(\frac{x}{\varepsilon}\Big)\, dx + \varepsilon \int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon \cdot w\Big(\frac{x}{\varepsilon}\Big)\nabla \varphi \, dx \\ &= \Lambda \int_\Omega h\Big(x, \frac{x}{\varepsilon}\Big)\varphi \frac{\partial w}{\partial x_1}\Big(\frac{x}{\varepsilon}\Big)\, dx + \varepsilon\Lambda \int_\Omega h\Big(x, \frac{x}{\varepsilon}\Big) w\Big(\frac{x}{\varepsilon}\Big)\frac{\partial \varphi}{\partial x_1}\, dx. \end{aligned} \tag{3.12}$$

Let us now consider what will happen if $\varepsilon \to 0$ in (3.12). First we note that both the second term in the left hand side and the second term in the right hand side of (3.12) tends to 0 as $\varepsilon \to 0$. From these two observations we conclude that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon \cdot \varphi \nabla w\Big(\frac{x}{\varepsilon}\Big)\, dx &= \lim_{\varepsilon\to 0} \Lambda \int_\Omega h\Big(x, \frac{x}{\varepsilon}\Big)\varphi \frac{\partial w}{\partial x_1}\Big(\frac{x}{\varepsilon}\Big)\, dx \\ &= \Lambda \int_\Omega \int_Y h(x,y)\varphi(x)\frac{\partial w}{\partial y_1}(y)\, dy dx. \end{aligned} \tag{3.13}$$

This together with (3.10) gives

$$\begin{aligned} &\int_\Omega \int_Y h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]\varphi(x) \cdot \nabla w(y)\, dy dx \\ &= \Lambda \int_\Omega \int_Y h(x,y)\varphi(x)\frac{\partial w}{\partial y_1}(y)\, dy dx. \end{aligned} \tag{3.14}$$

Since $\varphi \in C_0^\infty(\Omega)$ is arbitrary we have that (for a.e. $x$) $p_1(x,y)$ is the solution of the periodic problem: Find $p_1 \in L^2(\Omega; W^{1,2}_{\text{per}}(Y))$ such that

$$\int_Y h^3(x,y)\,[\nabla p_0(x) + \nabla_y p_1(x,y)] \cdot \nabla w(y)\, dy = \Lambda \int_Y h(x,y)\frac{\partial w}{\partial y_1}(y)\, dy \tag{3.15}$$

for any $w \in C^\infty_{\text{per}}(Y)$; or

$$\int_Y h^3 \nabla_y p_1 \cdot \nabla w \, dy = \Lambda \int_Y h \frac{\partial w}{\partial y_1}\, dy - \frac{\partial p_0}{\partial x_1}\int_Y h^3 \frac{\partial w}{\partial y_1}\, dy - \frac{\partial p_0}{\partial x_2}\int_Y h^3 \frac{\partial w}{\partial y_2}\, dy$$

for any $w \in C^\infty_{\mathrm{per}}(Y)$. By linearity it is clear that $p_1$ is of the form

$$p_1(x,y) = v_1(x,y)\frac{\partial p_0}{\partial x_1} + v_2(x,y)\frac{\partial p_0}{\partial x_2} + v_3(x,y), \tag{3.16}$$

where $v_n \in L^2(\Omega; W^{1,2}_{\mathrm{per}}(Y))$, $n = 1,2,3$, and solves a corresponding periodic problem (see (3.3)–(3.5))

$$\begin{aligned}
\int_Y h^3 \nabla_y v_1 \cdot \nabla w \, dy &= -\int_Y h^3 \frac{\partial w}{\partial y_1}\, dy, \quad \forall\, w \in C^\infty_{\mathrm{per}}(Y),\\
\int_Y h^3 \nabla_y v_2 \cdot \nabla w \, dy &= -\int_Y h^3 \frac{\partial w}{\partial y_2}\, dy, \quad \forall\, w \in C^\infty_{\mathrm{per}}(Y),\\
\int_Y h^3 \nabla_y v_3 \cdot \nabla w \, dy &= \Lambda \int_Y h \frac{\partial w}{\partial y_1}\, dy, \quad \forall\, w \in C^\infty_{\mathrm{per}}(Y).
\end{aligned}$$

Substitution of (3.16) into the equation (3.11) gives the desired result

$$\int_\Omega B(x)\nabla p_0 \cdot \nabla\phi \, dx = \int_\Omega c(x)\cdot\nabla\phi\, dx.$$

Thus the theorem is proved for a subsequence. As we will see in Section 4, $B(x)$ is symmetric and there exists a constant $k > 0$ such that

$$k^{-1}\,|\xi|^2 \le B(x)\xi\cdot\xi \le k\,|\xi|^2\,.$$

From this it follows by the Lax-Milgram lemma that the homogenized equation (3.8) has a unique solution and thus that the theorem holds for the whole sequence.

We remark that from the point of homogenization it is nothing special with the boundary condition $p_\varepsilon = 0$, which is implicitly embedded in (3.1). Moreover, the continuity assumption on the global film thickness $h_0$ may be relaxed.

## 4 Properties of the Homogenized Matrix *B*

By linearity the homogenized matrix $B(x)$ satisfies

$$B(x)\xi = \int_Y h^3(x,y)(\xi + \nabla v_\xi(y))dy, \quad \forall\,\xi\in\mathbb{R}^2,$$

where $v_\xi \in W^{1,2}_{\mathrm{per}}(Y)$ is the solution of the periodic problem:

$$\int_Y h^3(x,y)(\xi+\nabla v_\xi(y))\cdot\nabla w(y)\,dy = 0, \quad \forall\, w\in W^{1,2}_{\mathrm{per}}(Y). \tag{4.1}$$

We observe that the periodic problem (4.1) actually is the Euler equation to the minimization problem:

$$\min_{v\in W^{1,2}_{\mathrm{per}}(Y)} \int_Y h^3(x,y)|\xi+\nabla v(y)|^2 dy.$$

Thus it follows by periodicity that

$$\begin{aligned}
B(x)\xi\cdot\xi &= \int_Y h^3(x,y)|\xi+\nabla v_\xi(y)|^2 dy\\
&= \min_{v\in W^{1,2}_{\mathrm{per}}(Y)} \int_Y h^3(x,y)\,|\xi+\nabla v(y)|^2\,dy. \qquad (4.2)
\end{aligned}$$

The assumptions on $h$ implies that there exist two constants $c_2 \geq c_1 > 0$ such that

$$\min_{v \in W^{1,2}_{\mathrm{per}}(Y)} \int_Y c_1 \left|\xi + \nabla v\right|^2 dy \leq B(x)\xi \cdot \xi \leq \min_{v \in W^{1,2}_{\mathrm{per}}(Y)} \int_Y c_2 \left|\xi + \nabla v\right|^2 dy. \tag{4.3}$$

By choosing $v = 0$ as a test function in the right hand side we obtain that

$$B(x)\xi \cdot \xi \leq c_2 |\xi|^2.$$

By (4.3) and homogeneity we get that

$$B(x)\xi \cdot \xi \geq \min_{v \in W^{1,2}_{\mathrm{per}}(Y)} \int_Y c_1 |\xi + \nabla v|^2 dy = c_1 |\xi|^2 \min_{v \in W^{1,2}_{\mathrm{per}}(Y)} \int_Y |\eta + \nabla v|^2 dy,$$

where $\eta = |\xi|^{-1}\xi$. The set $\{\eta : \eta = |\xi|^{-1}\xi\}$ is compact in $\mathbb{R}^2$. Hence the strictly positive continuous function $m$ defined as

$$m(\eta) \stackrel{\text{def}}{=} \min_{v \in W^{1,2}_{\mathrm{per}}(Y)} \int_Y |\eta + \nabla v|^2 \, d\mu$$

attains its minimum, denoted by $m_0$. Hence

$$B(x)\xi \cdot \xi \geq c_1 m_0 |\xi|^2.$$

The homogenized matrix is symmetric. Indeed, if $v_\xi$ and $v_\eta$ are the solutions of the periodic problems

$$\int_Y h^3[\xi + \nabla_y v_\xi] \cdot \nabla w \, dy = 0, \quad \forall\, w \in W^{1,2}_{\mathrm{per}}(Y),$$
$$\int_Y h^3[\eta + \nabla_y v_\eta] \cdot \nabla w \, dy = 0, \quad \forall\, w \in W^{1,2}_{\mathrm{per}}(Y),$$

then

$$\begin{aligned} B(x)\xi \cdot \eta &= \int_Y h^3(x,y)(\xi + \nabla v_\xi(y)) \cdot (\eta + \nabla v_\eta(y)) dy \\ &= \int_Y (\xi + \nabla v_\xi(y)) \cdot h^3(x,y)(\eta + \nabla v_\eta(y)) dy = \xi \cdot B(x)\eta. \end{aligned}$$

## 5 Corrector Results

By Theorem 3.1 the sequence of solutions $(p_\varepsilon)$ of the Reynolds equations (3.1) converges weakly to $p_0$ in $W^{1,2}_0(\Omega)$ and the sequence $\nabla p_\varepsilon$ two-scale converges weakly to $\nabla p_0(x) + \nabla_y p_1(x,y)$, where $p_0$ is the solution of the homogenized equation (3.8) and $p_1$ is given by the relation (3.9). Since the imbedding of $W^{1,2}_0(\Omega)$ in $L^2(\Omega)$ is compact we have that $p_\varepsilon$ converges to $p_0$ strongly in $L^2(\Omega)$. For the gradients we only have weak convergence of $\nabla p_\varepsilon$ to $\nabla p_0$ in $L^2(\Omega)^2$. To improve this convergence we have to add an extra term, a so called corrector, which take care of the oscillations. We will now see that such corrector results can be obtained by using two-scale convergence.

**Theorem 5.1** *If* $\nabla_y p_1(x,y)$ *is an admissible test function, then*

$$\nabla p_\varepsilon - \nabla p_0 - \nabla p_1\left(\cdot, \frac{\cdot}{\varepsilon}\right) \to 0 \quad \text{in } [L^2(\Omega)]^2.$$

**Proof** By the condition $h(x,y) \geq \alpha > 0$ and the Hölder inequality it follows

$$\begin{aligned}
&\alpha \int_\Omega \Big| \nabla p_\varepsilon - \nabla p_0 - \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big|^2 dx \\
&\leq \int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\Big(\nabla p_\varepsilon - \nabla p_0 - \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big) \cdot \Big(\nabla p_\varepsilon - \nabla p_0 - \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)\, dx \\
&= \int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon \cdot \nabla p_\varepsilon \, dx - 2\int_\Omega \nabla p_\varepsilon \cdot h^3\Big(x, \frac{x}{\varepsilon}\Big)\Big(\nabla p_0 + \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)dx \\
&\quad + \int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\Big(\nabla p_0 + \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big) \cdot \Big(\nabla p_0 + \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)dx. \qquad (5.1)
\end{aligned}$$

Let us now study the convergence as $\varepsilon \to 0$ for the three terms in the right hand side of (5.1) separately.

**Term 1** Choose $p_\varepsilon$ as a test function in (3.2). Since $h$ is an admissible test function and $\nabla p_\varepsilon \xrightharpoonup{2} \nabla p_0 + \nabla_y p_1$ it follows

$$\int_\Omega h^3\Big(x, \frac{x}{\varepsilon}\Big)\nabla p_\varepsilon \cdot \nabla p_\varepsilon \, dx = \Lambda \int_\Omega h\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial p_\varepsilon}{\partial x_1}\, dx \to \Lambda \int_\Omega \int_Y h\Big(\frac{\partial p_0}{\partial x_1} + \frac{\partial p_1}{\partial y_1}\Big)dydx. \qquad (5.2)$$

**Term 2** By assumption $\nabla_y p_1$ is an admissible test function. Hence

$$\begin{aligned}
&\int_\Omega \nabla p_\varepsilon \cdot h^3\Big(x, \frac{x}{\varepsilon}\Big)\Big(\nabla p_0 + \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)dx \\
&\to \int_\Omega \int_Y (\nabla p_0 + \nabla_y p_1) \cdot h^3(\nabla p_0 + \nabla_y p_1)dydx. \qquad (5.3)
\end{aligned}$$

**Term 3** Again we use that $\nabla_y p_1$ is an admissible test function and obtain

$$\begin{aligned}
&\int_\Omega h_\varepsilon^3\Big(\nabla p_0 + \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big) \cdot \Big(\nabla p_0 + \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big)dx \\
&\to \int_\Omega \int_Y h^3(\nabla p_0 + \nabla_y p_1) \cdot (\nabla p_0 + \nabla_y p_1)dydx. \qquad (5.4)
\end{aligned}$$

From (5.1)–(5.4) it follows

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \alpha \int_\Omega \Big| \nabla p_\varepsilon - \nabla p_0 - \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big|^2 dx \\
&\leq \Lambda \int_\Omega \int_Y h\Big(\frac{\partial p_0}{\partial x_1} + \frac{\partial p_1}{\partial y_1}\Big)dydx - \int_\Omega \int_Y h^3(\nabla p_0 + \nabla_y p_1) \cdot (\nabla p_0 + \nabla_y p_1)dydx. \qquad (5.5)
\end{aligned}$$

We have that $L^2(\Omega; W^{1,2}_{\text{per}}(Y))$ is the closure in $L^2(\Omega \times Y)$ of the linear span of vectors $\varphi(x)w(y)$, where $\varphi \in C_0^\infty(\Omega)$ and $w \in C^\infty_{\text{per}}(Y)$. Thus $p_1$ may be chosen as a test function in (3.14). This fact together with (3.11) implies that the right hand side of (5.5) is equal to zero. Hence

$$\lim_{\varepsilon \to 0} \int_\Omega \Big| \nabla p_\varepsilon - \nabla p_0 - \nabla_y p_1\Big(x, \frac{x}{\varepsilon}\Big)\Big|^2 dx = 0$$

and the proof is completed.

We remark that if $p_1$, $\nabla_x p_1$ and $\nabla_y p_1$ are admissible, then

$$p_\varepsilon(x) - p_0(x) - \varepsilon p_1\Big(x, \frac{x}{\varepsilon}\Big) \to 0 \quad \text{in } W^{1,2}(\Omega).$$

We also recall the relation (3.16), i.e., that $p_1$ is of the form

$$p_1(x,y) = v_1(x,y)\frac{\partial p_0}{\partial x_1} + v_2(x,y)\frac{\partial p_0}{\partial x_2} + v_3(x,y),$$

where $v_1$, $v_2$ and $v_3$ are the solutions of the respective periodic problem (3.3)–(3.5). Hence $\nabla_y p_1$ is an admissible test function if e.g. $\nabla p_0 \in L^{2s}(\Omega)^2$, $\nabla_y v_1, \nabla_y v_2 \in L^{2t}_{\text{per}}(Y; C(\overline{\Omega}))^2$ and $v_3(x,y)$ is admissible, with $1 \le s,t \le \infty$ and such that $\frac{1}{s} + \frac{1}{t} = 1$.

## 6 Transversal and Longitudinal Roughness

Generally, one has to solve the periodic problems (3.3)–(3.5) by some numerical method to find the homogenized matrix $B$ in (3.6) and the homogenized vector $c$ in (3.7). However, for $h$ of the form $h(x,y) = h_0(x) + h_1(y_1)$ (i.e. the roughness $h_1$ is independent of $y_2$) it is possible to find explicit formulas for the homogenized matrix $B(x)$ and the homogenized vector $c(x)$ without solving the periodic problems.

Consider the first local problem (3.3):

$$\int_Y h^3(x,y_1)\nabla_y v_1 \cdot \nabla w\, dy = -\int_Y h^3(x,y_1)\frac{\partial w}{\partial y_1}\, dy, \quad \forall\, w \in C^\infty_{\text{per}}(Y).$$

Clearly $v_1 = v_1(x,y_1)$. This means that

$$\int_Y h^3(x,y_1)\Big(1 + \frac{\partial v_1}{\partial y_1}\Big)\frac{\partial w}{\partial y_1}\, dy = 0, \quad \forall\, w \in C^\infty_{\text{per}}(Y),$$

which implies that

$$h^3(x,y_1)\Big(1 + \frac{\partial v_1}{\partial y_1}\Big) = k_1(x) \tag{6.1}$$

This together with periodicity gives

$$0 = \int_Y \frac{\partial v_1}{\partial y_1} dy = k_1(x)\int_Y h^{-3}(x,y_1)dy - 1. \tag{6.2}$$

From (6.1) and (6.2) together with the fact that $v_1 = v_1(x,y_1)$ gives

$$\begin{aligned} b_{11}(x) &\stackrel{\text{def}}{=} \int_Y h^3\Big(1 + \frac{\partial v_1}{\partial y_1}\Big)\, dy = \Big(\int_Y h^{-3}\, dy\Big)^{-1} dy, \\ b_{21}(x) &\stackrel{\text{def}}{=} \int_Y h^3 \frac{\partial v_1}{\partial y_2}\, dy = 0. \end{aligned}$$

Let us now consider the second periodic problem (3.4):

$$\int_Y h^3(x,y_1)\nabla_y v_2 \cdot \nabla w\, dy = -\int_Y h^3(x,y_1)\frac{\partial w}{\partial y_2}\, dy, \quad \forall\, w \in C^\infty_{\text{per}}(Y).$$

The right hand side is equal to 0, which implies that $v_2 = 0$. Hence, by (3.6)

$$b_{12}(x) \stackrel{\text{def}}{=} \int_Y h^3 \frac{\partial v_2}{\partial y_1}\, dy = 0, \quad b_{22}(x) \stackrel{\text{def}}{=} \int_Y h^3\Big(1 + \frac{\partial v_2}{\partial y_2}\Big)\, dy = \int_Y h^3\, dy.$$

Finally, consider the third periodic problem (3.5):

$$\int_Y h^3(x,y_1)\nabla_y v_3 \cdot \nabla w\, dy = \Lambda\int_Y h(x,y_1)\frac{\partial w}{\partial y_1}\, dy, \quad \forall\, w \in C^\infty_{\text{per}}(Y).$$

Clearly $v_3$ only depends on $y_1$. Thus for any $w \in C^\infty_{\text{per}}(Y)$ such that $w$ only depends on $y_1$

$$\int_Y h^3(x,y_1)\frac{\partial v_3}{\partial y_1}\cdot\frac{\partial w}{\partial y_1}\,dy = \Lambda\int_Y h(x,y_1)\frac{\partial w}{\partial y_1}\,dy.$$

Hence

$$\Lambda h(x,y_1) - h^3(x,y_1)\frac{\partial v_3}{\partial y_1} = k_2(x). \tag{6.3}$$

By periodicity

$$0 = \int_Y \frac{\partial v_3}{\partial y_1}dy = -k_2(x)\int_Y h^{-3}(x,y_1)\,dy + \Lambda\int_Y h^{-2}(x,y_1)\,dy. \tag{6.4}$$

From (6.3) and (6.4) it follows that the homogenized vector $c$ (3.7) is

$$\begin{aligned} c_1(x) &\overset{\text{def}}{=} \int_Y \Big(\Lambda h - h^3\frac{\partial v_3}{\partial y_1}\Big)dy = \Lambda\int_Y h^{-2}\,dy\Big(\int_Y h^{-3}\,dy\Big)^{-1}, \\ c_2(x) &\overset{\text{def}}{=} \int_Y -h^3\frac{\partial v_3}{\partial y_2}\,dy = 0. \end{aligned}$$

The one dimensional analogue is obvious. In the same way we may consider longitudinal roughness, i.e., $h$ of the form $h(x,y) = h_0(x) + h_1(y_2)$. Then we obtain that the elements in the homogenized matrix $B$ are

$$b_{11}(x) = \int_Y h^3\,dy, \quad b_{22}(x) = \Big(\int_Y h^{-3}\,dy\Big)^{-1}dy, \quad b_{12} = b_{21} = 0,$$

and the elements in the homogenized vector $c$ are

$$c_1(x) = \Lambda\int_Y h\,dy \quad \text{and} \quad c_2(x) = 0.$$

## 7 Numerical Results and Illustrations

Below we give two numerical examples one in one dimension and one in two dimensions, which illustrates the convergence in the homogenization process.

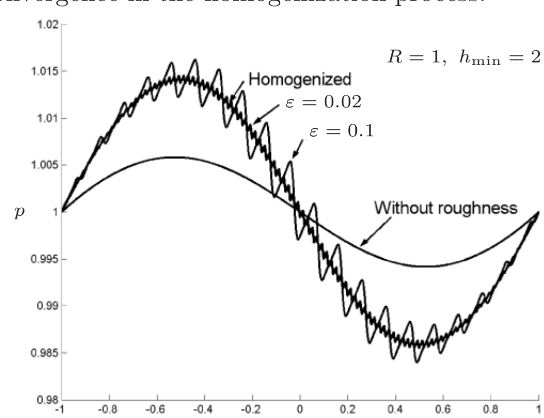


Figure 1 Illustration of the homogenization process

**Example 7.1** Consider the homogenization of the 1-dimensional Reynolds equation (3.1). Assume that $\Omega = (-1, 1)$, the pressure on the boundary is 1 and $h(x, y) = h_0(x) + h_1(y)$, where $h_0 = h_{\min} + \frac{1}{2R}x^2$ and $h_1(y) = \sin(2\pi y)$. In Figure 1 the convergence in the homogenization process is illustrated. In Figure 2, $B^{1/3}$ and $h_0$ are plotted for 4 different values of $h_{\min}$. We observe that for small values of $h_{\min}$ the influence of the surface roughness is more significant.

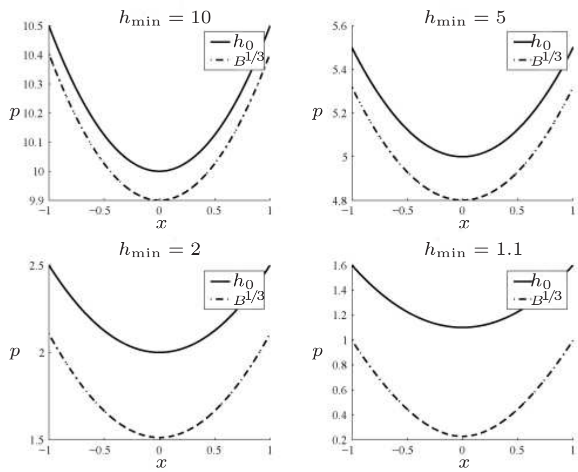


Figure 2 The pressure with and without roughness for 4 different $h_{\min}$

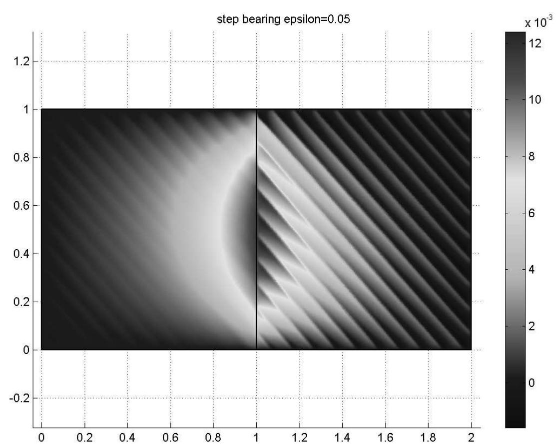


Figure 3 Pressure distribution of a step bearing, $\varepsilon = 0.05$

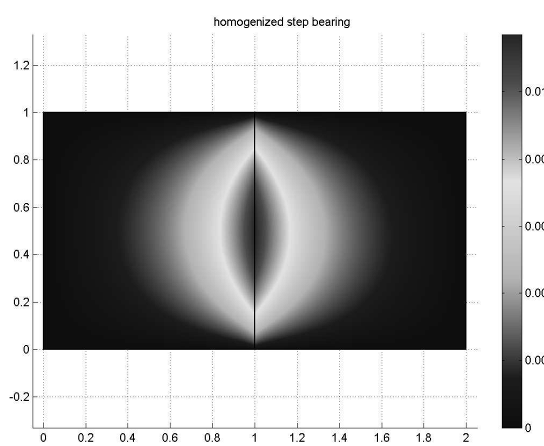


Figure 4 Pressure distribution of a homogenized step bearing

**Example 7.2** (Step Bearing) Let $\Omega = (0,2) \times (0,1)$, $\Lambda = 1$ and

$$h_0(x) = \begin{cases} 4, & x_1 \leq 1, \\ 2, & x_1 > 1 \end{cases} \quad \text{and} \quad h_1(y) = \sin(\pi(y_1 + y_2)).$$

Then a numerical computation gives that homogenized matrix $B(x)$ is equal to $B_1$ for $x_1 \leq 1$ and $B_2$ for $x_1 > 1$, where

$$B_1 = \begin{pmatrix} 61.14 & -1.30 \\ -1.30 & 61.14 \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 6.77 & -0.53 \\ -0.53 & 6.77 \end{pmatrix}$$

and the homogenized vector $c(x)$ is equal to $c_1$ for $x_1 \leq 1$ and $c_2$ for $x_1 > 1$, where

$$c_1 = \begin{pmatrix} 3.82 \\ -0.18 \end{pmatrix} \quad \text{and} \quad c_2 = \begin{pmatrix} 1.67 \\ -0.33 \end{pmatrix}.$$

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