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Proof of the Volume Conjecture for Whitehead Doubles of a Family of Torus Knots**

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Abstract A technique to compute the colored Jones polynomials of satellite knots, illustrated by the Whitehead doubles of knots, is presented. Then the author proves the volume conjecture for Whitehead doubles of a family of torus knots and shows some interesting observations.

Keywords Volume conjecture, Whitehead double, Torus knots, Whitehead link
2000 MR Subject Classification 57M25, 57N10

1 Introduction

The volume conjecture was proposed by Kashaev and reformulated and refined by H. Murakami and J. Murakami as follows.

Conjecture 1.1 (See [2, 5]) *For any knot K,*

$$2\pi \lim_{N\to\infty} \frac{\log \left| J_{K,N}(e^{\frac{2\pi\sqrt{-1}}{N}}) \right|}{N} = v_3 \|S^3 \setminus K\|, \tag{1.1}$$

where $J_{K,N}$ is the (normalized) colored Jones polynomial of K, $\|S^3 \setminus K\|$ is the simplicial volume of the complement of K and v_3 is the volume of the ideal regular tetrahedron.

Recall that $v_3\|S^3 \setminus K\|$ is nothing but the sum of the hyperbolic volumes of hyperbolic pieces in the JSJ-decomposition of the complement of K. In Kashaev's original form, the knot K is hyperbolic and the equation is in terms of the quantum dilogarithm invariant and the hyperbolic volume of the complement of K.

The conjecture is marvellous in the sense that it reveals the topological meaning of the quantum invariants of knots which is quite unobvious from definition. However, it also turns out to be rather hard to be proved. Till now, besides positive numerical evidences (see [1, 6]) for some hyperbolic knots, only the cases of torus knots (see [3]) and the simplest hyperbolic knot, the figure 8 knot (see [2]) have been verified.

In view of the compatible behavior of both sides of the conjectured equation (1.1) under connected sum

$$J_{K_1\sharp K_2,N} = J_{K_1,N} \cdot J_{K_2,N}, \tag{1.2}$$
$$\|S^3 \setminus K_1\sharp K_2\| = \|S^3 \setminus K_1\| + \|S^3 \setminus K_2\|, \tag{1.3}$$

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the volume conjecture, in fact, may be reduced to the consideration of prime knots. By Thurston's hyperbolization theorem (see [7]), the prime knots further fall into three families: torus knots, hyperbolic knots and satellite knots.

In this article, we deal with the conjecture by examining a special case of the third family, the Whitehead doubles of torus knots. The approach is emphasized on the relation between the colored Jones polynomial of a satellite knot and those of the associated companion knot and pattern link. In particular, we show a technique to compute the colored Jones polynomial of satellite knots by cutting and gluing method.

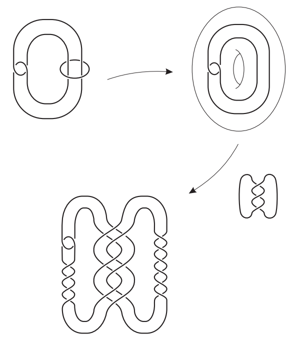

Figure 1

A Whitehead double of a knot K is a knot obtained as follows. Remove the regular neighborhood of one component of the Whitehead link from S^3 and thus get a knot inside a torus. Then knot the torus in the shape of a knot K.

Note that, when K is nontrivial, a Whitehead double K' of K is a satellite knot whose complement contains an obvious essential torus T^2. Cutting along the torus, we get

$$(S^3 \setminus K') \setminus T^2 \cong (S^3 \setminus \text{Whitehead link}) \cup (S^3 \setminus K). \tag{1.4}$$

Thus

$$\|S^3 \setminus K'\| = \|S^3 \setminus \text{Whitehead link}\| + \|S^3 \setminus K\|. \tag{1.5}$$

In particular, if K is a nontrivial torus knot, the complement of K is Seifert fibred and the complement of the Whitehead link is hyperbolic, hence

$$v_3\|S^3 \setminus K'\| = \text{vol}(S^3 \setminus \text{Whitehead link}). \tag{1.6}$$

The article proceeds as follows. First, we compute the colored Jones polynomials of the twisted Whitehead links and the Whitehead doubles of knots in Section 2. Next, as a warming-up we prove in the next two consecutive sections the following two theorems, of which the former one is, in fact, the volume conjecture for twisted Whitehead links and both extend the estimation (1.1) to the second order.

Theorem 1.1 *For every twisted Whitehead link L, we have*

$$2\pi \log \left|J_{L,N}(e^{\frac{2\pi\sqrt{-1}}{N}})\right| = \text{vol}(S^3 \setminus L) \cdot N + 3\pi \log N + O(1), \quad \textit{as } N \to \infty. \tag{1.7}$$

Theorem 1.2 *For every nontrivial torus knot $T(p,q)$ with $q = 2$, we have*

$$2\pi \log \left|J_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}})\right| = 3\pi \log N + O(1), \quad \textit{as } N \to \infty. \tag{1.8}$$

Then we prove the main theorem in Section 5 and show some observations in the final section.

Theorem 1.3 *If K is a Whitehead double of a nontrivial torus knot $T(p,q)$ with $q=2$, then*

$$2\pi \log \left| J_{K,N}(e^{\frac{2\pi\sqrt{-1}}{N}}) \right| = v_3 \|S^3 \setminus K\| \cdot N + 4\pi \log N + O(1), \quad \textit{as } N \to \infty. \tag{1.9}$$

In particular, the volume conjecture is true for K.

Remark 1.1 In their proof of the volume conjecture for torus knots, Kashaev and Tirkkonen [3] derived the following estimation

$$2\pi \log \left| J_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}}) \right| = O(\log N). \tag{1.10}$$

But improving the estimation to (1.8) requires a nonvanishing proposition on number theory (see Proposition 4.1) to which both Theorem 1.2 and Theorem 1.3 are reduced in this article. With a technical condition $q=2$, we proved the nonvanishing proposition in Section 4. A complete proof has been beyond the scope of the article. We only mention here that our technique can be sharpened to prove the nonvanishing proposition, hence both theorems, at least for the cases that both p, q are odd or one of them is a power of 2.

Remark 1.2 It is noteworthy that the coefficient "4π" of the second term in the asymptotic expansion (1.9) disagrees with the observation due to Hikami [1]

$$2\pi \log \left| J_{K,N}(e^{\frac{2\pi\sqrt{-1}}{N}}) \right| = v_3 \|S^3 \setminus K\| \cdot N + 3\pi \log N + O(1) \tag{1.11}$$

for many prime knots K.

2 Computation of Colored Jones Polynomial

In this section, we compute the colored Jones polynomials of the twisted Whitehead link $\mathrm{WL}(r)$ and the Whitehead double $\mathrm{WD}(K,r)$ of a knot K.

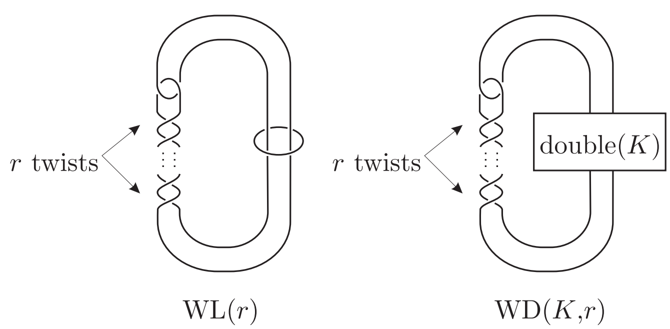

Figure 2

In Figure 2, double(K) denotes the (2,2)-tangle obtained by doubling the knot K to a link with zero linking number and then removing a pair of parallel segments.

Our trick is cutting the link diagrams into (2,2)-tangles and gluing the tangle invariants together.

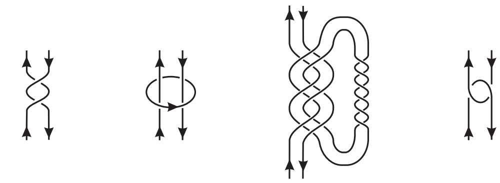

Figure 3

Colored Jones polynomial is also defined for tangles, but, instead of a Laurent polynomial of t, it is in general a module homomorphism of $U_q(\mathrm{sl}_2)$ (choose $t = q^2$). Especially, the colored Jones polynomial of a (2,2)-tangle is a module homomorphism

$$V_N \otimes V_N \to V_N \otimes V_N, \tag{2.1}$$

where V_N is the N dimensional irreducible representation of $U_q(\mathrm{sl}_2)$.

Note that the tensor product admits the decomposition

$$V_N \otimes V_N = \bigoplus_{n=0}^{N-1} V_{2n+1}. \tag{2.2}$$

A straightforward calculation shows that the (framing independent, unnormalized) colored Jones polynomials of the tangles are

$$\widetilde{J}_{\mathrm{twist},N} = \bigoplus_{n=0}^{N-1} t^{n(n+1)} \cdot \mathrm{id}_{V_{2n+1}}, \tag{2.3}$$

$$\widetilde{J}_{\mathrm{belt},N} = \bigoplus_{n=0}^{N-1} \frac{t^{\frac{N(2n+1)}{2}} - t^{-\frac{N(2n+1)}{2}}}{t^{\frac{2n+1}{2}} - t^{-\frac{2n+1}{2}}} \cdot \mathrm{id}_{V_{2n+1}}, \tag{2.4}$$

$$\widetilde{J}_{\mathrm{double}(K),N} = \bigoplus_{n=0}^{N-1} J_{K,2n+1} \cdot \mathrm{id}_{V_{2n+1}}, \tag{2.5}$$

$$\widetilde{J}_{\mathrm{clasp},N} = \bigoplus_{n=0}^{N-1} \xi_{N,n} \cdot \mathrm{id}_{V_{2n+1}}, \tag{2.6}$$

where

$$\xi_{N,n} = t^{\frac{N^2-1}{2} + \frac{N(N-1)}{2}} \sum_{i=0}^{N-1-n} t^{-N(i+n)} \prod_{j=1}^{n} \frac{(1 - t^{N-i-j})(1 - t^{i+j})}{1 - t^j}. \tag{2.7}$$

Combining the tangle invariants together, one has

$$J_{\mathrm{WL}(r),N} = \sum_{n=0}^{N-1} \frac{t^{\frac{2n+1}{2}} - t^{-\frac{2n+1}{2}}}{t^{\frac{N}{2}} - t^{-\frac{N}{2}}} \cdot t^{rn(n+1)} \cdot \xi_{N,n} \cdot \frac{t^{\frac{N(2n+1)}{2}} - t^{-\frac{N(2n+1)}{2}}}{t^{\frac{2n+1}{2}} - t^{-\frac{2n+1}{2}}}, \tag{2.8}$$

$$J_{\mathrm{WD}(K,r),N} = \sum_{n=0}^{N-1} \frac{t^{\frac{2n+1}{2}} - t^{-\frac{2n+1}{2}}}{t^{\frac{N}{2}} - t^{-\frac{N}{2}}} \cdot t^{rn(n+1)} \cdot \xi_{N,n} \cdot J_{K,2n+1}. \tag{2.9}$$

Note that, in the expression of $J_{\mathrm{WD}(K,r),N}$, the factor $J_{K,2n+1}$ is contributed by the companion knot K and the other part is precisely obtained from the expression of $J_{\mathrm{WL}(r),N}$ by removing the factor contributed by the belt tangle.

3 Proof of Theorem 1.1

Let L denote the twisted Whitehead link WL(r). Setting $t = e^{\frac{2\pi\sqrt{-1}}{N}}$, we have

$$J_{L,N}(e^{\frac{2\pi\sqrt{-1}}{N}}) = -t^{-\frac{1}{2}} \sum_{n=0}^{N-1} (2n+1) t^{rn(n+1)} \sum_{i=0}^{N-1-n} \prod_{j=1}^{n} \frac{(1-t^{-i-j})(1-t^{i+j})}{1-t^j}$$
$$= -e^{-\frac{\pi\sqrt{-1}}{N}} \sum_{n=0}^{N-1} (2n+1) a_n^{4r-1} \sum_{i=0}^{N-1-n} S_{n,i}, \tag{3.1}$$

where

$$a_n = e^{\frac{n(n+1)}{2N}\pi\sqrt{-1} - \frac{n}{2}\pi\sqrt{-1}} = e^{\frac{n(n+1-N)}{2N}\pi\sqrt{-1}}, \tag{3.2}$$
$$S_{n,i} = \prod_{j=1}^{n} \frac{4\sin^2 \frac{(i+j)\pi}{N}}{2\sin\frac{j\pi}{N}}. \tag{3.3}$$

First, we prepare a lemma to estimate the norm factor $S_{n,i}$. Put

$$s_n = -\sum_{j=1}^{n} \log\left|2\sin\frac{j\pi}{N}\right| \tag{3.4}$$

and let

$$L(x) = -\int_0^x \log|2\sin u| du \tag{3.5}$$

be the Lobachevsky function.

Lemma 3.1 *For $0 < \alpha < 1$, we have uniform estimations*

$$s_m - s_n = \frac{N}{\pi} L\Big(\frac{m\pi}{N}\Big) - \frac{N}{\pi} L\Big(\frac{n\pi}{N}\Big) + O(N^{-1})(m-n) \tag{3.6}$$

on $\frac{\alpha}{2}N < n < m < (1-\frac{\alpha}{2})N$,

$$s_n = \frac{N}{\pi} L\Big(\frac{n\pi}{N}\Big) - \frac{1}{2}\log n + O(1) \tag{3.7}$$

on $0 < n < \alpha N$ and

$$s_n = \frac{N}{\pi} L\Big(\frac{n\pi}{N}\Big) - \frac{1}{2}\log(N-n) + O(1) \tag{3.8}$$

on $(1-\alpha)N < n < N$.

Proof We have

$$-\log\left|2\sin\frac{j\pi}{N}\right| + \frac{N}{\pi}\int_{\frac{(j-1)\pi}{N}}^{\frac{j\pi}{N}} \log|2\sin u| du = -\log\left|2\sin\frac{j\pi}{N}\right| + \frac{N}{\pi}\int_0^{\frac{\pi}{N}} \log\left|2\sin\Big(\frac{j\pi}{N} - u\Big)\right| du$$
$$= \frac{N}{\pi}\int_0^{\frac{\pi}{N}} \log\left|\frac{\sin(\frac{j\pi}{N} - u)}{\sin\frac{j\pi}{N}}\right| du. \tag{3.9}$$

Since

$$\log\left|\frac{\sin(x-u)}{\sin x}\right| = O(u) \tag{3.10}$$

uniformly on $x \in [\frac{\alpha}{2}\pi, (1-\frac{\alpha}{2})\pi]$ as $u \to 0$, the first estimation follows as

$$\begin{aligned} s_m - s_n - \frac{N}{\pi}L\Big(\frac{m\pi}{N}\Big) + \frac{N}{\pi}L\Big(\frac{n\pi}{N}\Big) &= \sum_{j=n+1}^{m} \frac{N}{\pi}\int_0^{\frac{\pi}{N}} \log\Big|\frac{\sin\big(\frac{j\pi}{N}-u\big)}{\sin\frac{j\pi}{N}}\Big|du \\ &= O(N^{-1})(m-n). \end{aligned} \tag{3.11}$$

Note that

$$\frac{\sin(x-u)}{x-u}\cdot\frac{x}{\sin x} = 1 + O(u). \tag{3.12}$$

Thus

$$\log\Big|\frac{\sin(x-u)}{\sin x}\Big| = \log\Big|\frac{x-u}{x}\Big| + O(u) \tag{3.13}$$

uniformly on $x \in [-\alpha\pi, \alpha\pi] \setminus \{0, u\}$ as $u \to 0$. It follows that

$$\begin{aligned} s_n - \frac{N}{\pi}L\Big(\frac{n\pi}{N}\Big) &= \sum_{j=1}^{n} \frac{N}{\pi}\int_0^{\frac{\pi}{N}} \log\Big|\frac{\sin(\frac{j\pi}{N}-u)}{\sin\frac{j\pi}{N}}\Big|du \\ &= \sum_{j=1}^{n} \frac{N}{\pi}\int_0^{\frac{\pi}{N}} \log\Big|\frac{\frac{j\pi}{N}-u}{\frac{j\pi}{N}}\Big|du + nO(N^{-1}) \\ &= \sum_{j=1}^{n}\Big(-(j-1)\log\frac{j-1}{j} - 1\Big) + O(1) \\ &= -\log\frac{n!}{n^n} - n + O(1) \end{aligned} \tag{3.14}$$

uniformly on $0 < n < \alpha N$. Thanks to the Sterling series

$$\log n! = n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log 2\pi + \cdots, \tag{3.15}$$

the second estimation holds.

To see the third estimation, one notices that

$$L(x) + L(\pi - x) = 0, \tag{3.16}$$

$$s_{n-1} + s_{N-n} = s_{N-1}. \tag{3.17}$$

In particular, we have

$$L\Big(\frac{\pi}{2}\Big) = 0 \tag{3.18}$$

and, by the second estimation,

$$s_{N-1} = s_{[\frac{N-1}{2}]} + s_{[\frac{N}{2}]} = -\log N + O(1). \tag{3.19}$$

Therefore,

$$\begin{aligned} s_n &= s_{N-1} - s_{N-n} - \log\Big|2\sin\frac{n\pi}{N}\Big| \\ &= -\log N + \frac{N}{\pi}L\Big(\frac{n\pi}{N}\Big) + \frac{1}{2}\log(N-n) - \log\frac{2(N-n)\pi}{N} + O(1) \\ &= \frac{N}{\pi}L\Big(\frac{n\pi}{N}\Big) - \frac{1}{2}\log(N-n) + O(1) \end{aligned} \tag{3.20}$$

uniformly on $(1-\alpha)N < n < N$.

From the second and the third estimations of above lemma, we have

$$\log S_{n,i} = -2s_{n+i} + 2s_i + s_n = \frac{N}{\pi} f\Big(\frac{n\pi}{N}, \frac{i\pi}{N}\Big) + O(\log N) \tag{3.21}$$

uniformly on $0 \le n, i, n+i < N$, where

$$f(x,y) = -2L(x+y) + 2L(y) + L(x). \tag{3.22}$$

The function $f(x,y)$ has a unique critical point $(\frac{\pi}{2}, \frac{\pi}{4})$ in the region $0 \le x, y, x+y \le \pi$, at which f reaches maximum

$$f\Big(\frac{\pi}{2}, \frac{\pi}{4}\Big) = 4L\Big(\frac{\pi}{4}\Big) \tag{3.23}$$

and expands as

$$f\Big(x + \frac{\pi}{2}, y + \frac{\pi}{4}\Big) = f\Big(\frac{\pi}{2}, \frac{\pi}{4}\Big) - (x^2 + 2xy + 2y^2) + \cdots . \tag{3.24}$$

Notice that the phase factor a_n is also steady near $\frac{N}{2}$. In what follows, the sum (3.1) is expected to be dominated by the sum whose index (n, i) is near $(\frac{N}{2}, \frac{N}{4})$. Indeed, this is the case as demonstrated by the next pair of lemmas.

Lemma 3.2 *For any $\frac{1}{2} < \delta < 1$, there exist $\epsilon > 0$ and $C > 0$ such that*

$$S_{n,i} < Ce^{-\epsilon N^{2\delta-1}} S_{[\frac{N}{2}],[\frac{N}{4}]} \tag{3.25}$$

for $|n - \frac{N}{2}| + |i - \frac{N}{4}| \ge N^\delta$.

Proof Since f has a unique critical point $(\frac{\pi}{2}, \frac{\pi}{4})$ in the region $0 \le x, y, x+y \le \pi$, we have

$$f(x,y) \le \max_{|x'-\frac{\pi}{2}|+|y'-\frac{\pi}{4}|=\pi N^{\delta-1}} f(x', y') \tag{3.26}$$

for $|x - \frac{\pi}{2}| + |y - \frac{\pi}{4}| \ge \pi N^{\delta-1}$. By (3.24), there exist $\epsilon > 0$ and $C' > 0$ such that

$$\max_{|x'-\frac{\pi}{2}|+|y'-\frac{\pi}{4}|=\pi N^{\delta-1}} f(x', y') < f\Big(\frac{\pi}{2}, \frac{\pi}{4}\Big) - 2\pi\epsilon(N^{\delta-1})^2 + C'. \tag{3.27}$$

Therefore, by (3.21) there exists $C'' > 0$ such that

$$\log S_{n,i} < \log S_{[\frac{N}{2}],[\frac{N}{4}]} - \epsilon N^{2\delta-1} + C'' \tag{3.28}$$

for $|n - \frac{N}{2}| + |i - \frac{N}{4}| \ge N^\delta$.

Lemma 3.3 *For any $\alpha \ge 0$, $\beta \in \mathbb{R}$ and $\frac{1}{2} < \delta < \frac{2}{3}$, there exists a nonzero constant $C \in \mathbb{C}$ such that*

$$\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|<N^\delta} (2n+1)^\alpha a_n^\beta S_{n,i} = CN^{\alpha+1} e^{-\frac{\beta N}{8}\pi\sqrt{-1}} S_{[\frac{N}{2}],[\frac{N}{4}]}(1 + O(N^{3\delta-2})). \tag{3.29}$$

Proof For simplicity, we use the notation $n' = n - \frac{N}{2}$, $i' = i - \frac{N}{4}$ in the proof. Note that

$$\begin{aligned}
&\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|<N^\delta} (2n+1)^\alpha e^{-\frac{\pi}{N}(n'^2+2n'i'+2i'^2)+\frac{\beta n'^2}{2N}\pi\sqrt{-1}} \\
&= \int_{|x|+|y|<N^{\delta-\frac{1}{2}}} N^{\alpha+1} e^{-\pi(x^2+2xy+2y^2)+\frac{\beta x^2}{2}\pi\sqrt{-1}} dxdy(1 + O(N^{\delta-1})) \\
&= \int_{\mathbb{R}^2} N^{\alpha+1} e^{-\pi(x^2+2xy+2y^2)+\frac{\beta x^2}{2}\pi\sqrt{-1}} dxdy(1 + O(N^{\delta-1})).
\end{aligned} \tag{3.30}$$

By (3.6) and (3.24), we have

$$\begin{aligned}\log S_{n,i}-\log S_{[\frac{N}{2}],[\frac{N}{4}]}&=\frac{N}{\pi}f\Big(\frac{n\pi}{N},\frac{i\pi}{N}\Big)-\frac{N}{\pi}f\Big(\frac{\pi}{2},\frac{\pi}{4}\Big)+O(N^{\delta-1})\\&=-\frac{\pi}{N}(n'^2+2n'i'+2i'^2)+O(N^{3\delta-2})\end{aligned}\tag{3.31}$$

uniformly on $|n-\frac{N}{2}|+|i-\frac{N}{4}|<N^\delta$. Moreover, on the same region we have the uniform estimation

$$a_n=e^{\frac{n(n+1-N)}{2N}\pi\sqrt{-1}}=e^{(\frac{n'^2}{2N}-\frac{N}{8}+\frac{n}{2N})\pi\sqrt{-1}}=e^{(\frac{n'^2}{2N}-\frac{N}{8}+\frac{1}{4})\pi\sqrt{-1}}(1+O(N^{\delta-1})).\tag{3.32}$$

Therefore, by (3.30),

$$\begin{aligned}\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|<N^\delta}(2n+1)^\alpha a_n^\beta S_{n,i}&=\int_{\mathbb{R}^2}e^{-\pi(x^2+2xy+2y^2)+\frac{\beta x^2}{2}\pi\sqrt{-1}}dxdy\\&\quad\cdot N^{\alpha+1}e^{\beta(-\frac{N}{8}+\frac{1}{4})\pi\sqrt{-1}}S_{[\frac{N}{2}],[\frac{N}{4}]}(1+O(N^{3\delta-2})).\end{aligned}\tag{3.33}$$

To conclude the lemma it suffices to choose

$$C=e^{\frac{\beta}{4}\pi\sqrt{-1}}\int_{\mathbb{R}^2}e^{-\pi(x^2+2xy+2y^2)+\frac{\beta x^2}{2}\pi\sqrt{-1}}dxdy.\tag{3.34}$$

It follows from Lemma 3.2 and Lemma 3.3 that, in the same notations,

$$\begin{aligned}\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|\ge N^\delta}(2n+1)a_n^{4r-1}S_{n,i}&=N^3e^{-\epsilon N^{2\delta-1}}S_{[\frac{N}{2}],[\frac{N}{4}]}O(1),\\\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|<N^\delta}(2n+1)a_n^{4r-1}S_{n,i}&=N^2S_{[\frac{N}{2}],[\frac{N}{4}]}e^{O(1)},\end{aligned}\tag{3.35}$$

so

$$\log\left|J_{L,N}(e^{\frac{2\pi\sqrt{-1}}{N}})\right|=\log(N^2S_{[\frac{N}{2}],[\frac{N}{4}]})+O(1).\tag{3.36}$$

From (3.7) we also have

$$\log S_{[\frac{N}{2}],[\frac{N}{4}]}=\frac{4N}{\pi}L\Big(\frac{\pi}{4}\Big)-\frac{1}{2}\log N+O(1).\tag{3.37}$$

Therefore,

$$2\pi\log\left|J_{L,N}(e^{\frac{2\pi\sqrt{-1}}{N}})\right|=8L\Big(\frac{\pi}{4}\Big)\cdot N+3\pi\log N+O(1)=\mathrm{vol}(S^3\setminus L)\cdot N+3\pi\log N+O(1)\tag{3.38}$$

as $N\to\infty$. In the last row, we used the fact

$$\mathrm{vol}(S^3\setminus L)=\mathrm{vol}(S^3\setminus\text{Whitehead link})=8L\Big(\frac{\pi}{4}\Big).\tag{3.39}$$

4 Proof of Theorem 1.2

The colored Jones polynomial of the torus knot $T(p,q)$ was calculated in [4] as

$$J_{T(p,q),n}=\frac{t^{-\frac{pq(n^2-1)}{4}}}{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}}t^{pk(qk+1)}(t^{qk+\frac{1}{2}}-t^{-qk-\frac{1}{2}}).\tag{4.1}$$

We put

$$A_{p,q}^{\pm}(N,k)=\sum_{j=1}^{pq-1}(\pm 1)^j e^{-\frac{Nj^2}{2pq}\pi\sqrt{-1}}j^{2k}\sin\frac{j\pi}{p}\sin\frac{j\pi}{q}. \tag{4.2}$$

Note that

$$A_{p,q}^{\pm}(N,k)=A_{p,q}^{\mp}(N+2pq,k)=A_{p,q}^{\pm}(N+4pq,k), \tag{4.3}$$

so

$$A_{p,q}^{(-1)^n}(N,k)=A_{p,q}^{+}(N+2npq,k),\quad A_{p,q}^{+}(N,k)=A_{p,q}^{(-1)^n}(N+2npq,k). \tag{4.4}$$

In [3], an estimation of $J_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}})$ was derived as

$$J_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}})=2e^{-pq\frac{N^2-1}{2N}\pi\sqrt{-1}}\frac{N^{\frac{3}{2}}}{(2pq)^{\frac{3}{2}}}e^{-(\frac{p}{q}+\frac{q}{p})\frac{\pi\sqrt{-1}}{2N}+\frac{\pi\sqrt{-1}}{4}}A_{p,q}^{(-1)^{N-1}}(N,1)+O(1). \tag{4.5}$$

In view of the periodicity of $A_{p,q}^{\pm}$, to establish the theorem it suffices to show that $A_{p,q}^{(-1)^{N-1}}(N,1)$ never vanishes if $q=2$, or equivalently by (4.4),

Proposition 4.1 *Let $p,q\geq 2$ be coprime integers with $q=2$. Then for every integer N,*

$$A_{p,q}^{+}(N,1)=\sum_{j=1}^{pq-1}e^{-\frac{Nj^2}{2pq}\pi\sqrt{-1}}j^2\sin\frac{j\pi}{p}\sin\frac{j\pi}{q}\neq 0. \tag{4.6}$$

The proof of the proposition is purely arguments on elementary algebraic number theory. In the following, we write $\zeta_n=e^{\frac{2\pi\sqrt{-1}}{n}}$ for each $n\in\mathbb{N}$. An algebraic number field means a finite extension of $\mathbb{Q}$ contained in $\mathbb{C}$.

For any finite extension E/K of field, one has a K-linear map $\mathrm{tr}_{E/K}:E\to K$, called the trace function, which values on $x\in E$ the trace of the K-linear transformation $\rho_x:E\to E$ given by $\rho_x(z)=xz$.

Lemma 4.1 *Let α be a prime, $k,l\in\mathbb{N}$ and K be an algebraic number field such that $K\cap\mathbb{Q}(\zeta_{\alpha^{k+l}})=\mathbb{Q}$. Then we have*

$$\mathrm{tr}_{K(\zeta_{\alpha^{k+l}})/K(\zeta_{\alpha^l})}(\zeta_{\alpha^{k+l}}^n)=\begin{cases}0, & \alpha^k\nmid n,\\ \alpha^k\cdot\zeta_{\alpha^{k+l}}^n, & \alpha^k\mid n.\end{cases} \tag{4.7}$$

Proof The field extension $K(\zeta_{\alpha^{k+l}})/K(\zeta_{\alpha^l})$ has a basis $\{\zeta_{\alpha^{k+l}}^i\mid 0\leq i<\alpha^k\}$ on which the diagonal of the matrix of $\zeta_{\alpha^{k+l}}^n$ consists of only 0 if $\alpha^k\nmid n$, or $\zeta_{\alpha^{k+l}}^n$ otherwise.

Lemma 4.2 *Let α be a prime and K be an algebraic number field such that $K\cap\mathbb{Q}(\zeta_\alpha)=\mathbb{Q}$. Then*

$$\sum_{j=0}^{\alpha-1}c_j\cdot\zeta_\alpha^j=0 \tag{4.8}$$

for $c_j\in K$ if and only if the c_j's are identical.

Proof On one hand, the field extension $K(\zeta_\alpha)/K$ has a basis $\{1,\zeta_\alpha,\zeta_\alpha^2,\cdots,\zeta_\alpha^{\alpha-2}\}$. On the other hand, we have $\sum_{j=0}^{\alpha-1}\zeta_\alpha^j=0$. Therefore, the sum vanishes if and only if the c_j's are identical to $c_{\alpha-1}$.

Thanks to the next lemma, we are able to eliminate the Gaussian exponential appearing in the expression of $A_{p,q}^{\pm}$.

Lemma 4.3 *Let α be an odd prime, $l \in \mathbb{N}$ and K be an algebraic number field such that $K \cap \mathbb{Q}(\zeta_{\alpha^l}) = \mathbb{Q}$. Assume that*

$$\sum_{j \in X} c_j \cdot \zeta_{\alpha^l}^{-Nj^2+2aj} = 0, \tag{4.9}$$

where X is a finite subset of $\mathbb{Z}$, $c_j \in K$ and $\alpha \nmid a$. Then we have

$$\sum_{j \in X:\ j \equiv 0 \mod \alpha^{l-1}} c_j \cdot \zeta_{\alpha^l}^{2aj} = 0, \tag{4.10}$$

if $\alpha \mid N$, or otherwise,

$$\sum_{j \in X:\ Nj \equiv a \mod \alpha^{[\frac{l+1}{2}]}} c_j = 0. \tag{4.11}$$

Proof If $\alpha \mid N$, taking the trace function of $K(\zeta_{\alpha^l})/K(\zeta_\alpha)$ on both sides of the equality assumed, we find from Lemma 4.1 that

$$\sum_{j \in X:\ j \equiv 0 \mod \alpha^{l-1}} c_j \cdot \zeta_{\alpha^l}^{2aj} = 0. \tag{4.12}$$

Otherwise, choose $b \in \mathbb{Z}$ such that $bN \equiv 1 \mod \alpha^l$. From the assumption, we have

$$\sum_{j \in X} c_j \cdot \zeta_{\alpha^l}^{-b(Nj-a)^2} = \zeta_{\alpha^l}^{-ba^2} \sum_{j \in X} c_j \cdot \zeta_{\alpha^l}^{-Nj^2+2aj} = 0. \tag{4.13}$$

Taking the trace function of $K(\zeta_{\alpha^l})/K(\zeta_\alpha)$ on both sides of the equality, we get

$$\sum_{k=0}^{\alpha-1} \Big(\sum_{j \in X:\ -b(Nj-a)^2 \equiv k\alpha^{l-1} \mod \alpha^l} c_j \Big) \cdot \zeta_\alpha^k = 0. \tag{4.14}$$

Since α is an odd prime, the congruence equation $x^2 \equiv -kN\alpha^{l-1} \mod \alpha^l$ has no solution for some $0 < k < \alpha$. It follows from Lemma 4.2 that the coefficient of ζ_α^k in the above sum identically vanishes. In particular,

$$\sum_{j \in X:\ -b(Nj-a)^2 \equiv 0 \mod \alpha^l} c_j = 0. \tag{4.15}$$

Hence the lemma follows.

Proof of Proposition 4.1 Assume that $A_{p,q}^+(N,1) = 0$ for

$$p = \alpha_1^{l_1} \alpha_2^{l_2} \cdots \alpha_r^{l_r} \cdot \beta_1^{k_1} \beta_2^{k_2} \cdots \beta_s^{k_s}, \tag{4.16}$$

where the α_i's and β_i's are distinct odd primes not dividing and dividing N, respectively. Rewrite $A_{p,q}^+(N,1) = 0$ as

$$-\frac{1}{4} \sum_{-pq<j<pq} j^2 \cdot \zeta_{4pq}^{-Nj^2+2qj} \cdot (\zeta_{2q}^j - \zeta_{2q}^{-j}) = 0 \tag{4.17}$$

and choose $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{8p})/\mathbb{Q})$ so that

$$\sigma(\zeta_{4pq}) = \zeta_{4q} \cdot \zeta_{\alpha_1^{l_1}} \cdots \zeta_{\alpha_r^{l_r}} \cdot \zeta_{\beta_1^{k_1}} \cdots \zeta_{\beta_s^{k_s}}. \tag{4.18}$$

Under the Galois action of σ, the equality becomes

$$-\frac{1}{4}\sum_{-pq<j<pq} j^2\cdot(\zeta_{4q}\cdot\zeta_{\alpha_1^{l_1}}\cdots\zeta_{\alpha_r^{l_r}}\cdot\zeta_{\beta_1^{k_1}}\cdots\zeta_{\beta_s^{k_s}})^{-Nj^2+2qj}\cdot(\zeta_{2q}^{pj}-\zeta_{2q}^{-pj})=0. \tag{4.19}$$

Put

$$\alpha=\alpha_1^{[\frac{l_1+1}{2}]}\cdots\alpha_r^{[\frac{l_r+1}{2}]},\quad \beta=\beta_1^{k_1-1}\cdots\beta_s^{k_s-1},\quad p'=\beta_1\cdots\beta_s. \tag{4.20}$$

It follows from Lemma 4.3 that

$$-\frac{1}{4}\sum_{j\in X} j^2\cdot(\zeta_{\beta_1^{k_1}}\cdots\zeta_{\beta_s^{k_s}})^{2qj}\cdot\zeta_{4q}^{-Nj^2+2qj}\cdot(\zeta_{2q}^{pj}-\zeta_{2q}^{-pj})=0, \tag{4.21}$$

where

$$X=\{-pq<j<pq\mid Nj\equiv q \mod \alpha,\ j\equiv 0 \mod \beta\}. \tag{4.22}$$

Now we apply the condition $q=2$. Notice that

$$\zeta_8^{-Nj^2+4j}\cdot(\zeta_4^{pj}-\zeta_4^{-pj})=\begin{cases}0, & 2\mid j,\\ -\zeta_8^{-N}\cdot(-1)^{\frac{j-1}{2}}(\zeta_4^p-\zeta_4^{-p}), & 2\nmid j.\end{cases} \tag{4.23}$$

Dropping a nonzero factor, (4.21) becomes

$$\sum_{j\in X:2\nmid j} j^2\cdot\zeta_{p'}^{\frac{4j}{\beta}}\cdot(-1)^{\frac{j-1}{2}}=0. \tag{4.24}$$

Choose $0\le j_0<2\alpha$ so that $N\beta j_0\equiv 2 \mod \alpha$ and $j_0\equiv 1 \mod 2$. Then the left hand side of above equality, up to a sign, is

$$\sum_{-\frac{p}{\alpha\beta}\le j<\frac{p}{\alpha\beta}}\beta^2(2\alpha j+j_0)^2\cdot\zeta_{p'}^{8\alpha j+4j_0}\cdot(-1)^j=\frac{8p\beta\zeta_{p'}^{4j_0}}{1+\zeta_{p'}^{8\alpha}}\Big(j_0-\frac{2\alpha}{1+\zeta_{p'}^{-8\alpha}}\Big). \tag{4.25}$$

Therefore, we must have $p'=1$ and $j_0=\alpha$. But from the choice of j_0, it follows that $\alpha=1$. Hence $p=1$, a contradiction.

This completes the proof of the proposition and hence Theorem 1.2.

5 Proof of Theorem 1.3

Let K denote the r-twisted Whitehead double of the torus knot $T(p,q)$. Then

$$J_{K,N}=\frac{1}{t^{\frac{N}{2}}-t^{-\frac{N}{2}}}\sum_{n=0}^{N-1}t^{rn(n+1)}\xi_{N,n}\widehat{J}_{T(p,q),2n+1}, \tag{5.1}$$

where

$$\widehat{J}_{T(p,q),n}=(t^{\frac{n}{2}}-t^{-\frac{n}{2}})J_{T(p,q),n}. \tag{5.2}$$

Setting $t=e^{\frac{2\pi\sqrt{-1}}{N}}$, one notices that the denominator $t^{\frac{N}{2}}-t^{-\frac{N}{2}}$ vanishes. Therefore, one has to apply the L'Hospital's rule, i.e., take derivative of both the denominator and the numerator. It follows that

$$J_{K,N}=\frac{-t^{-\frac{1}{2}}}{-N}\sum_{n=0}^{N-1}a_n^{4r-1}\sum_{i=0}^{N-1-n}S_{n,i}\Big(b_{n,i}\widehat{J}_{T(p,q),2n+1}+t\frac{d}{dt}\widehat{J}_{T(p,q),2n+1}\Big), \tag{5.3}$$

where

$$b_{n,i} = rn(n+1) - N(i+n) + \sum_{j=1}^{n}\Big(-\frac{N-i-j}{1-t^{-N+i+j}} - \frac{i+j}{1-t^{-i-j}} + \frac{j}{1-t^{-j}}\Big) \tag{5.4}$$

and $a_n, S_{n,i}$ are the same as in Section 3.

Below, we follow the approach used in [3] to derive an estimation of $\widehat{J}_{T(p,q),n}$ and $t\frac{d}{dt}\widehat{J}_{T(p,q),n}$ in the form of (4.5). For any complex number h with $\mathrm{Im}(h) > 0$, one has the integral formula

$$\widehat{J}_{T(p,q),n}(e^h) = e^{-pq(n^2-1)\frac{h}{4}}\Big(\frac{pq}{\pi h}\Big)^{\frac{1}{2}} e^{-(\frac{p}{q}+\frac{q}{p})\frac{h}{4}} \int_C dz e^{pq(nz-\frac{z^2}{h})}\tau(z), \tag{5.5}$$

where the contour C is given by the line $e^{\frac{\pi\sqrt{-1}}{4}}\mathbb{R}$ and

$$\tau(z) = \frac{(e^{pz}-e^{-pz})(e^{qz}-e^{-qz})}{e^{pqz}-e^{-pqz}}. \tag{5.6}$$

Lemma 5.1 *For $h = \frac{2\pi\sqrt{-1}}{N}$, we have*

$$\frac{d^k}{dh^k}\int_C dz e^{pq(nz-\frac{z^2}{h})}\tau(z) = -4\pi\sqrt{-1}\frac{1}{pq}\Big(\frac{N^2}{4pq}\Big)^k A_{p,q}^{(-1)^{n-1}}(N,k) + O(N^{2k-\frac{1}{2}}) \tag{5.7}$$

uniformly on $|n-N| < \frac{N}{2pq}$.

Proof Put $z_0 = \frac{n}{2}h = \frac{n}{N}\pi\sqrt{-1}$. We have

$$\int_{C+z_0} dz e^{pq(nz-\frac{z^2}{h})} z^{2k}\tau(z) = e^{pq\frac{z_0^2}{h}}\int_C dz e^{-pq\frac{z^2}{h}}(z+z_0)^{2k}\tau(z+z_0) = O(N^{-\frac{1}{2}}) \tag{5.8}$$

uniformly on $|n-N| < \frac{N}{2pq}$, since the function $z^{2k}\tau(z)$ is bounded on the region

$$z \in \Big\{e^{\frac{\pi\sqrt{-1}}{4}}x + y\pi\sqrt{-1}\,\Big|\, x, y\in\mathbb{R},\ |y-1| < \frac{1}{2pq}\Big\}. \tag{5.9}$$

Counting the residues of the integrand at $\frac{j\pi\sqrt{-1}}{pq}$, $0 < j < pq$, we also have

$$\Big(\int_C - \int_{C+z_0}\Big) dz e^{pq(nz-\frac{z^2}{h})} z^{2k}\tau(z) = -4\Big(\frac{\pi\sqrt{-1}}{pq}\Big)^{2k+1} A_{p,q}^{(-1)^{n-1}}(N,k). \tag{5.10}$$

Therefore,

$$\begin{aligned}\frac{d^k}{dh^k}\int_C dz e^{pq(nz-\frac{z^2}{h})}\tau(z) &= \Big(\frac{pq}{h^2}\Big)^k \int_C dz e^{pq(nz-\frac{z^2}{h})} z^{2k}\tau(z)\\ &= -4\pi\sqrt{-1}\frac{1}{pq}\Big(\frac{N^2}{4pq}\Big)^k A_{p,q}^{(-1)^{n-1}}(N,k) + O(N^{2k-\frac{1}{2}})\end{aligned} \tag{5.11}$$

uniformly on $|n-N| < \frac{N}{2pq}$.

Lemma 5.2 *For $t = e^{\frac{2\pi\sqrt{-1}}{N}}$, we have*

$$\widehat{J}_{T(p,q),n} = O(1), \tag{5.12}$$

$$t\frac{d}{dt}\widehat{J}_{T(p,q),n} = -2e^{-pq\frac{n^2-1}{2N}\pi\sqrt{-1}}\frac{N^{\frac{5}{2}}}{(2pq)^{\frac{3}{2}}} e^{-(\frac{p}{q}+\frac{q}{p})\frac{\pi\sqrt{-1}}{2N}+\frac{\pi\sqrt{-1}}{4}} A_{p,q}^{(-1)^{n-1}}(N,1) + O(N^2) \tag{5.13}$$

uniformly on $|n-N| < \frac{N}{2pq}$.

Proof From (5.5) and Lemma 5.1, we have

$$\widehat{J}_{T(p,q),n} = -4e^{-pq\frac{n^2-1}{2N}\pi\sqrt{-1}} \frac{N^{\frac{1}{2}}}{(2pq)^{\frac{1}{2}}} e^{-(\frac{p}{q}+\frac{q}{p})\frac{\pi\sqrt{-1}}{2N}+\frac{\pi\sqrt{-1}}{4}} A_{p,q}^{(-1)^{n-1}}(N,0) + O(1). \tag{5.14}$$

It follows from the periodicity of $A^{\pm}_{p,q}$ and the identity

$$\widehat{J}_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}}) = 0 \cdot J_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}}) = 0 \tag{5.15}$$

that $A_{p,q}^{(-1)^{N-1}}(N,0)$, hence by (4.4) $A^{\pm}_{p,q}(N,0)$, identically vanishes. So, the leading term of the right hand side of (5.14) is zero. Then, applying Lemma 5.1 to the derivative of (5.5), one obtains the second estimation.

Now we conclude the proof of the theorem. It is clear that

$$b_{n,i}\widehat{J}_{T(p,q),2n+1} + t\frac{d}{dt}\widehat{J}_{T(p,q),2n+1} = O(N^3)O(N) + O(N^3) = O(N^4) \tag{5.16}$$

uniformly on $0 \leq n, i, n+i < N$. Moreover, for $0 < \alpha < 1$, since the function

$$\frac{x}{1-e^{-2\sqrt{-1}x}} = \frac{e^{\sqrt{-1}x}x}{2\sqrt{-1}\sin x} \tag{5.17}$$

is bounded on $x \in [0, \alpha\pi]$, we have

$$b_{n,i} = O(N^2) \tag{5.18}$$

uniformly on $0 < n, i, n+i < \alpha N$. It follows from Lemma 5.2 that

$$b_{n,i}\widehat{J}_{T(p,q),2n+1} = O(N^2), \tag{5.19}$$

$$t\frac{d}{dt}\widehat{J}_{T(p,q),2n+1} = -2(a_n)^{-4pq}\frac{N^{\frac{5}{2}}}{(2pq)^{\frac{3}{2}}} e^{-(\frac{p}{q}+\frac{q}{p})\frac{\pi\sqrt{-1}}{2N}+\frac{\pi\sqrt{-1}}{4}} A^{+}_{p,q}(N,1) + O(N^2) \tag{5.20}$$

uniformly on $|n-\frac{N}{2}| + |i-\frac{N}{4}| < \frac{N}{4pq}$.

Therefore, in the case that $q = 2$, by Lemma 3.2, Lemma 3.3 and Proposition 4.1, in the same notations we have

$$\begin{aligned}
&\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|\geq N^{\delta}} a_n^{4r-1}S_{n,i}\Big(b_{n,i}\widehat{J}_{T(p,q),2n+1} + t\frac{d}{dt}\widehat{J}_{T(p,q),2n+1}\Big) = N^6 e^{-\epsilon N^{2\delta-1}} S_{[\frac{N}{2}],[\frac{N}{4}]}O(1), \\
&\sum_{|n-\frac{N}{2}|+|i-\frac{N}{4}|< N^{\delta}} a_n^{4r-1}S_{n,i}\Big(b_{n,i}\widehat{J}_{T(p,q),2n+1} + t\frac{d}{dt}\widehat{J}_{T(p,q),2n+1}\Big) = N^{\frac{7}{2}} S_{[\frac{N}{2}],[\frac{N}{4}]}e^{O(1)},
\end{aligned} \tag{5.21}$$

and hence

$$\begin{aligned}
2\pi\log\big|J_{K,N}(e^{\frac{2\pi\sqrt{-1}}{N}})\big| &= 2\pi\log(N^{\frac{5}{2}}S_{[\frac{N}{2}],[\frac{N}{4}]}) + O(1) = 8L\Big(\frac{\pi}{4}\Big)\cdot N + 4\pi\log N + O(1) \\
&= v_3\|S^3\setminus K\|\cdot N + 4\pi\log N + O(1), \quad \text{as } N\to\infty.
\end{aligned} \tag{5.22}$$

6 Concluding Remarks

Although the proof of Theorem 1.3 depends on the simple nature of the Whitehead doubles of torus knots, the approach still works for the satellite knots to which the colored Jones polynomials of the associated companion knot and pattern link satisfy certain mild conditions. Meanwhile, one has to deal with several problems.

For example, as shown in expression (2.9), although the volume conjecture itself is only concerned with the value of the colored Jones polynomial $J_{K,N}$ at the N-th root of unity, the values at other roots of unity become crucial once the satellite knots of K are involved.

A more challenging problem is due to the estimations (5.19) and (5.20), which have enabled us to neglect the term $b_{n,i}\widehat{J}_{T(p,q),2n+1}$ in the sum (5.3). Note that the derivative of the polynomial

$$\widehat{J}_{K,N} = (t^{\frac{N}{2}} - t^{-\frac{N}{2}})J_{K,N} \tag{6.1}$$

is related to $J_{K,N}$ by the identity

$$t\frac{d}{dt}\widehat{J}_{K,N}(e^{\frac{2\pi\sqrt{-1}}{N}}) = -NJ_{K,N}(e^{\frac{2\pi\sqrt{-1}}{N}}). \tag{6.2}$$

Therefore, the term $t\frac{d}{dt}\widehat{J}_{T(p,q),2n+1}$ in the sum (5.3) indeed plays the role of $J_{T(p,q),N}(e^{\frac{2\pi\sqrt{-1}}{N}})$. Hence, it is quite natural to see the term $b_{n,i}\widehat{J}_{T(p,q),2n+1}$ is suppressed. Following this observation, when the Whitehead doubles of general knots are considered, it is reasonable to expect that a similar suppression happens.

Conjecture 6.1 *For every nontrivial knot K, we have*

$$\frac{\widehat{J}_{K,2n+1}(e^{\frac{2\pi\sqrt{-1}}{N}})}{t\frac{d}{dt}\widehat{J}_{K,2n+1}(e^{\frac{2\pi\sqrt{-1}}{N}})} = o(N^{-2}) \tag{6.3}$$

uniformly on $|n - \frac{N}{2}| < N^{\delta}$ for some $\frac{1}{2} < \delta < \frac{2}{3}$.

Note that the conjecture excludes the unknot, for which the statement of the conjecture is obviously false. Indeed, the Whitehead doubles of the unknot are no longer satellite knots but the so called twist knots (including unknot, trefoil, figure 8, etc.), whose complements always admit a volume strictly smaller than that of Whitehead link. In the sequel, the conjecture has an immediate consequence: colored Jones polynomial detects the unknot.

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