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Weak Solutions for the Vlasov-Poisson Initial-Boundary Value Problem with Bounded Electric Field

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Abstract The aim of this work is to construct weak solutions for the three dimensional Vlasov-Poisson initial-boundary value problem with bounded electric field. The main ingredient consists of estimating the change in momentum along characteristics of regular electric fields inside bounded spatial domains. As direct consequences, the propagation of the momentum moments and the existence of weak solution satisfying the balance of total energy are obtained.

Keywords Vlasov-Poisson equations, Vlasov-Maxwell equations, Weak solutions
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1 Introduction

The Vlasov equation gives a kinetic description of the motion of charged particles under the action of the electro-magnetic field in the collisionless case. This equation is coupled to the Maxwell equations for the electro-magnetic field; we obtain the Vlasov-Maxwell system. When the magnetic field is neglected, the system obtained is called the Vlasov-Poisson system.

Consider Ω an open bounded subset of $\mathbb{R}^3_x$ with regular boundary $\partial\Omega$. We introduce the notations $\Sigma = \partial\Omega \times \mathbb{R}^3_p$, $\Sigma_R = \partial\Omega \times B_R$, where $B_R = \{p \in \mathbb{R}^3_p \mid |p| \le R\}$ and

$$\Sigma^{\pm} = \{(x,p) \in \partial\Omega \times \mathbb{R}^3_p \mid \pm (v(p)\cdot n(x)) > 0\}, \quad \Sigma^{\pm}_R = \Sigma^{\pm} \cap \Sigma_R, \tag{1.1}$$

where $n(x)$ is the unit outward normal to $\partial\Omega$ at x and $v(p)$ is the velocity associated with some energy function $\mathcal{E}(p)$ by $v(p) = \nabla_p \mathcal{E}(p),\ \forall\, p \in \mathbb{R}^3_p$. The functions to be considered are

$$\mathcal{E}(p) = \frac{|p|^2}{2m}, \quad v(p) = \frac{p}{m} \tag{1.2}$$

for the classical case and

$$\mathcal{E}(p) = mc_0^2\Big(\Big(1+\frac{|p|^2}{m^2c_0^2}\Big)^{1/2} - 1\Big), \quad v(p) = \frac{p}{m}\Big(1+\frac{|p|^2}{m^2c_0^2}\Big)^{-1/2} \tag{1.3}$$

for the relativistic case, where m is the mass of particles, c_0 is the light speed in the vacuum. We denote by $f(t,x,p)$ the particles distribution depending on the time $t \in]0,T[$, position $x \in \Omega$ and momentum $p \in \mathbb{R}^3_p$ and by $F(t,x,p)$ the electro-magnetic force

$$F(t,x,p) = q(E(t,x) + v(p) \wedge B(t,x)), \quad (t,x,p) \in]0,T[\times \Omega \times \mathbb{R}^3_p, \tag{1.4}$$

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where (E, B) is the electro-magnetic field and q is the charge of particles. The Vlasov-Maxwell system is given by

$$\partial_t f + v(p) \cdot \nabla_x f + F(t,x,p) \cdot \nabla_p f = 0, \quad (t,x,p) \in]0,T[\times \Omega \times \mathbb{R}^3_p, \tag{1.5}$$

$$\partial_t E - c_0^2 \cdot \text{rot } B = -\frac{j}{\varepsilon_0}, \quad \partial_t B + \text{rot } E = 0, \quad \text{div } E = \frac{\rho}{\varepsilon_0}, \ \text{div } B = 0, \ (t,x) \in]0,T[\times \Omega, \tag{1.6}$$

where $\rho(t,x) = q \int_{\mathbb{R}^3_p} f(t,x,p)\, dp$, $j(t,x) = q \int_{\mathbb{R}^3_p} v(p) f(t,x,p)\, dp$ are the charge and current densities respectively, ε_0 is the permittivity of the vacuum, μ_0 is the permeability of the vacuum ($\varepsilon_0 \cdot \mu_0 \cdot c_0^2 = 1$). The above equations are completed with the initial conditions:

$$f(0,x,p) = f_0(x,p), \quad (x,p) \in \Omega \times \mathbb{R}^3_p, \quad E(0,x) = E_0(x), \quad B(0,x) = B_0(x), \quad x \in \Omega, \tag{1.7}$$

and the boundary conditions:

$$f(t,x,p) = g(t,x,p), \quad (t,x) \in]0,T[\times \Sigma^-, \tag{1.8}$$

$$n \wedge E(t,x) + c_0 \cdot n \wedge (n \wedge B(t,x)) = h(t,x), \quad (t,x) \in]0,T[\times \partial\Omega. \tag{1.9}$$

Some other boundary conditions can be analyzed. When we neglect the magnetic field, $B = 0$, the electric field derives from a potential $E = -\nabla_x \Phi$, the electric force is given by $F(t,x) = -q\nabla_x \Phi$ and we obtain the Vlasov-Poisson system:

$$\partial_t f + v(p) \cdot \nabla_x f + F(t,x) \cdot \nabla_p f = 0, \quad (t,x,p) \in]0,T[\times \Omega \times \mathbb{R}^3_p, \tag{1.10}$$

$$-\Delta_x \Phi = \frac{\rho}{\varepsilon_0}, \quad (t,x) \in]0,T[\times \Omega, \tag{1.11}$$

$$f(0,x,p) = f_0(x,p), \quad (x,p) \in \Omega \times \mathbb{R}^3_p, \quad f(t,x,p) = g(t,x,p), \quad (t,x) \in]0,T[\times \Sigma^-, \tag{1.12}$$

$$\Phi(t,x) = \varphi_0(t,x), \quad (t,x) \in]0,T[\times \partial\Omega. \tag{1.13}$$

This model can be derived from the relativistic Vlasov-Maxwell system by letting $c_0 \to +\infty$ (see [1, 2]).

Various results were obtained for the free space Vlasov-Poisson system. Weak solutions were constructed by Arseneev [3], Horst and Hunze [4]. The existence of classical solutions has been studied by Ukai and Okabe [5], Horst [6], Batt [7], Pfaffelmoser [8]. The existence of global classical solutions for the Vlasov-Poisson equations was proved by Bardos and Degond [9], Schaeffer [10, 11]. The propagation of the moments for the three dimensional Vlasov-Poisson system was studied by Lions and Perthame in [12]. The existence of global weak solution for the Vlasov-Maxwell system in three dimension was obtained by Diperna and Lions [13]. Results for the relativistic case were proved by Glassey and Schaeffer [14, 15], Glassey and Strauss [16, 17], Klainerman and Staffilani [18], Bouchut, Golse and Pallard [19].

Results for the initial-boundary value problemwere obtained by Ben Abdallah [20] for the Vlasov-Poisson system in three dimension and Guo [21] for the Vlasov-Maxwell system. The stationary problem for the Vlasov-Poisson equations was studied by Greengard and Raviart [22] in one dimension and by Poupaud [23] in three dimension for the Vlasov-Maxwell system. An asymptotic analysis of the Vlasov-Poisson system was done by Degond and Raviart [24] in the case of the plane diode. The regularity of the solutions for the Vlasov-Maxwell system has been studied by Guo [25]. Results for the time periodic case can be found in [26, 27].

The aim of this paper is to construct weak solutions for the three dimensional Vlasov-Poisson initial-boundary value problemwith bounded electric field. As usual we start by analyzing a regularized system for which the existence of solution follows by a fixed point method. Next we find uniform a priori bounds for these solutions by using the physical conservation laws, under the natural hypotheses that

$$\int_\Omega \int_{\mathbb{R}^3_p} (1+\mathcal{E}(p)) f_0(x,p)\ dxdp + \int_\Omega |\nabla_x \Phi(0,x)|^2\ dx$$
$$+ \int_0^T \int_{\Sigma^-} |(v(p)\cdot n(x))|(1+\mathcal{E}(p)) g\ dtd\sigma dp < +\infty$$

and φ_0 is smooth. Finally we construct a weak solution by taking a weak limit of the sequence of smooth solutions (see Theorem 5.1 for exact statements). Of course, such a construction is standard (see [20]). The new results of this work consists of establishing L^∞ bounds for the electric field (see Subsection 4.2) and the derivation of some important consequences. One of the crucial points is to observe that the change in momentum along characteristics inside a bounded spatial domain can be estimated in term of the L^∞ norm of the electric field. This idea has been already used in [26]. For example, in the classical case we prove that for all characteristic

$$\frac{dX}{ds} = \frac{P(s)}{m}, \quad \frac{dP}{ds} = qE(s, X(s)),$$

we have

$$|P(s_1) - P(s_2)| \le 2 \cdot (2 \cdot |q| \cdot \|E\|_{L^\infty} \cdot m \cdot \mathrm{diam}(\Omega))^{\frac{1}{2}}$$

for all $s_{\mathrm{in}} \le s_1 \le s_2 \le s_{\mathrm{out}}$ (here $s_{\mathrm{in}}, s_{\mathrm{out}}$ denote the incoming and outgoing times, respectively). Combining the above result with Sobolev inequalities and standard bounds for the total mass and energy yields a L^∞ estimate for the electric field. As direct consequences of the L^∞ bound for the electric field we mention the propagation of the momentum moments and also the existence of weak solutions (f, E) for the Vlasov-Poisson system with particle distribution f compactly supported in momentum when the initial-boundary conditions have compact support in momentum. Another consequence is that the weak solution obtained as limit of smooth solutions exactly verifies the energy conservation law (generally by weak limit only inequalities are preserved). For example, if the potential vanishes on the boundary we construct a weak solution (f, E) satisfying

$$\frac{d}{dt}\Big\{ \int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f\ dxdp + \frac{\varepsilon_0}{2} \int_\Omega |E|^2 dx \Big\} + \int_\Sigma (v(p)\cdot n(x)) \mathcal{E}(p) \gamma f d\sigma dp = 0, \quad \text{a.e. } t \in]0, T[,$$

where γf is the trace of f on Σ.

The content of this paper is organized as follows. We recall some standard definitions and results about the Vlasov problem. We remind the notion of weak/mild solution for this problem with initial-boundary conditionsor only boundary conditions (the time periodic case). We state the momentum change lemma for the classical and relativistic cases (the details of proofs can be found in Appendix) and we apply the above lemma in order to construct weak solutions uniformly compactly supported in momentum for the Vlasov problem with initial-boundary conditionsor time periodic boundary conditions. In Section 3 we prove the existence of weak solution for a regularized Vlasov-Poisson system by using a fixed point method. In the next section we establish a priori estimates for the total energy and the L^∞ norm of the electric field, uniformly with respect to the regularization parameters. In the last section we construct solutions for the Vlasov-Poisson system by weak stability arguments. We end this paper with some properties of the solutions constructed above. We present also the time periodic case.

2 The Vlasov Equation

In this section we recall the basic definitions and results on the Vlasov equation. For the completeness of the presentation we consider the case of electro-magnetic forces. Later on the magnetic field will be neglected in order to study the Vlasov-Poisson system. We assume that the electro-magnetic field is given and bounded. We introduce the notion of weak solution for the initial-boundary value problem:

$$\partial_t f + v(p)\cdot\nabla_x f + F(t,x,p)\cdot\nabla_p f = 0, \quad (t,x,p)\in]0,+\infty[\times\Omega\times\mathbb{R}^3_p, \tag{2.1}$$

$$f(0,x,p) = f_0(x,p), \quad (x,p)\in\Omega\times\mathbb{R}^3_p, \tag{2.2}$$

$$f(t,x,p) = g(t,x,p), \quad (t,x,p)\in]0,+\infty[\times\Sigma^-. \tag{2.3}$$

Remark 2.1 Note that in both classical and relativistic case we have $\nabla_x\cdot v(p) = 0,\ \nabla_p\cdot F = 0$ and thus (2.1) can be written also:

$$\partial_t f + \nabla_x\cdot(v(p)f) + \nabla_p\cdot(F(t,x,p)f) = 0, \quad (t,x,p)\in]0,+\infty[\times\Omega\times\mathbb{R}^3_p.$$

Definition 2.1 *Assume that* $E,B\in L^\infty(]0,T_1[\times\Omega)^3,\ f_0\in L^1(\Omega\times B_R),\ (v(p)\cdot n(x))g\in L^1(]0,T_1[\times\Sigma^-_R),\ \forall T_1>0,\ \forall R>0$. *We say that* $f\in L^1(]0,T_1[\times\Omega\times B_R),\ \forall T_1>0,\ \forall R>0$ *is a weak solution for the problem* (2.1)–(2.3) *iff*

$$-\int_0^\infty\!\!\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\partial_t\varphi + v(p)\cdot\nabla_x\varphi + F(t,x,p)\cdot\nabla_p\varphi)\,dtdxdp$$

$$= \int_\Omega\int_{\mathbb{R}^3_p} f_0(x,p)\varphi(0,x,p)\,dxdp - \int_0^{+\infty}\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)\,dtd\sigma dp \tag{2.4}$$

for all test function which belongs to $\mathcal{T}_w = \{\varphi\in C^1_c([0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p)\mid \varphi|_{[0,+\infty[\times\Sigma^+} = 0\}$.

Suppose now that $E,B\in L^\infty_{\text{loc}}(]0,+\infty[;W^{1,\infty}(\Omega))^3$ and introduce the characteristic equations:

$$\frac{dX}{ds} = v(P(s;t,x,p)), \quad \frac{dP}{ds} = F(s,X(s;t,x,p),P(s;t,x,p)), \quad s_{\text{in}}(t,x,p)\le s\le s_{\text{out}}(t,x,p),$$

with the conditions $X(s=t;t,x,p) = x,\ P(s=t;t,x,p) = p$. Here $s_{\text{in}}(t,x,p),\ s_{\text{out}}(t,x,p)$ denote the incoming, respectively outgoing time, given by

$$s_{\text{in}}(t,x,p) = \max\{0,\sup\{s\le t\mid X(s;t,x,p)\in\partial\Omega\}\},$$
$$s_{\text{out}}(t,x,p) = \inf\{s\ge t\mid X(s;t,x,p)\in\partial\Omega\}.$$

The mild formulation follows now formally by solving

$$-\partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi = \psi, \quad (t,x,p)\in]0,+\infty[\times\Omega\times\mathbb{R}^3_p,$$

with the boundary condition $\varphi|_{[0,+\infty[\times\Sigma^+} = 0$, which gives, after integration along the characteristic curves,

$$\varphi_\psi(t,x,p) = \int_t^{s_{\rm out}(t,x,p)} \psi(s, X(s;t,x,p), P(s;t,x,p))ds.$$

Definition 2.2 *Assume that* $E, B \in L^\infty_{\rm loc}(]0,+\infty[; W^{1,\infty}(\Omega))^3, f_0 \in L^1(\Omega\times B_R), (v(p)\cdot n(x))g \in L^1(]0,T_1[\times\Sigma_R^-),\ \forall T_1>0,\ \forall R>0$. *We say that* $f\in L^1(]0,T_1[\times\Omega\times B_R),\ \forall T_1>0,\ \forall R>0$ *is a mild solution for* (2.1)–(2.3) *iff*

$$\begin{aligned}&\int_0^{+\infty}\!\!\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp\\ &= \int_\Omega\int_{\mathbb{R}^3_p} f_0(x,p)\varphi_\psi(0,x,p)dxdp - \int_0^{+\infty}\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi_\psi(t,x,p)dtd\sigma dp \qquad (2.5)\end{aligned}$$

for all test function which belongs to $\mathcal{T}_m = \{\psi\in C^0_c([0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p)\}$.

Note that for all $\psi\in\mathcal{T}_m$ the function φ_ψ has compact support in $[0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p$ and is bounded. Thus the above definition makes sense. Indeed suppose that $\psi = \psi\cdot\mathbf{1}_{\{0\le t\le T_1\}}\cdot\mathbf{1}_{\{|p|\le R\}}$. Therefore when $t>T_1$ we have $\varphi_\psi = 0$ and for $t\le T_1$,

$$\varphi_\psi(t,x,p) = \int_t^{\min\{T_1, s_{\rm out}(t,x,p)\}} \psi(s,X(s;t,x,p),P(s;t,x,p))ds.$$

By writing for $t\le s\le \min\{T_1, s_{\rm out}(t,x,p)\}$

$$\frac12|P(s;t,x,p)|^2 = \frac12|p|^2 + \int_t^s qE(\tau,X(\tau))\cdot P(\tau)d\tau \ge \frac12|p|^2 - \int_t^s |q|\cdot\|E\|_{L^\infty}\cdot|P(\tau)|d\tau,$$

we deduce by using Bellman's lemma that $|P(s;t,x,p)|\ge|p|-(s-t)\cdot|q|\cdot\|E\|_{L^\infty}\ge|p|-T_1\cdot|q|\cdot\|E\|_{L^\infty}$ and thus we have $\varphi_\psi(t,x,p)=0$ if $|p|>R+T_1\cdot|q|\cdot\|E\|_{L^\infty}$. Moreover, we have also that $\|\varphi_\psi\|_{L^\infty}\le T_1\cdot\|\psi\|_{L^\infty}$.

Remark 2.2 It is well known that the mild solution is unique and is given by

$$\begin{aligned}&f(t,x,p) = f_0(X(0;t,x,p),P(0;t,x,p)), && \text{if } s_{\rm in}(t,x,p)=0,\\ &f(t,x,p) = g(s_{\rm in}, X(s_{\rm in};t,x,p), P(s_{\rm in};t,x,p)), && \text{if } s_{\rm in}(t,x,p)>0.\end{aligned}$$

Remark 2.3 We check easily that the mild solution is also a weak solution. Moreover, the mild solution verifies the following Green formula:

$$\begin{aligned}&-\int_0^{T_1}\!\!\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\partial_t\varphi + v(p)\cdot\nabla_x\varphi + F(t,x,p)\cdot\nabla_p\varphi)dtdxdp\\ &+\int_\Omega\int_{\mathbb{R}^3_p}\gamma f(T_1,x,p)\varphi(T_1,x,p)dxdp + \int_0^{T_1}\!\!\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f(t,x,p)\varphi(t,x,p)dtd\sigma dp\\ &= \int_\Omega\int_{\mathbb{R}^3_p} f_0(x,p)\varphi(0,x,p)dxdp - \int_0^{T_1}\!\!\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi(t,x,p)dtd\sigma dp, \qquad (2.6)\end{aligned}$$

$\forall\varphi\in C^1_c([0,+\infty[\times\overline{\Omega}\times\mathbb{R}^3_p),\ \forall T_1>0$, where the traces $\gamma f(T_1,\cdot,\cdot), \gamma^+ f$ are defined as in Remark 2.2 and belong to $L^1(\Omega\times B_R)$, respectively $L^1(]0,T_1[\times\Sigma^+_R),\ \forall R>0,\ \forall T_1>0$.

Remark 2.4 By using the Remark 2.2 we check easily that the mild solution f verifies

$$\min\left\{\inf_{\Omega\times\mathbb{R}^3_p} f_0, \inf_{]0,+\infty[\times\Sigma^-} g\right\} \le f \le \max\left\{\sup_{\Omega\times\mathbb{R}^3_p} f_0, \sup_{]0,+\infty[\times\Sigma^-} g\right\},$$

with the same inequalities for the traces $\gamma f(T_1,\cdot,\cdot),\ \gamma^+ f$. In particular, if $f_0 \ge 0,\ g \ge 0$ then $f \ge 0,\ \gamma^+ f \ge 0,\ \gamma f(T_1,\cdot,\cdot) \ge 0,\ \forall T_1 > 0$.

2.1 The momentum change in the classical case

In this section we set $\mathcal{E}(p) = \frac{|p|^2}{2m},\ v(p) = \frac{p}{m},\ \forall p \in \mathbb{R}^3_p$. In this case, the characteristic system is given by

$$\frac{dX}{ds} = \frac{P(s)}{m}, \quad \frac{dP}{ds} = q(E(s,X(s)) + \frac{P(s)}{m} \wedge B(s,X(s))), \quad s_{\text{in}} \le s \le s_{\text{out}}, \tag{2.7}$$

where the electro-magnetic field is regular $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$. We state the momentum change lemma for the classical case. The details of the proof can be found in Appendix.

Lemma 2.1 *Assume that $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ and consider $(X(s), P(s)),\ s_{\text{in}} \le s \le s_{\text{out}}$ an arbitrary solution for (2.7). Denote by D_{cla} the quantity*

$$D_{\text{cla}} = (2|q| \cdot \|E\|_\infty \cdot m \cdot \text{diam}(\Omega))^{1/2} + 2 \cdot |q| \cdot \|B\|_\infty \cdot \text{diam}(\Omega).$$

Then

(1) *if there is $t \in [s_{\text{in}}, s_{\text{out}}]$ such that $|P(t)| > D_{\text{cla}}$, then we have*

$$s_{\text{out}} - s_{\text{in}} \le 4 \cdot \frac{\text{diam}(\Omega)}{|v(P(t))|} \le 4m \cdot \frac{\text{diam}(\Omega)}{D_{\text{cla}}}, \quad \textit{and} \quad |P(s) - P(t)| \le D_{\text{cla}}, \quad \forall\, s_{\text{in}} \le s \le s_{\text{out}};$$

(2) *for all $s_{\text{in}} \le s_1 \le s_2 \le s_{\text{out}}$ we have $|P(s_1) - P(s_2)| \le 2D_{\text{cla}}$.*

The Lemma 2.1 holds true in two dimensional spatial domain $\Omega \subset \mathbb{R}^2_x$ for orthogonal electric and magnetic fields $E = (E_1, E_2, 0),\ B = (0, 0, B_3)$. In this case, the system of characteristics is given by

$$\frac{dX_1}{ds} = \frac{P_1(s)}{m}, \quad \frac{dP_1}{ds} = q\Big(E_1(s, X_1(s), X_2(s)) + \frac{P_2(s)}{m} \cdot B_3(s, X_1(s), X_2(s))\Big),$$
$$\frac{dX_2}{ds} = \frac{P_2(s)}{m} \quad \frac{dP_2}{ds} = q\Big(E_2(s, X_1(s), X_2(s)) - \frac{P_1(s)}{m} \cdot B_3(s, X_1(s), X_2(s))\Big).$$

Remark also that in the purely electric case ($B = 0$) Lemma 2.1 is valid in any dimension.

2.2 The momentum change lemma in the relativistic case

We analyze also the relativistic case. In this section we set $\mathcal{E}(p) = mc_0^2\left(\left(1 + \frac{|p|^2}{(mc_0)^2}\right)^{\frac{1}{2}} - 1\right)$ with the corresponding velocity $v(p) = \frac{p}{m} \cdot \left(1 + \frac{|p|^2}{(mc_0)^2}\right)^{-1/2}$. We start with the purely electric system of characteristics which is given by

$$\frac{dX}{ds} = \frac{P(s)}{m}\Big(1 + \frac{|P(s)|^2}{m^2 c_0^2}\Big)^{-1/2}, \quad \frac{dP}{ds} = qE(s, X(s)), \quad s_{\text{in}} \le s \le s_{\text{out}}. \tag{2.8}$$

We will analyze (2.8) for any dimension $N \ge 1$. We state the momentum change lemma for the relativistic case (see Appendix for the details of the proof).

Lemma 2.2 *Assume that* $E \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^N$ *and consider* $(X(s), P(s))$, $s_{\text{in}} \le s \le s_{\text{out}}$ *an arbitrary solution for* (2.8). *Denote by* $D^{\text{ele}}_{\text{rel}}$ *the quantity*

$$D^{\text{ele}}_{\text{rel}} = mc_0\sqrt{\beta(1+\beta)}, \quad \text{with } \beta = \frac{4\sqrt{N}\cdot \text{diam}(\Omega)\cdot |q| \cdot \|E\|_\infty}{mc_0^2}.$$

Then

(1) *if there is* $t \in [s_{\text{in}}, s_{\text{out}}]$ *such that* $|P(t)| > D^{\text{ele}}_{\text{rel}}$, *then*

$$s_{\text{out}} - s_{\text{in}} \le 4 \cdot \frac{\text{diam}(\Omega)}{|v(P(t))|} \quad \text{and} \quad |P(s) - P(t)| \le D^{\text{ele}}_{\text{rel}}, \quad \forall\, s_{\text{in}} \le s \le s_{\text{out}};$$

(2) *for all* $s_{\text{in}} \le s_1 \le s_2 \le s_{\text{out}}$ *we have* $|P(s_1) - P(s_2)| \le 2D^{\text{ele}}_{\text{rel}}$.

Consider now the relativistic characteristic system with $N = 3$:

$$\frac{dX}{ds} = v(P(s)), \quad \frac{dP}{ds} = q(E(s, X(s)) + v(P(s)) \wedge B(s, X(s))), \quad s_{\text{in}} \le s \le s_{\text{out}}. \tag{2.9}$$

By observing that $|q(E + v(p) \wedge B)| \le |q| \cdot (\|E\|_\infty + c_0 \cdot \|B\|_\infty)$ we deduce also

Lemma 2.3 *Assume that* $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ *and consider* $(X(s), P(s))$, $s_{\text{in}} \le s \le s_{\text{out}}$ *an arbitrary solution for* (2.9). *Then the conclusions of Lemma* 2.2 *hold true with*

$$D_{\text{rel}} = mc_0\sqrt{\beta_1(1+\beta_1)}, \quad \text{with } \beta_1 = \frac{4\sqrt{3}\cdot |q| \cdot \text{diam}(\Omega) \cdot (\|E\|_\infty + c_0\|B\|_\infty)}{mc_0^2}.$$

2.3 Estimate of the momentum support for the initial-boundary value problem

Generally we will assume that the electro-magnetic field is bounded $(E, B) \in L^\infty(]0, +\infty[\times\Omega)^6$ and that the initial-boundary conditionsare compactly supported in momentum, uniformly in t, x: $\exists R > 0$ such that $f_0(x, p) = 0$, $\forall\, (x, p) \in \Omega \times \mathbb{R}^3_p$, $|p| > R$ and $g(t, x, p) = 0$, $\forall\, (t, x, p) \in]0, +\infty[\times \Sigma^-$, $|p| > R$. In this case, at least for regular electro-magnetic field it is easy to see that f has compact support in momentum, uniformly with respect to $(t, x) \in]0, T_1[\times\Omega$, $\forall\, T_1 > 0$. Indeed, by using the characteristic equations

$$\frac{dX}{ds} = v(P(s)), \quad \frac{dP}{ds} = F(s, X(s), P(s)),$$

we deduce that

$$\frac{1}{2}\frac{d}{ds}|P(s)|^2 = q \cdot E(s, X(s)) \cdot P(s),$$

and by Bellman's lemma we obtain that the change of the momentum norm along any characteristic included in $]0, T_1[\times\Omega \times \mathbb{R}^3_p$ is bounded by $T_1 \cdot |q| \cdot \|E\|_{L^\infty}$ and thus we have

$$f = f \cdot \mathbf{1}_{\{|p| \le R_1\}}, \quad (t, x, p) \in]0, T_1[\times\Omega \times \mathbb{R}^3_p, \quad \forall\, T_1 > 0, \tag{2.10}$$

where $R_1 = R + T_1 \cdot |q| \cdot \|E\|_{L^\infty}$. The situation is very different for boundary value problems (for example stationary or time periodic problems). In this case we do not know if the solution of the Vlasov equation remains compactly supported in momentum (think that the life time of the characteristics inside the bounded domain Ω can be arbitrarily large). The natural question arising from the above observations is: can we deduce that $f = f \cdot \mathbf{1}_{\{|p| \le R_1\}}$ with R_1 not depending on $(t,x) \in]0,+\infty[\times\Omega$ respectively $(t,x) \in \mathbb{R}_t\times\Omega$? The motivation for finding globally in time estimate for the momentum support comes for numerical considerations. Clearly, if a bound R_1 of the momentum support is available, the computation domain can be restricted to $\Omega \times B_{R_1}$.

Theorem 2.1 *Assume that* $E, B \in L^\infty(]0,+\infty[;W^{1,\infty}(\Omega))^3$, $f_0 \in L^1(\Omega\times\mathbb{R}^3_p)$, $(v(p)\cdot n(x))g \in L^1(]0,T_1[\times\Sigma^-)$, $\forall T_1 > 0$ *with* $f_0 = f_0\cdot \mathbf{1}_{\{|p|\le R\}}$, $g = g\cdot\mathbf{1}_{\{|p|\le R\}}$, *for some* $R>0$. *Then the mild solution for* (2.1)–(2.3) *is compactly supported in momentum uniformly in* $(t,x) \in]0,+\infty[\times\Omega$ *and we have*

$$f = f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma^+ f = \gamma^+ f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma f(T_1,\cdot,\cdot) = \gamma f(T_1,\cdot,\cdot)\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \forall T_1>0,$$

where $R_1 = R + 2D_{\mathrm{cla/rel}}$.

Proof Take $p \in \mathbb{R}^3_p$ with $|p| > R_1$. By Lemmas 2.1 and 2.3 we deduce that $|P(s;t,x,p) - p| \le 2D_{\mathrm{cla/rel}}$, $\forall s_{\mathrm{in}} \le s \le t$ and therefore $|P(s;t,x,p)| \ge |p| - |P(s;t,x,p) - p| > R_1 - 2D_{\mathrm{cla/rel}} = R$, $\forall s_{\mathrm{in}} \le s \le t$. By Remark 2.2 we deduce that $f(t,x,p) = 0$. The same arguments apply for the traces $\gamma^+ f, \gamma f(T_1,\cdot,\cdot)$, $\forall T_1 > 0$.

We can construct also weak solutions for (2.1)–(2.3) with compact support in momentum.

Theorem 2.2 *Assume that* $E, B \in L^\infty(]0,T_1[;\Omega)^3$, $|f_0|^r \in L^1(\Omega\times\mathbb{R}^3_p)$, $(v(p)\cdot n(x))|g|^r \in L^1(]0,T_1[\times\Sigma^-)$, *for some* $T_1 > 0$, $1 < r < +\infty$ *with* $f_0 = f_0\cdot\mathbf{1}_{\{|p|\le R\}}$, $g = g\cdot\mathbf{1}_{\{|p|\le R\}}$. *Then there is a weak solution for* (2.1)–(2.3) *on* $]0,T_1[\times\Omega\times\mathbb{R}^3_p$ *such that*

$$f = f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma^+ f = \gamma^+ f\cdot\mathbf{1}_{\{|p|\le R_1\}},\quad \gamma f(T_1,\cdot,\cdot) = \gamma f(T_1,\cdot,\cdot)\cdot\mathbf{1}_{\{|p|\le R_1\}},$$

where $R_1 = R + 2D_{\mathrm{cla/rel}}$.

Proof Regularize the electro-magnetic field by convolution in respect to x (extend E, B by 0 outside Ω). Denote by f_ε the mild solution for (2.1)–(2.3) corresponding to the regularized field $E_\varepsilon, B_\varepsilon$. As in [28] we obtain

$$\partial_t|f_\varepsilon|^r + v(p)\cdot\nabla_x|f_\varepsilon|^r + F_\varepsilon\cdot\nabla_p|f_\varepsilon|^r = 0,$$

where $F_\varepsilon = q(E_\varepsilon(t,x) + v(p)\wedge B_\varepsilon(t,x))$. After integration on $]0,T_1[\times\Omega\times\mathbb{R}^3_p$ we find

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^3_p}|\gamma f_\varepsilon|^r(T_1,x,p)dxdp + \int_0^{T_1}\!\!\int_{\Sigma^+}(v(p)\cdot n(x))|\gamma^+ f_\varepsilon|^r(t,x,p)dtd\sigma dp\\ &= \int_\Omega\int_{\mathbb{R}^3_p}|f_0|^r(x,p)dxdp - \int_0^{T_1}\!\!\int_{\Sigma^-}(v(p)\cdot n(x))|g|^r(t,x,p)dtd\sigma dp,\end{aligned}$$

which gives uniform estimates in L^r for $\varepsilon > 0$:

$$\begin{aligned}&\sup_{0\le t\le T_1}\int_\Omega\int_{\mathbb{R}^3_p}|\gamma f_\varepsilon|^r(t,x,p)dxdp + \int_0^{T_1}\!\!\int_{\Sigma^+}(v(p)\cdot n(x))|\gamma^+ f_\varepsilon|^r(t,x,p)dtd\sigma dp\\ &\le 2\Big(\int_\Omega\int_{\mathbb{R}^3_p}|f_0|^r(x,p)dxdp - \int_0^{T_1}\!\!\int_{\Sigma^-}(v(p)\cdot n(x))|g|^r(t,x,p)dtd\sigma dp\Big).\end{aligned}$$

We can extract subsequences $f_{\varepsilon_k} \rightharpoonup f$ weakly in $L^r(]0,T_1[\times\Omega\times\mathbb{R}^3_p)$, $\gamma f_{\varepsilon_k}(T_1,\cdot,\cdot) \rightharpoonup \gamma f(T_1,\cdot,\cdot)$ weakly in $L^r(\Omega\times\mathbb{R}^3_p)$, $\gamma^+ f_{\varepsilon_k} \rightharpoonup \gamma^+ f$ weakly in $L^r(]0,T_1[\times\Sigma^+,(v(p)\cdot n(x))dtd\sigma dp)$. By standard arguments we deduce that f is a weak solution for (2.1)–(2.3) associated to the electro-magnetic field (E,B) with traces $\gamma^+ f,\ \gamma f(T_1,\cdot,\cdot)$. On the other hand, for $|p| > R_1 = R + 2D_{\text{cla/rel}} \geq R + 2D^{\varepsilon_k}_{\text{cla/rel}} = R_1^{\varepsilon_k}$ we have $f_{\varepsilon_k} = 0,\ \gamma f_{\varepsilon_k}(T_1) = 0,\ \gamma^+ f_{\varepsilon_k} = 0$ and by weak limit we deduce that $\int_0^{T_1}\int_\Omega\int_{\mathbb{R}^3_p} f\psi dtdxdp = \lim_{k\to+\infty}\int_0^{T_1}\int_\Omega\int_{\mathbb{R}^3_p} f_{\varepsilon_k}\psi dtdxdp = 0,\ \forall\,\psi\in C^0_c([0,T_1]\times\overline{\Omega}\times(\mathbb{R}^3_p - B_{R_1}))$ which implies that $f = 0$ a.e. in $]0,T_1[\times\Omega\times(\mathbb{R}^3_p - B_{R_1})$ or $\operatorname{supp} f \subset]0,T_1[\times\Omega\times B_{R_1}$. Similarly we deduce that $\operatorname{supp}\gamma^+ f \subset]0,T_1[\times\Sigma^+_{R_1}$ and $\operatorname{supp}\gamma f(T_1,\cdot,\cdot) \subset \Omega\times B_{R_1}$. Note that if $E,B\in L^\infty(]0,+\infty[\times\Omega)^3$, then $R_1 = R + 2D_{\text{cla/rel}}$ does not depend on T_1 and therefore the solution is compactly supported in momentum uniformly with respect to $T_1 > 0$.

Remark 2.5 By using the Remark 2.4 we can prove that the conclusion of the above theorem holds also in the case $r = +\infty$.

2.4 Estimate of the momentum support for the time periodic problem

An application of the momentum change lemma could be the estimate of the momentum support for time periodic solutions of the Vlasov problem. First we introduce the perturbed time periodic Vlasov problem

$$\alpha f + \partial_t f + v(p)\cdot\nabla_x f + F(t,x,p)\cdot\nabla_p f = 0,\quad (t,x,p)\in\mathbb{R}_t\times\Omega\times\mathbb{R}^3_p, \tag{2.11}$$

with the boundary condition

$$f(t,x,p) = g(t,x,p),\quad (t,x,p)\in\mathbb{R}_t\times\Sigma^-, \tag{2.12}$$

where this time g,E,B are supposed T periodic in time, $T>0,\ \alpha>0$ fixed. The definition of T periodic weak solution is given by

Definition 2.3 *Assume that $E,B\in L^\infty(\mathbb{R}_t\times\Omega)^3$ and g are T periodic with $(v(p)\cdot n(x))g\in L^1(]0,T[\times\Sigma^-_R),\ \forall\,R>0$. We say that $f\in L^1(]0,T[\times\Omega\times B_R),\ \forall\,R>0$ is a T periodic weak solution for the problem* (2.11), (2.12) *iff*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^3_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi)dtdxdp$$
$$= -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi dtd\sigma dp$$

for all test function which belongs to

$$\mathcal{T}^{\text{per}}_w = \{\varphi\in C^1(\mathbb{R}_t\times\overline{\Omega}\times\mathbb{R}^3_p)\,|\,\exists R>0:\varphi=\varphi\cdot\mathbf{1}_{\{|p|\leq R\}},\ \varphi|_{\mathbb{R}_t\times\Sigma^+} = 0,\ \varphi(\cdot+T)=\varphi\}.$$

Note also that in the periodic case the definition for s_{in} is

$$s_{\text{in}}(t,x,p) = \sup\{s\leq t\,|\,X(s;t,x,p)\in\partial\Omega\}.$$

It may happen that $s_{\text{in}} = -\infty$. Let us give now the definition for time periodic mild solution.

Definition 2.4 *Assume that* $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$ *and* g *are* T *periodic with* $(v(p) \cdot n(x))g \in L^1(]0, T[\times \Sigma_R^-)$, $\forall R > 0$. *We say that* $f \in L^1(]0, T[\times \Omega \times B_R)$, $\forall R > 0$ *is a* T *periodic mild solution for* (2.11), (2.12) *iff*

$$\int_0^T\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi_\psi^\alpha(t,x,p)dtd\sigma dp$$

for all test function which belongs to

$$\mathcal{T}_m^{\mathrm{per}} = \{\psi \in C^0(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)\,|\,\exists R > 0 : \psi = \psi \cdot \mathbf{1}_{\{|p|\le R\}},\ \psi(\cdot + T) = \psi\},$$

where

$$\varphi_\psi^\alpha(t,x,p) = \int_t^{s_{\mathrm{out}}(t,x,p)} e^{-\alpha(s-t)}\psi(s, X(s;t,x,p), P(s;t,x,p))ds.$$

Remark 2.6 Observe that by Lemmas 2.1 and 2.3, the function φ_ψ^α has also compact support in momentum (if $\psi = \psi \cdot \mathbf{1}_{\{|p|\le R\}}$ then $\varphi_\psi^\alpha = \varphi_\psi^\alpha \cdot \mathbf{1}_{\{|p|\le R+2D_{\mathrm{cla/rel}}\}}$) and that for $\alpha > 0$ the function φ_ψ^α is bounded: $\|\varphi_\psi^\alpha\|_\infty \le \frac{\|\psi\|_\infty}{\alpha}$. Therefore the above definition makes sense.

Remark 2.7 In this case, the mild solution is given by $f(t,x,p) = 0$ if $s_{\mathrm{in}} = -\infty$ and $f(t,x,p) = e^{-\alpha(t-s_{\mathrm{in}})}g(s_{\mathrm{in}}, X(s_{\mathrm{in}};t,x,p), P(s_{\mathrm{in}};t,x,p))$ if $s_{\mathrm{in}} > -\infty$.

Remark 2.8 The mild T periodic solution is also a T periodic weak solution and verifies the following Green formula

$$\begin{aligned}&\int_0^T\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi)dtdxdp\\ &= -\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\varphi dtd\sigma dp - \int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi dtd\sigma dp\end{aligned}$$

for all $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$, compactly supported in momentum and T periodic, where the trace function $\gamma^+ f$ is defined as in Remark 2.7.

Remark 2.9 Suppose that g is bounded. Then the T periodic mild solution of problem (2.11), (2.12) verifies

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \le \|g\|_\infty.$$

In particular, if $g \ge 0$ then $f, \gamma^+ f \ge 0$.

Theorem 2.3 *Assume that* $\alpha > 0$, $E, B \in L^\infty(\mathbb{R}_t; W^{1,\infty}(\Omega))^3$, $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ *are* T *periodic with* $g = g \cdot \mathbf{1}_{\{|p|\le R\}}$ *for some* $R > 0$. *Then the* T *periodic mild solution* f *for* (2.11), (2.12) *verifies*

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \le \|g\|_\infty,\ f = f\cdot \mathbf{1}_{\{|p|\le R_1\}},\ \gamma^+ f = \gamma^+ f \cdot \mathbf{1}_{\{|p|\le R_1\}},$$

with $R_1 = R + 2D_{\mathrm{cla/rel}}$.

Proof Take $\psi \in C^0(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}^3_p)$, T periodic, with compact support in momentum in $\mathbb{R}^3_p - B_{R_1}$. By the mild formulation, we have

$$\int_0^T\int_\Omega \int_{\mathbb{R}^3_p} f(t,x,p)\psi(t,x,p)dtdxdp = -\int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g(t,x,p)\varphi_\psi^\alpha(t,x,p)dtd\sigma dp.$$

If $|p| > R$, then $g = 0$ and $g \cdot \varphi_\psi^\alpha = 0$. If $|p| \le R$, then by Lemmas 2.1 and 2.3 we deduce that $|P(s)| \le |p| + 2D_{\text{cla/rel}} \le R_1$ and thus $\varphi_\psi^\alpha = 0$ or $g \cdot \varphi_\psi^\alpha = 0$. We deduce that $\int_0^T\int_\Omega\int_{\mathbb{R}_p^3} f\psi dtdxdp = 0$, or $\operatorname{supp} f \subset \mathbb{R}_t \times \Omega \times B_{R_1}$. Now, by using the Green formula we have

$$\int_0^T\int_\Omega\int_{\mathbb{R}_p^3} f(t,x,p)(\alpha\varphi - \partial_t\varphi - v(p)\cdot\nabla_x\varphi - F(t,x,p)\cdot\nabla_p\varphi)dtdxdp$$
$$= -\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\varphi dtd\sigma dp - \int_0^T\int_{\Sigma^-}(v(p)\cdot n(x))g\varphi dtd\sigma dp$$

for any function $\varphi \in C^1(\mathbb{R}_t \times \overline{\Omega} \times \mathbb{R}_p^3)$, T periodic, with compact support in momentum in $\mathbb{R}_p^3 - B_{R_1}$. Therefore we have $\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\varphi dtd\sigma dp = 0$ which implies that $\operatorname{supp}\gamma^+ f \subset \mathbb{R}_t \times \Sigma_{R_1}^+$.

By regularization we can prove the existence of T periodic weak solution with compact support in momentum.

Theorem 2.4 *Assume that* $\alpha = 0$, $E, B \in L^\infty(\mathbb{R}_t \times \Omega)^3$, $g \in L^\infty(\mathbb{R}_t \times \Sigma^-)$ *are* T *periodic with* $g = g \cdot \mathbf{1}_{\{|p|\le R\}}$ *for some* $R > 0$. *Then there is a* T *periodic weak solution* f *for* (2.11), (2.12) *which verifies*

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \le \|g\|_\infty,\ f = f\cdot\mathbf{1}_{\{|p|\le R_1\}},\ \gamma^+ f = \gamma^+ f\cdot\mathbf{1}_{\{|p|\le R_1\}},$$

with $R_1 = R + 2D_{\text{cla/rel}}$.

Proof Regularize the electro-magnetic field and take f_ε the T periodic mild solutions constructed in the previous theorem with $\alpha = \varepsilon$ and the electro-magnetic field $(E_\varepsilon, B_\varepsilon)$. We have

$$\max\{\|f_\varepsilon\|_\infty, \|\gamma^+ f_\varepsilon\|_\infty\} \le \|g\|_\infty, \quad f_\varepsilon = f_\varepsilon\cdot\mathbf{1}_{\{|p|\le R_1\}}, \quad \gamma^+ f_\varepsilon = \gamma^+ f_\varepsilon\cdot\mathbf{1}_{\{|p|\le R_1\}}$$

since

$$R_1^\varepsilon = R + 2D_{\text{cla/rel}}^\varepsilon \le R + 2D_{\text{cla/rel}} = R_1.$$

We can extract sequences such that $f_{\varepsilon_k} \rightharpoonup f$ weakly $*$ in $L^\infty(\mathbb{R}_t \times \Omega \times \mathbb{R}_p^3)$, $\gamma^+ f_{\varepsilon_k} \rightharpoonup \gamma^+ f$ weakly $*$ in $L^\infty(\mathbb{R}_t \times \Sigma^+)$. By passing to the limit for $k \to \infty$ in the weak formulation, we deduce that f is the periodic weak solution corresponding to the electro-magnetic field (E, B) and $\varepsilon = 0$. Also by passing to the limit in the Green formula for $k \to +\infty$ we deduce that $\gamma^+ f$ is the trace of f. By weak $*$ limit we have

$$\max\{\|f\|_\infty, \|\gamma^+ f\|_\infty\} \le \liminf_{k\to+\infty}\max\{\|f_{\varepsilon_k}\|_\infty, \|\gamma^+ f_{\varepsilon_k}\|_\infty\} \le \|g\|_\infty$$

and also $f = f\cdot\mathbf{1}_{\{|p|\le R_1\}}$ and $\gamma^+ f = \gamma^+ f\cdot\mathbf{1}_{\{|p|\le R_1\}}$.

3 The Regularized Vlasov-Poisson System

We consider $\Omega \subset \mathbb{R}_x^N$ an open, regular bounded set. We denote by $E_0 = -\nabla_x\Phi_0$ the exterior electric field

$$-\Delta_x\Phi_0(t,x) = 0, \quad (t,x) \in]0,T[\times\Omega, \quad \Phi_0(t,x) = \varphi_0(t,x), \quad (t,x) \in]0,T[\times\partial\Omega.$$

In this section, we construct solutions for the following regularized Vlasov-Poisson system (classical or relativistic case)

$$\begin{cases}\partial_t f + v(p)\cdot\nabla_x f + qE_\varepsilon\cdot\nabla_p f = 0, \quad (t,x,p)\in]0,T[\times\Omega\times\mathbb{R}^N_p,\\ f(0,x,p) = f_0(x,p),\ \ (x,p)\in\Omega\times\mathbb{R}^N_p, \quad f(t,x,p) = g(t,x,p),\ \ (t,x,p)\in]0,T[\times\Sigma^-,\\ -(1-\alpha\Delta_x)^{2m}\Delta_x\Phi = \dfrac{\rho_\varepsilon}{\varepsilon_0}, \quad (t,x)\in]0,T[\times\Omega,\\ \Phi = \Delta_x\Phi = \cdots = \Delta_x^{2m}\Phi = 0, \quad (t,x)\in]0,T[\times\partial\Omega,\end{cases} \tag{3.1}$$

where $E_\varepsilon = \overline{E}*\zeta_\varepsilon$, $\overline{E}$ is the extension by 0 outside $]0,T[\times\Omega$ of $E = -\nabla_x\Phi - \nabla_x\Phi_0$ and $\zeta_\varepsilon(t,x) = \frac{1}{\varepsilon^{N+1}}\zeta(\frac{t}{\varepsilon},\frac{x}{\varepsilon})$ is a mollifier, i.e., $\zeta\in C_c^\infty(\mathbb{R}\times\mathbb{R}^N_x)$, $\zeta\geq 0$, $\int_{\mathbb{R}^{N+1}}\zeta(s,y)\,dsdy = 1$ and $\alpha,\varepsilon > 0$ are small parameters. Regularized systems of this type have been used in previous works (see [20]). We recall here the following result

Lemma 3.1 *Let $\rho\in L^p(\Omega)$ for some $1<p<+\infty$ and suppose that $\partial\Omega$ is smooth. Then the solution Φ of the regularized Poisson problem*

$$-(1-\alpha\Delta_x)^{2m}\Delta_x\Phi = \frac{\rho}{\varepsilon_0}, \qquad x\in\Omega,$$
$$\Phi = \Delta_x\Phi = \cdots = \Delta_x^{2m}\Phi = 0, \quad x\in\partial\Omega$$

verifies

$$\|\Phi\|_{W^{4m+2,p}(\Omega)} \leq C(p,\alpha,\Omega)\cdot\|\rho\|_{L^p(\Omega)}, \quad \|\Phi\|_{W^{2,p}(\Omega)} \leq C(p,\Omega)\cdot\|\rho\|_{L^p(\Omega)}.$$

We prove the existence of solution for the regularized Vlasov-Poisson system. For the sake of the presentation we give a sketch of the proof. For more details the reader can refer to [20]. We consider the set $\chi = L^2(]0,T[;H^1(\Omega))$ and define the application $\mathcal{F}:\chi\to\chi$ by

$$\Phi\to E = -\nabla_x\Phi - \nabla_x\Phi_0 \to E_\varepsilon = \overline{E}*\zeta_\varepsilon \to f \to \rho = q\int_{\mathbb{R}^N_p} f\,dp \to \rho_\varepsilon \to \Phi_1 = \mathcal{F}\Phi,$$

where

- f is the mild solution of the Vlasov problem associated with the regularized field

$$E_\varepsilon(t,x) = -\int_0^T\int_\Omega(\nabla_x\Phi(s,y) + \nabla_x\Phi_0(s,y))\zeta_\varepsilon(t-s,x-y)\,dsdy;$$

- ρ_ε is the regularized charge density

$$\rho_\varepsilon = \int_0^T\int_\Omega\rho(s,y)\zeta_\varepsilon(t-s,x-y)dsdy;$$

- Φ_1 is the solution of the regularized Poisson problem associated with the charge density ρ_ε.

Proposition 3.1 *Under the hypotheses $f_0\geq 0$, $g\geq 0$, $M_0 + M^- := \int_\Omega\int_{\mathbb{R}^N_p} f_0(x,p)\,dxdp + \int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|g(t,x,p)\,dtd\sigma dp < +\infty$, $\varphi_0\in L^2(]0,T[;H^{\frac{1}{2}}(\partial\Omega))$ we have*

$$\mathcal{F}(\chi)\subset\{\Phi\in L^2(]0,T[;H^1(\Omega))\mid \|\Phi\|_{L^2(]0,T[;H^1(\Omega))}\leq M_\varepsilon\},$$

where $M_\varepsilon = C(\Omega)\cdot\frac{T}{\varepsilon_0}\cdot(M_0+M^-)\cdot\|\zeta\|_{L^2(\mathbb{R}^{N+1})}\cdot\varepsilon^{-\frac{N+1}{2}}$.

Proof As usual we have

$$\int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\, dxdp + \int_0^t \int_{\Sigma^+} (v(p)\cdot n(x))\gamma^+ f(s,x,p)\, dsd\sigma dp$$
$$= \int_\Omega \int_{\mathbb{R}^N_p} f_0(x,p)\, dxdp + \int_0^t \int_{\Sigma^-} |(v(p)\cdot n(x))| g\, dsd\sigma dp,$$

and therefore $\|f\|_{L^1(]0,T[\times\Omega\times\mathbb{R}^N_p)} \le T\cdot(M_0+M^-)$. We have the inequalities

$$\|\Phi_1\|_{L^2(]0,T[;H^1(\Omega))} \le C(\Omega)\left\|\frac{\rho_\varepsilon}{\varepsilon_0}\right\|_{L^2(]0,T[\times\Omega)} \le \frac{C(\Omega)}{\varepsilon_0}\cdot\|\rho\|_{L^1(]0,T[\times\Omega)}\cdot\|\zeta_\varepsilon\|_{L^2}$$
$$\le C(\Omega)\cdot\frac{T}{\varepsilon_0}\cdot(M_0+M^-)\cdot\|\zeta\|_{L^2(\mathbb{R}^{N+1})}\cdot\varepsilon^{-\frac{N+1}{2}}.$$

It is easily seen that $\mathcal{F}$ is continuous with respect to the weak topology of $L^2(]0,T[;H^1(\Omega))$.

Proposition 3.2 *Assume that*

$$0\le f_0\in L^\infty(\Omega\times\mathbb{R}^N_p),\quad 0\le g\in L^\infty(\mathbb{R}_t\times\Sigma^-),$$
$$\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f_0(x,p)\,dxdp+\int_0^T\int_{\Sigma^-}|(v(p)\cdot n(x))|(1+\mathcal{E}(p))g(t,x,p)\,dtd\sigma dp<\infty,$$
$$\varphi_0\in L^2(]0,T[;H^{\frac12}(\partial\Omega)).$$

Then the application $\mathcal{F}$ is continuous with respect to the weak topology of $L^2(]0,T[;H^1(\Omega))$.

Eventually the Schauder fixed point theorem implies the existence of a weak solution (f,Φ_s) for

$$\begin{cases}\partial_t f+v(p)\cdot\nabla_x f+q(\overline{E}*\zeta)\cdot\nabla_p f=0, & (t,x,p)\in]0,T[\times\Omega\times\mathbb{R}^N_p,\\ f(0,x,p)=f_0(x,p),\ (x,p)\in\Omega\times\mathbb{R}^N_p, & f(t,x,p)=g(t,x,p),\ (t,x,p)\in]0,T[\times\Sigma^-,\\ -(1-\alpha\Delta_x)^{2m}\Delta_x\Phi_s=\dfrac{\overline{\rho}*\zeta}{\varepsilon_0},\ (t,x)\in]0,T[\times\Omega, & E=-\nabla_x\Phi_s-\nabla_x\Phi_0,\ (t,x)\in]0,T[\times\Omega,\\ \Phi_s=\Delta_x\Phi_s=\cdots=\Delta^{2m}\Phi_s=0, & (t,x)\in]0,T[\times\partial\Omega.\end{cases}$$

Following the idea of [20] we can pass to the limit for $\varepsilon\searrow 0$ when $\alpha>0$ is fixed. We obtain

Proposition 3.3 *Assume that $\Omega\subset\mathbb{R}^N_x$ is open and bounded, with $\partial\Omega$ smooth. Consider $p_0=\frac{2N}{2N-1}$, $p_0'=2N$, $(\frac{1}{p_0}+\frac{1}{p_0'}=1)$ and m such that $W^{4m,p_0}(\Omega)\to L^\infty(\Omega)$ is continuous $(\frac{1}{p_0}-\frac{4m}{N}<0)$. We suppose also that the initial-boundary conditionsverify $0\le f_0\in L^\infty(\Omega\times\mathbb{R}^N_p)$, $0\le g\in L^\infty(]0,T[\times\Sigma^-)$, $\exists R>0$ such that $f_0=f_0\cdot\mathbf{1}_{\{|p|\le R\}}$, $g=g\cdot\mathbf{1}_{\{|p|\le R\}}$, $\varphi_0\in L^\infty(]0,T[;W^{4m+2-\frac{1}{p_0},p_0}(\partial\Omega))$, $\partial_t\varphi_0\in L^\infty(]0,T[;W^{4m+1-\frac{1}{p_0},p_0}(\partial\Omega))$. Then there is at least one solution for the Vlasov problem (classical or relativistic case) coupled to the regularized Poisson problem*

$$\begin{cases}\partial_t f+v(p)\cdot\nabla_x f+q(-\nabla_x\Phi_s-\nabla_x\Phi_0)\cdot\nabla_p f=0, & (t,x,p)\in]0,T[\times\Omega\times\mathbb{R}^N_p,\\ f(0,x,p)=f_0(x,p),\ (x,p)\in\Omega\times\mathbb{R}^N_p, & f(t,x,p)=g(t,x,p),\ (t,x,p)\in]0,T[\times\Sigma^-,\\ -(1-\alpha\Delta_x)^{2m}\Delta_x\Phi_s=\dfrac{\rho}{\varepsilon_0}, & (t,x)\in]0,T[\times\Omega,\\ \Phi_s=\Delta_x\Phi_s=\cdots=\Delta^{2m}\Phi_s=0, & (t,x)\in]0,T[\times\partial\Omega.\end{cases}\tag{3.2}$$

The particle densities f, $\gamma^+ f$ have compact support in momentum and the self consistent potential Φ_s verifies $\partial_t \Phi_s \in L^\infty(]0,T[;W^{1,\infty}(\Omega))$, $\nabla_x \Phi_s \in L^\infty(]0,T[;W^{1,\infty}(\Omega))^N$. In particular, the electric field $E = -\nabla_x \Phi_s - \nabla_x \Phi_0$ belongs to $W^{1,\infty}(]0,T[\times\Omega)^N$.

Proof The proof follows by standard arguments (see [20]). The main idea is to estimate the L^∞ norm of the electric field uniformly with respect to $\varepsilon > 0$, when $\alpha > 0$ is fixed. Denote by $(f_\varepsilon, \Phi_{s,\varepsilon})$ the solutions of (3.1) constructed above. First, since the initial-boundary conditions have momentum support contained in $B(0,R)$, we deduce that f_ε has momentum support contained in $B(0,R_1)$, with $R_1 = R + |q| \cdot T \cdot (\|\nabla_x \Phi_0\|_{L^\infty} + \|\nabla_x \Phi_{s,\varepsilon}\|_{L^\infty})$. We deduce that $\|\rho_\varepsilon\|_{L^\infty} \le C \cdot (1 + \|\nabla_x \Phi_{s,\varepsilon}\|^N_{L^\infty})$. By elliptic regularity result (see Lemma 3.1) we can write

$$\begin{aligned}
\|\nabla_x \Phi_{s,\varepsilon}\|_{L^\infty} &\le C \cdot \|\Phi_{s,\varepsilon}\|_{L^\infty(]0,T[;W^{4m+2,p_0}(\Omega))} \le C \cdot \|\rho_\varepsilon\|_{L^\infty(]0,T[;L^{p_0}(\Omega))} \\
&\le C \cdot \|\rho_\varepsilon\|^{\frac{1}{p_0}}_{L^\infty(]0,T[;L^1(\Omega))} \cdot \|\rho_\varepsilon\|^{\frac{1}{p_0'}}_{L^\infty(]0,T[;L^\infty(\Omega))} \\
&\le C \cdot (1 + \|\nabla_x \Phi_{s,\varepsilon}\|^N_{L^\infty})^{\frac{1}{p_0'}}, \qquad (3.3)
\end{aligned}$$

which gives the desired estimate for the L^∞ norm of the electric field $E_\varepsilon = -\nabla_x \Phi_{s,\varepsilon} - \nabla_x \Phi_0$. The existence of solution follows by passing to the limit for $\varepsilon \searrow 0$ in (3.1). For the other statements use the inclusion $W^{4m,p_0}(\Omega) \to L^\infty(\Omega)$, the elliptic regularity result and the continuity equation $\partial_t \rho + \operatorname{div}_x j = 0$.

4 A Priori Estimates

In this section we establish uniform estimates with respect to $\alpha > 0$ for the solutions of (3.2). Firstly we recall the classical estimates for the total mass and energy. Secondly we deduce also an estimate for the L^∞ norm of the electric field. We assume that the hypotheses of Proposition 3.3 are verified and we denote by (f, Φ_s) the solution of (3.2). We recall that $\partial_t \Phi_0,\ \partial_t \Phi_s \in L^\infty(]0,T[;W^{1,\infty}(\Omega))$, $\nabla_x \Phi_0,\ \nabla_x \Phi_s \in L^\infty(]0,T[;W^{1,\infty}(\Omega))^N$ and $f, \gamma^+ f$ have compact support in momentum.

4.1 The mass and energy estimates

We introduce the notations

$$\begin{aligned}
&M_0 := \int_\Omega \int_{\mathbb{R}^N_p} f_0(x,p)\, dxdp, \quad M(t) := \int_\Omega \int_{\mathbb{R}^N_p} f(t,x,p)\, dxdp, \\
&M^\pm(t) := \int_{\Sigma^\pm} |(v(p) \cdot n(x))| \gamma^\pm f(t,x,p)\, d\sigma dp, \quad M^\pm := \int_0^T M^\pm(t)\, dt, \\
&K_0 := \int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f_0(x,p)\, dxdp, \quad K(t) := \int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f(t,x,p)\, dxdp, \\
&K^\pm(t) := \int_{\Sigma^\pm} |(v(p) \cdot n(x))| \mathcal{E}(p) \gamma^\pm f(t,x,p)\, d\sigma dp, \quad K^\pm := \int_0^T K^\pm(t)\, dt, \\
&V_s(t) := \frac{1}{2} \int_\Omega \rho(t,x) \Phi_s(t,x)\, dx, \quad V_0(t) := \frac{1}{2} \int_\Omega \rho(t,x) \Phi_0(t,x)\, dx.
\end{aligned}$$

The estimate for the total mass follows by using the weak formulation of the Vlasov problem with the test function $\theta = 1$

$$\frac{d}{dt}M(t) + M^+(t) = M^-(t), \quad t \in]0,T[. \tag{4.1}$$

We deduce that

$$M(t) + \int_0^t M^+(s)\, ds = M_0 + \int_0^t M^-(s)\, ds, \quad t \in]0,T[, \tag{4.2}$$

which implies

$$\sup_{0 \le t \le T} \{M(t)\} + M^+ \le 2(M_0 + M^-). \tag{4.3}$$

The estimate for the total energy follows by using the test functions $\mathcal{E}(p)$ and $q\Phi_s$. We have

$$\frac{d}{dt}K(t) + K^+(t) = K^-(t) + \int_\Omega E(t,x) \cdot j(t,x)\, dx, \quad t \in]0,T[. \tag{4.4}$$

We deduce that

$$K(t) + \int_0^t K^+(s)\, ds = K_0 + \int_0^t K^-(s)\, ds + \int_0^t \int_\Omega E(s,x) \cdot j(s,x)\, dsdx, \quad t \in]0,T[. \tag{4.5}$$

By using as test function the potential Φ_s one gets

$$\frac{d}{dt}\int_\Omega \rho(t,x)\Phi_s(t,x)\, dx = \int_\Omega \{\rho(t,x)\partial_t \Phi_s + j(t,x) \cdot \nabla_x \Phi_s\}\, dx, \quad t \in]0,T[. \tag{4.6}$$

By using the regularized Poisson equation, after multiplication by Φ_s and integration by parts we obtain

$$V_s(t) = \frac{1}{2}\int_\Omega \rho(t,x)\Phi_s(t,x)\, dx = \frac{\varepsilon_0}{2}\int_\Omega |(1-\alpha\Delta_x)^m \nabla_x \Phi_s|^2\, dx, \tag{4.7}$$

and we deduce that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}\rho(t,x)\Phi_s(t,x)\, dx &= \varepsilon_0 \int_\Omega (1-\alpha\Delta_x)^m \nabla_x \Phi_s \cdot (1-\alpha\Delta_x)^m \nabla_x \partial_t \Phi_s\, dx \\ &= \int_\Omega \rho(t,x)\partial_t \Phi_s\, dx.\end{aligned} \tag{4.8}$$

Now, by combining (4.6) and (4.8) we have

$$\frac{d}{dt}V_s(t) = \int_\Omega j(t,x) \cdot \nabla_x \Phi_s\, dx, \quad t \in [0,T]. \tag{4.9}$$

Finally, by using (4.4), (4.9) one gets

$$\frac{d}{dt}\{K(t) + V_s(t)\} + K^+(t) = K^-(t) - \int_\Omega \nabla_x \Phi_0 \cdot j(t,x)\, dx, \quad t \in]0,T[, \tag{4.10}$$

which implies

$$\begin{aligned}K(t) + V_s(t) + \int_0^t K^+(s)\, ds &= K_0 + V_s(0) + \int_0^t K^-(s)\, ds \\ &\quad - \int_0^t \int_\Omega \nabla_x \Phi_0(s,x) \cdot j(s,x)\, dsdx, \quad t \in]0,T[.\end{aligned} \tag{4.11}$$

By interpolation inequalities we have

$$\Big| \int_\Omega \nabla_x \Phi_0 \cdot j(s,x)\, dx \Big| \le \|\nabla_x \Phi_0(s)\|_{L^\infty} \cdot \|j(s)\|_{L^1(\Omega)} \le C \cdot \|\nabla_x \Phi_0(s)\|_{L^\infty} \cdot \|j(s)\|_{L^\beta(\Omega)}$$
$$\le C \cdot \|\nabla_x \Phi_0(s)\|_{L^\infty} \cdot (M(s) + K(s))^{\frac{1}{\beta}},$$

where $\beta = \frac{N+2}{N+1}$ in the classical case and $\beta = \frac{N+1}{N}$ in the relativistic case. From (4.2), (4.11) we obtain

$$M(t) + K(t) + V_s(t) + \int_0^t \{M^+(s) + K^+(s)\}\, ds$$
$$\le M_0 + K_0 + V_s(0) + \int_0^t \{M^-(s) + K^-(s)\}\, ds$$
$$+ C \cdot \|\nabla_x \Phi_0\|_{L^\infty} \cdot \int_0^t (M(s) + K(s))^{\frac{1}{\beta}}\, ds, \tag{4.12}$$

which implies easily that there is a constant depending on the initial-boundary conditionsand T but not on the size of the momentum support R and α such that

$$\sup_{0 \le t \le T} \{M(t) + K(t) + V_s(t)\} + M^+ + K^+ \le C(M_0, K_0, V_s(0), M^-, K^-, \|\nabla_x \Phi_0\|_{L^\infty}, T). \tag{4.13}$$

4.2 The L^∞ estimate for the electric field

We want to estimate uniformly with respect to $\alpha > 0$ the L^∞ norm of the electric field $E = -\nabla_x \Phi_s - \nabla_x \Phi_0$, where (f, Φ_s) is solution of (3.2). In the one dimensional case such a bound follows immediately from the estimate (4.13). Consider now the cases $N \ge 2$. We assume that there are non-increasing functions $F_0, G : [0, +\infty[\to \mathbb{R}^+$ such that

$$f_0(x,p) \le F_0(|p|), \quad \forall\, (x,p) \in \Omega \times \mathbb{R}^N_p, \quad g(t,x,p) \le G(|p|), \quad \forall\, (t,x,p) \in]0,T[\times \Sigma^-, \tag{4.14}$$

$$\widetilde{M}_0 := \int_{\mathbb{R}^N_p} F_0(|p|)\, dp + \int_{\mathbb{R}^N_p} G(|p|)\, dp < +\infty. \tag{4.15}$$

Roughly speaking, the above hypotheses say that the initial-boundary conditionshave charge densities in L^∞:

$$\rho_0(x) = \int_{\mathbb{R}^N_p} f_0(x,p)\, dp \le \int_{\mathbb{R}^N_p} F_0(|p|)\, dp, \qquad x \in \Omega,$$
$$\rho^-(t,x) = \int_{(v(p)\cdot n(x))<0} g(t,x,p)\, dp \le \int_{\mathbb{R}^N_p} G(|p|)\, dp, \quad (t,x) \in]0,T[\times \Omega.$$

Note that E is smooth and therefore f can be calculated by using characteristics. The idea is to separate the charge density into two parts corresponding to small and large momentum and to use the momentum change lemma which says that $|P(s_1) - P(s_2)| \le 2D_{\mathrm{cal/rel}}, \forall\, s_{\mathrm{in}} \le s_1 \le s_2 \le s_{\mathrm{out}}$, where $D_{\mathrm{cla}} \sim \|E\|^{\frac{1}{2}}_{L^\infty}$ and $D_{\mathrm{rel}} \sim \|E\|_{L^\infty}$. Let us decompose

$$\rho(t,x) = \rho_1 + \rho_2 = q \int_{\mathbb{R}^N_p} f(t,x,p) \mathbf{1}_{\{|p| \le 4D\}}\, dp + q \int_{\mathbb{R}^N_p} f(t,x,p) \mathbf{1}_{\{|p| > 4D\}}\, dp,$$

with $D = D_{\text{cla/rel}}$ and estimate separately ρ_1, ρ_2. For $\eta > 0$ we can write

$$q^{-1}\rho_1(t,x) = \int_{|p|\le 4D} f^{\frac{1}{N+\eta}} \cdot |p|^{\frac{r}{N+\eta}} \cdot f^{\frac{1}{(N+\eta)'}} \cdot |p|^{-\frac{r}{N+\eta}}\, dp$$
$$\le \Big(\int_{|p|\le 4D} f(t,x,p) \cdot |p|^r \, dp \Big)^{\frac{1}{N+\eta}} \cdot \Big(\int_{|p|\le 4D} f(t,x,p) \cdot |p|^{-\frac{r\cdot (N+\eta)'}{N+\eta}} \, dp \Big)^{\frac{1}{(N+\eta)'}},$$

where $\frac{1}{N+\eta} + \frac{1}{(N+\eta)'} = 1$, $r = 2$ in the classical case and $r = 1$ in the relativistic case. We deduce that

$$\int_\Omega (|q|^{-1}\rho_1(t,x))^{N+\eta}\, dx \le C \cdot \|f\|_{L^\infty}^{\frac{N+\eta}{(N+\eta)'}} \cdot D^{[N-\frac{r\cdot(N+\eta)'}{N+\eta}]\cdot \frac{N+\eta}{(N+\eta)'}} \cdot \int_\Omega \int_{\mathbb{R}^N_p} (1+\mathcal{E}(p)) f(t,x,p)\, dxdp,$$

which implies, by the estimate (4.13),

$$\|\rho_1(t)\|_{L^{N+\eta}} \le C \cdot \|f\|_{L^\infty}^{\frac{1}{(N+\eta)'}} \cdot D^{[N-\frac{r\cdot(N+\eta)'}{N+\eta}]\cdot\frac{1}{(N+\eta)'}} \cdot (M(t)+K(t))^{\frac{1}{N+\eta}} \le C \cdot D^{[\frac{N}{(N+\eta)'} - \frac{r}{N+\eta}]}.$$

Notice that the above estimate is valid for $\eta > 0$ such that $\frac{N}{(N+\eta)'} - \frac{r}{N+\eta} > 0$. For ρ_2 it is possible to find a L^∞ bound. We have

$$q^{-1}\rho_2(t,x) = \int_{|p|>4D} f(t,x,p)\, dp$$
$$= \int_{|p|>4D} f_0(X(0;t,x,p), P(0;t,x,p)) \cdot \mathbf{1}_{\{s_{\text{in}}(t,x,p)=0\}} dp$$
$$+ \int_{|p|>4D} g(s_{\text{in}}(t,x,p), X(s_{\text{in}};t,x,p), P(s_{\text{in}};t,x,p)) \cdot \mathbf{1}_{\{s_{\text{in}}(t,x,p)>0\}} dp.$$

By using the momentum change lemma we have $|P(s;t,x,p)| \ge |p| - 2D$, $\forall s_{\text{in}}(t,x,p) \le s \le t$ and therefore we have the inequalities

$$q^{-1}\rho_2 \le \int_{|p|>4D} F_0(|p|-2D)dp + \int_{|p|>4D} G(|p|-2D)dp$$
$$\le C \cdot \int_{4D}^{+\infty} \{F_0(u-2D)\cdot u^{N-1} + G(u-2D)\cdot u^{N-1}\} du$$
$$= C \cdot \int_{2D}^{+\infty} \{F_0(w) + G(w)\} \cdot (2D+w)^{N-1} dw$$
$$\le C \cdot \int_{2D}^{+\infty} \{F_0(w) + G(w)\} \cdot (2\cdot w)^{N-1} dw$$
$$\le C \cdot \int_{\mathbb{R}^N_p} \{F_0(|p|) + G(|p|)\}\, dp = C \cdot \widetilde{M}_0 < +\infty.$$

The L^∞ bound for E follows by Sobolev inequalities and Lemma 3.1

$$\|\nabla_x \Phi_s(t)\|_{L^\infty(\Omega)} \le \|\nabla_x \Phi_s(t)\|_{W^{1,N+\eta}(\Omega)} \le \|\Phi_s(t)\|_{W^{2,N+\eta}(\Omega)} \le C \cdot \|\rho(t)\|_{L^{N+\eta}(\Omega)}$$
$$\le C \cdot \|\rho_1(t)\|_{L^{N+\eta}(\Omega)} + C \cdot \|\rho_2(t)\|_{L^{N+\eta}(\Omega)}$$
$$\le C \cdot D^{[\frac{N}{(N+\eta)'} - \frac{r}{N+\eta}]} + C.$$

In the classical case we have $D \sim \|E\|_{L^\infty}^{\frac{1}{2}}$, $r = 2$ and thus we deduce that

$$\|E\|_{L^\infty(]0,T[\times\Omega)} \le \|\nabla_x \Phi_0\|_{L^\infty(]0,T[\times\Omega)} + C(T)\cdot\left(1 + \|E\|_{L^\infty(]0,T[\times\Omega)}^{\frac{1}{2}[\frac{N}{(N+\eta)'} - \frac{2}{N+\eta}]}\right),$$

which gives an L^∞ bound for E as soon as there is $\eta > 0$ such that $0 < \frac{1}{2}[\frac{N}{(N+\eta)'} - \frac{2}{N+\eta}] < 1$, or $N(N+\eta) > N+2$ and $N - 2 < \frac{N+2}{N+\eta}$. This is possible for $N \in \{2, 3\}$. In the relativistic case we have $D \sim \|E\|_{L^\infty}$, $r = 1$ and

$$\|E\|_{L^\infty(]0,T[\times\Omega)} \le \|\nabla_x \Phi_0\|_{L^\infty(]0,T[\times\Omega)} + C(T)\cdot\left(1 + \|E\|_{L^\infty(]0,T[\times\Omega)}^{[\frac{N}{(N+\eta)'} - \frac{1}{N+\eta}]}\right),$$

which gives an L^∞ bound for E if there is $\eta > 0$ such that $0 < \frac{N}{(N+\eta)'} - \frac{1}{N+\eta} < 1$, or $N(N+\eta) > N+1$ and $N - 1 < \frac{N+1}{N+\eta}$. This is possible for $N = 2$. Note that once we have a bound for the L^∞ norm of E we can estimate the L^∞ norm of the charge density $\|\rho\|_{L^\infty} \le \|\rho_1\|_{L^\infty} + \|\rho_2\|_{L^\infty}$. It is sufficient to estimate ρ_1. We have

$$|\rho_1(t,x)| = |q| \cdot \int_{|p|\le 4D} f(t,x,p)\,dp \le C\cdot D^N \cdot \|f\|_{L^\infty} \le C,$$

since $D \sim \|E\|_{L^\infty}^{\frac{1}{2}}$ in the classical case, $D \sim \|E\|_{L^\infty}$ in the relativistic case and E is bounded. Similar computations show that $\partial_t \Phi_s$ belongs to $L^\infty(]0,T[\times\Omega)$. For this we need to assume that the current densities of the initial-boundary conditions belong to L^∞

$$\widetilde{M_1} := \int_{\mathbb{R}^N_p} F_0(|p|)|v(p)|\,dp + \int_{\mathbb{R}^N_p} G(|p|)|v(p)|\,dp < +\infty. \tag{4.16}$$

Note also that in the relativistic case (4.15) implies (4.16). Indeed, by using elliptic regularity results and the continuity equation $\partial_t \rho + \mathrm{div}_x j = 0$ we have

$$\begin{aligned}\|\partial_t \Phi_s(t)\|_{L^\infty(\Omega)} &\le C\cdot\|\partial_t \Phi_s(t)\|_{W^{1,N+\eta}(\Omega)} \le C\cdot\|\partial_t \rho(t)\|_{W^{-1,N+\eta}(\Omega)}\\ &= C\cdot\|\mathrm{div}_x j(t)\|_{W^{-1,N+\eta}(\Omega)} \le C\cdot\|j(t)\|_{L^{N+\eta}(\Omega)}.\end{aligned} \tag{4.17}$$

As before we decompose

$$j(t,x) = j_1 + j_2 = q\int_{\mathbb{R}^N_p} v(p) f(t,x,p)\mathbf{1}_{\{|p|\le 4D\}}\,dp + q\int_{\mathbb{R}^N_p} v(p) f(t,x,p)\mathbf{1}_{\{|p|> 4D\}}\,dp. \tag{4.18}$$

For the first current density we can write

$$|j_1(t,x)| \le |q|\cdot\|f\|_{L^\infty}\cdot\int_{\mathbb{R}^N_p} |v(p)|\mathbf{1}_{\{|p|\le 4D\}}\,dp \le C. \tag{4.19}$$

For the second current density we have

$$\begin{aligned}q^{-1} j_2(t,x) &= \int_{|p|>4D} v(p) f(t,x,p)\,dp\\ &= \int_{|p|>4D} v(p) f_0(X(0;t,x,p), P(0;t,x,p))\cdot\mathbf{1}_{\{s_{\mathrm{in}}(t,x,p)=0\}}\,dp\\ &\quad + \int_{|p|>4D} v(p) g(s_{\mathrm{in}}(t,x,p), X(s_{\mathrm{in}};t,x,p), P(s_{\mathrm{in}};t,x,p))\cdot\mathbf{1}_{\{s_{\mathrm{in}}(t,x,p)>0\}}\,dp.\end{aligned} \tag{4.20}$$

We deduce that

$$|q^{-1} \cdot j_2(t,x)| \leq \int_{|p|>4D} |v(p)| \cdot F_0(|p|-2D)\ dp + \int_{|p|>4D} |v(p)| \cdot G(|p|-2D)\ dp. \tag{4.21}$$

In the classical case $v(p) = \frac{p}{m}$ and therefore we have

$$\begin{aligned} |q^{-1} \cdot j_2(t,x)| &\leq C \int_{4D}^{+\infty} \{F_0(u-2D) + G(u-2D)\} \cdot u^N\ du \\ &= C \int_{2D}^{+\infty} \{F_0(u) + G(u)\} \cdot (u+2D)^N\ du \\ &\leq C \cdot \int_{\mathbb{R}^N_p} \{F_0(|p|) + G(|p|)\} \cdot |p|\ dp = C \cdot \widetilde{M}_1. \end{aligned} \tag{4.22}$$

In the relativistic case we write

$$|q^{-1} \cdot j_2(t,x)| \leq c_0 \cdot \int_{|p|>4D} \{F_0(|p|-2D) + G(|p|-2D)\} \cdot\ dp \leq C \cdot \widetilde{M}_0. \tag{4.23}$$

We deduce from (4.18), (4.19), (4.22), (4.23) that $j \in L^\infty(]0,T[\times\Omega)$. By using now (4.17) we obtain that $\partial_t \Phi_s \in L^\infty(]0,T[\times\Omega)$.

5 The Vlasov-Poisson System

We can prove now the existence of weak solution for the Vlasov-Poisson system.

Theorem 5.1 *Assume that $\Omega \subset \mathbb{R}^N_x$ is open and bounded, with $\partial\Omega$ smooth. We suppose that the initial-boundary conditions verify*

(i) $0 \leq f_0 \in L^\infty(\Omega \times \mathbb{R}^N_p),\ 0 \leq g \in L^\infty(]0,T[\times\Sigma^-)$;

(ii) $$\begin{aligned} M_0 + K_0 + M^- + K^- + V_{s,0} = &\int_\Omega \int_{\mathbb{R}^N_p} (1+\mathcal{E}(p)) f_0\ dxdp \\ &+ \int_0^T \int_{\Sigma^-} |(v(p)\cdot n(x))| \cdot (1+\mathcal{E}(p)) g\ dtd\sigma dp \\ &+ \frac{q}{2} \int_\Omega \int_{\mathbb{R}^N_p} f_0 \Phi_{s,0}(x)\ dxdp < +\infty \end{aligned}$$

(here $\Phi_{s,0}(\cdot)$ is the solution for $-\Delta_x \Phi_{s,0} = \frac{\rho_0(x)}{\varepsilon_0},\ x \in \Omega,\ \Phi_{s,0}(x) = 0,\ x \in \partial\Omega$);

(iii) *$\nabla_x \Phi_0$ belongs to $L^\infty(]0,T[\times\Omega)^N$ (here Φ_0 is the solution of $-\Delta_x \Phi_0(t,x) = 0,\ (t,x) \in]0,T[\times\Omega,\ \Phi_0(t,x) = \varphi_0(t,x),\ (t,x) \in]0,T[\times\partial\Omega$).*

Then there is at least one weak solution $(f, \Phi = \Phi_s + \Phi_0)$ for the Vlasov-Poisson system verifying

$$0 \leq f \leq \max\{\|f_0\|_{L^\infty}, \|g\|_{L^\infty}\}, \quad 0 \leq \gamma^+ f \leq \max\{\|f_0\|_{L^\infty}, \|g\|_{L^\infty}\}, \tag{5.1}$$

$$\begin{aligned} &\operatorname*{ess\,sup}_{0<t<T} \Big\{ \int_\Omega \int_{\mathbb{R}^N_p} (1+\mathcal{E}(p)) f(t,x,p)\ dxdp + \frac{\varepsilon_0}{2} \int_\Omega |\nabla_x \Phi_s(t,x)|^2\ dx \Big\} \\ &+ \int_0^T \int_{\Sigma^+} (v(p)\cdot n(x))(1+\mathcal{E}(p)) \gamma^+ f\ dtd\sigma dp \\ &\leq C(M_0, K_0, V_s(0), M^-, K^-, \|\nabla_x \Phi_0\|_{L^\infty}, T). \end{aligned} \tag{5.2}$$

Moreover, in the classical case with $N \in \{2,3\}$ *or in the relativistic case with* $N=2$ *if there are non-increasing functions* $F_0, G : [0,+\infty[\to [0,+\infty[$ *such that*

(iv) $f_0(x,p) \le F_0(|p|),\ \forall (x,p) \in \Omega \times \mathbb{R}^N_p,\ g(t,x,p) \le G(|p|),\ \forall (t,x,p) \in]0,T[\times\Sigma^-$,

(v) $\widetilde{M_0} = \int_{\mathbb{R}^N_p} F_0(|p|)\,dp + \int_{\mathbb{R}^N_p} G(|p|)\,dp < +\infty$,

then $E \in L^\infty(]0,T[\times\Omega)^N$, $\rho \in L^\infty(]0,T[\times\Omega)$. *If*

(vi) $\partial_t \Phi_0 \in L^\infty(]0,T[\times\Omega)$,

(vii) $\widetilde{M_1} = \int_{\mathbb{R}^N_p} |v(p)| \cdot F_0(|p|)\,dp + \int_{\mathbb{R}^N_p} |v(p)| \cdot G(|p|)\,dp < +\infty$,

then $\partial_t \Phi \in L^\infty(]0,T[\times\Omega)$, $j \in L^\infty(]0,T[\times\Omega)^N$.

Proof We truncate the initial-boundary conditions by taking $f_{0,R} = f_0 \cdot \mathbf{1}_{\{|p|\le R\}}$, $g_R = g \cdot \mathbf{1}_{\{|p|\le R\}}$ and regularize the potential on the boundary so that

$$\varphi_{0,\alpha} \in L^\infty(]0,T[; W^{4m+2-\frac{1}{p_0},p_0}(\partial\Omega)), \quad \partial_t\varphi_{0,\alpha} \in L^\infty(]0,T[; W^{4m+1-\frac{1}{p_0},p_0}(\partial\Omega))$$

(here $p_0 = \frac{2N}{2N-1}$, $p_0' = 2N$, $\frac{1}{p_0} - \frac{4m}{N} < 0$), $\|\nabla_x\Phi_{0,\alpha}\|_{L^\infty} \le \|\nabla_x\Phi_0\|_{L^\infty}$, $\nabla_x\Phi_{0,\alpha} \rightharpoonup \nabla_x\Phi_0$ weakly $*$ in $L^\infty(]0,T[\times\Omega)^N$ as $\alpha \searrow 0$, $\nabla_x\Phi_{0,\alpha} \to \nabla_x\Phi_0$ strongly in $L^p(]0,T[\times\Omega)^N$, $1 \le p < +\infty$ as $\alpha \searrow 0$. We denote by $(f_\alpha, \Phi_\alpha = \Phi_{s,\alpha} + \Phi_{0,\alpha})$ the solution of (3.2) constructed at Proposition 3.3. We have for all $\alpha > 0$

$$\begin{aligned} M_\alpha(t) + K_\alpha(t) + V_{s,\alpha}(t) + &\int_0^t \{M_\alpha^+(s) + K_\alpha^+(s)\}ds \\ \le M_{0,\alpha} + K_{0,\alpha} + V_{s,\alpha}(0) + &\int_0^t \{M_\alpha^-(s) + K_\alpha^-(s)\}ds \\ + C \cdot \|\nabla_x\Phi_0\|_{L^\infty} \cdot &\int_0^t (M_\alpha(s) + K_\alpha(s))^{\frac{1}{\beta}}\,ds, \quad 0 \le t \le T, \end{aligned} \tag{5.3}$$

with $\beta = \frac{N+2}{N+1}$ in the classical case and $\beta = \frac{N+1}{N}$ in the relativistic case. Consider $(\alpha_k)_k$ a sequence such that $\lim_{k\to+\infty} \alpha_k = 0$ and keep $R > 0$ fixed. Obviously we have $M_{0,\alpha_k} \le M_0$, $K_{0,\alpha_k} \le K_0$, $M^-_{\alpha_k}(s) \le M^-(s)$, $K^-_{\alpha_k}(s) \le K^-(s)$, $\forall\, 0 \le s \le T$. Observe that

$$V_{s,\alpha_k}(0) = \frac{1}{2}\int_\Omega \rho_{0,R}(x)\Phi^R_{s,0}(x)dx =: V^R_{s,0},$$

where $-\Delta_x\Phi^R_{s,0} = \frac{\rho_{0,R}(x)}{\varepsilon_0}$, $x \in \Omega$, $\Phi^R_{s,0}(x) = 0$, $x \in \partial\Omega$. Note also that $0 \le q^{-1}\rho_{0,R} \le q^{-1}\rho_0$ and by the maximum principle we have $0 \le q^{-1}\Phi^R_{s,0} \le q^{-1}\Phi_{s,0}$, $x \in \Omega$, where $-\Delta_x\Phi_{s,0} = \frac{\rho_0}{\varepsilon_0}$, $x \in \Omega$, $\Phi_{s,0}(x) = 0$, $x \in \partial\Omega$. Finally one gets

$$\begin{aligned} V_{s,\alpha_k}(0) &= \frac{1}{2}\int_\Omega \rho_{0,R}(x)\Phi^R_{s,0}(x)dx \le \frac{1}{2}\int_\Omega \rho_0(x)\Phi_{s,0}(x)dx \\ &= \frac{1}{2}\int_\Omega\int_{\mathbb{R}^N_p} f_0(x,p)\Phi_{s,0}(x)\,dxdp = V_{s,0} < +\infty, \quad \forall R > 0. \end{aligned} \tag{5.4}$$

From the inequality (5.3) written for $\alpha = \alpha_k$ we deduce that

$$\begin{aligned} &\limsup_{k\to+\infty} \Big\{ \sup_{0\le t\le T} \{M_{\alpha_k}(t) + K_{\alpha_k}(t) + V_{s,\alpha_k}(t)\} + M^+_{\alpha_k} + K^+_{\alpha_k} \Big\} \\ &\le C(M_0, K_0, V_{s,0}, M^-, K^-, \|\nabla_x\Phi_0\|_{L^\infty}, T). \end{aligned} \tag{5.5}$$

Observe also that we have the following estimates: $(\rho_{\alpha_k})_k$ is bounded in $L^\infty(]0,T[;L^\gamma(\Omega))$, $(j_{\alpha_k})_k$ is bounded in $L^\infty(]0,T[;L^\beta(\Omega))$, $(\Phi_{s,\alpha_k})_k$ is bounded in $L^\infty(]0,T[;W^{2,\gamma}(\Omega))$, $(\partial_t\Phi_{s,\alpha_k})_k$ is bounded in $L^\infty(]0,T[;W^{1,\beta}(\Omega))$, with $\gamma = \frac{N+2}{N} > \frac{N+2}{N+1} = \beta$ in the classical case and $\gamma = \frac{N+1}{N} = \beta$ in the relativistic case. After extraction of subsequences if necessary we deduce that

$$f_{\alpha_k} \rightharpoonup f, \qquad \text{weakly} * \text{ in } L^\infty(]0,T[\times\Omega\times\mathbb{R}^N_p),$$
$$\gamma^+ f_{\alpha_k} \rightharpoonup \gamma^+ f, \quad \text{weakly} * \text{ in } L^\infty(]0,T[\times\Sigma^+).$$

By using also a result due to Aubin [29] we can assume that

$$\nabla_x\Phi_{s,\alpha_k} \to \nabla_x\Phi_s, \quad \text{strongly in } L^2(]0,T[;L^\gamma(\Omega)). \tag{5.6}$$

By using the above convergence we can pass easily to the limit for $k \to +\infty$ in the Vlasov equation and we deduce that f is a weak solution for

$$\partial_t f + v(p)\cdot\nabla_x f + q(-\nabla_x\Phi_s - \nabla_x\Phi_0)\cdot\nabla_p f = 0, \quad (t,x,p)\in]0,T[\times\Omega\times\mathbb{R}^N_p,$$
$$f(0,x,p) = f_{0,R}(x,p), \quad (x,p)\in\Omega\times\mathbb{R}^N_p, \quad f(t,x,p) = g_R(t,x,p), \quad (t,x,p)\in]0,T[\times\Sigma^-.$$

Moreover, the trace of f on $]0,T[\times\Sigma^+$ is $\gamma^+ f$. The passing to the limit for $k\to+\infty$ in the regularized Poisson equation follows immediately by observing that $\rho_{\alpha_k} \rightharpoonup \rho = q\int_{\mathbb{R}^N_p} f(t,x,p)\,dp$ weakly in $L^1(]0,T[\times\Omega)$. Indeed, for $R_1>0$, $k\ge 1$ we have

$$\int_0^T\!\!\int_\Omega\int_{|p|>R_1} f_{\alpha_k}\,dtdxdp \le \frac{1}{R_1}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p}|p|\cdot f_{\alpha_k}dtdxdp$$
$$\le \frac{C}{R_1}\int_0^T\!\!\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f_{\alpha_k}dtdxdp \le \frac{C}{R_1},$$

and the weak L^1 convergence of $(\rho_{\alpha_k})_k$ follows from the weak $*$ L^∞ convergence of $(f_k)_k$. The estimates (5.1), (5.2) follow by standard arguments. Note that these estimates are uniform with respect to $R>0$ and thus it is possible to pass to the limit for $R\to+\infty$ in order to solve the Vlasov-Poisson equations with the initial-boundary conditions f_0 and g. The L^∞ bounds for $\nabla_x\Phi, \partial_t\Phi, \rho$ and j follow by using the L^∞ estimates proved in Subsection 4.2 for smooth solutions (f_α,Φ_α) and by passing to the limit for $\alpha\searrow 0$, $R\to+\infty$ weakly $*$ in L^∞.

In the following let us give some immediate properties of the solution constructed above.

Proposition 5.1 *Under the hypotheses* (i)–(v) *of Theorem* 5.1*, the weak solution constructed before satisfies*

(1) *the application* $t\to\int_\Omega\int_{\mathbb{R}^N_p} f\,dxdp$ *is absolutely continuous for* $t\in[0,T]$ *and*

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^N_p} f\,dxdp + \int_{\Sigma^+}(v(p)\cdot n(x))\gamma^+ f\,d\sigma dp$$
$$= \int_{\Sigma^-}|(v(p)\cdot n(x))|g\,d\sigma dp, \quad a.e.\ t\in]0,T[\ ; \tag{5.7}$$

(2) *the application* $t \to \int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f \; dxdp + \frac{\varepsilon_0}{2} \int_\Omega |\nabla_x \Phi_s|^2 \; dx$ *is absolutely continuous for* $t \in [0,T]$ *and*

$$\frac{d}{dt}\Big\{ \int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f \; dxdp + \frac{\varepsilon_0}{2} \int_\Omega |\nabla_x \Phi_s|^2 \; dx \Big\} + \int_{\Sigma^+} (v(p)\cdot n(x)) \mathcal{E}(p) \gamma^+ f \; d\sigma dp$$
$$= - \int_\Omega \nabla_x \Phi_0 \cdot j \; dx + \int_{\Sigma^-} |(v(p)\cdot n(x))| \mathcal{E}(p) g \; d\sigma dp, \quad a.e. \; t \in]0,T[. \tag{5.8}$$

Proof Indeed, recall that the weak solution (f,E) was obtained as $(f,E) = \lim_{R\to+\infty} (f_R, E_R)$ with $(f_R, E_R) = (f_R, -\nabla_x \Phi_s^R - \nabla_x \Phi_0) = \lim_{\alpha \searrow 0} (f_{\alpha,R}, E_{\alpha,R})$, where $(f_{\alpha,R}, E_{\alpha,R})$ is a solution of (3.2) with the initial-boundary conditions $f_{0,R}, g_R, \varphi_{0,\alpha}$ (observe that (f_R, E_R) is the solution of the Vlasov-Poisson system with the initial-boundary conditions $f_{0,R}, g_R, \varphi_0$). For the moment we keep $R > 0$ fixed and write the analogous of (5.7), (5.8) for the smooth solutions $(f_{\alpha,R}, E_{\alpha,R}) = (f_\alpha, E_\alpha)$ which are uniformly compactly supported in momentum with respect to $\alpha > 0$

$$\frac{d}{dt} \int_\Omega \int_{\mathbb{R}^N_p} f_\alpha \; dxdp + \int_{\Sigma^+} (v(p)\cdot n(x)) \gamma^+ f_\alpha \; d\sigma dp$$
$$= \int_{\Sigma^-} |(v(p)\cdot n(x))| g_R \; d\sigma dp, \quad \text{a.e. } t \in]0,T[. \tag{5.9}$$

Similarly the application $t \to \int_\Omega \int_{\mathbb{R}^N_p} \mathcal{E}(p) f_\alpha \; dxdp + \frac{q}{2} \int_\Omega \int_{\mathbb{R}^N_p} f_\alpha(t,x,p) \Phi_{s,\alpha}(t,x) \; dxdp$ is absolutely continuous for $t \in [0,T]$ and

$$\frac{d}{dt}\Big\{ \int_\Omega \int_{\mathbb{R}^N_p} \Big(\mathcal{E}(p) + \frac{q}{2} \Phi_{s,\alpha}(t,x)\Big) f_\alpha \; dxdp \Big\} + \int_{\Sigma^+} (v(p)\cdot n(x)) \mathcal{E}(p) \gamma^+ f_\alpha \; d\sigma dp$$
$$= - \int_\Omega \nabla_x \Phi_{0,\alpha} \cdot j_\alpha \; dx + \int_{\Sigma^-} |(v(p)\cdot n(x))| \mathcal{E}(p) g_R \; d\sigma dp, \quad \text{a.e. } t \in]0,T[. \tag{5.10}$$

By passing to the limit for $\alpha \searrow 0$ in (5.9) we deduce that

$$\frac{d}{dt} \int_\Omega \int_{\mathbb{R}^N_p} f_R \; dxdp + \int_{\Sigma^+} (v(p)\cdot n(x)) \gamma^+ f_R \; d\sigma dp$$
$$= \int_{\Sigma^-} |(v(p)\cdot n(x))| g_R \; d\sigma dp, \quad \text{a.e. } t \in]0,T[. \tag{5.11}$$

The passing to the limit for $\alpha \searrow 0$ in (5.10) is a little more complicated. For $\theta \in \mathcal{D}([0,T[)$ we have

$$- \theta(0) \int_\Omega \int_{\mathbb{R}^N_p} \Big(\mathcal{E}(p) + \frac{q}{2} \Phi^R_{s,0}(x)\Big) f_{0,R} \; dxdp - \int_0^T \int_\Omega \int_{\mathbb{R}^N_p} \theta'(t) \Big(\mathcal{E}(p) + \frac{q}{2} \Phi_{s,\alpha}(t,x)\Big) f_\alpha dtdxdp$$
$$+ \int_0^T \int_{\Sigma^+} \theta(t) (v(p)\cdot n(x)) \mathcal{E}(p) \gamma^+ f_\alpha \; dtd\sigma dp$$
$$= \int_0^T \int_{\Sigma^-} \theta(t) |(v(p)\cdot n(x))| \mathcal{E}(p) g_R \; dtd\sigma dp - \int_0^T \int_\Omega \theta(t) \nabla_x \Phi_{0,\alpha} \cdot j_\alpha(t,x) \; dtdx. \tag{5.12}$$

Since $(f_\alpha)_{\alpha>0}$ are uniformly compactly supported in momentum we deduce also that

$$\lim_{\alpha\searrow 0}\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\theta'(t)\mathcal{E}(p)f_\alpha dtdxdp=\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\theta'(t)\mathcal{E}(p)f_R dtdxdp,$$

$$\lim_{\alpha\searrow 0}\int_0^T\int_{\Sigma^+}\theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_\alpha\, dtd\sigma dp=\int_0^T\int_{\Sigma^+}\theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_R\, dtd\sigma dp.$$

In order to pass to the limit in the term $\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\theta(t)\nabla_x\Phi_{0,\alpha}\cdot j_\alpha dtdxdp$ we can combine the weak convergence $j_\alpha \rightharpoonup j_R$ weakly in $L^1(]0,T[\times\Omega)^N$, the uniform bound of j_α in $L^\infty(]0,T[;L^\beta(\Omega))^N$ and the strong convergence $\nabla_x\Phi_{0,\alpha}\to\nabla_x\Phi_0$ strongly in $L^r(]0,T[\times\Omega)^N$, $\forall\, 1<r<+\infty$ (for example $r=\beta'$). In order to pass to the limit in the term

$$\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\theta'(t)\Phi_{s,\alpha}qf_\alpha dtdxdp=\int_0^T\int_\Omega\theta'(t)\Phi_{s,\alpha}\rho_\alpha dtdx,$$

combine the weak convergence $\rho_\alpha \rightharpoonup \rho_R$ in $L^1(]0,T[\times\Omega)$, the uniform bounds of ρ_α,Φ_α in $L^\infty(]0,T[\times\Omega)$ and the strong convergence $\Phi_{s,\alpha}\to\Phi_s^R$ in $L^2(]0,T[;W^{1,\gamma}(\Omega))$. After passing to the limit in (5.12) we deduce that

$$\begin{aligned}&-\theta(0)\int_\Omega\int_{\mathbb{R}^N_p}\Big(\mathcal{E}(p)+\frac{q}{2}\Phi^R_{s,0}(x)\Big)f_{0,R}\,dxdp-\int_0^T\int_\Omega\int_{\mathbb{R}^N_p}\theta'(t)\Big(\mathcal{E}(p)+\frac{q}{2}\Phi^R_s(t,x)\Big)f_R dtdxdp\\&+\int_0^T\int_{\Sigma^+}\theta(t)(v(p)\cdot n(x))\mathcal{E}(p)\gamma^+ f_R\,dtd\sigma dp\\&=\int_0^T\int_{\Sigma^-}\theta(t)|(v(p)\cdot n(x))|\mathcal{E}(p)g_R\,dtd\sigma dp-\int_0^T\int_\Omega\theta(t)\nabla_x\Phi_0\cdot j_R(t,x)\,dtdx.\end{aligned}\tag{5.13}$$

In order to prove (5.7), (5.8) we need to pass to the limit for $R\to+\infty$ in (5.11), (5.13). The proof is similar and is left to the reader. Note that $(f_R)_{R>0}$ is not anymore uniformly compactly supported in momentum but we can prove that

$$\sup_{0\le t\le T}\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f_R\cdot\mathbf{1}_{\{|p|>R_1\}}\,dxdp\to 0,\quad\text{as } R_1\to+\infty,\tag{5.14}$$

$$\int_0^T\int_{\Sigma^+}(v(p)\cdot n(x))(1+\mathcal{E}(p))\gamma^+ f_R\cdot\mathbf{1}_{\{|p|>R_1\}}\,dtd\sigma dp\to 0,\quad\text{as } R_1\to+\infty\tag{5.15}$$

uniformly with respect to the solution f_R. For this take $\chi\in C_c^\infty([0,+\infty[)$, $0\le\chi\le 1$, $\chi(u)=1, 0\le u\le\frac{1}{2}$, $\chi(u)=0, u\ge 1$ and multiply the Vlasov equation by $(1-\chi_{R_1}(|p|))\cdot(1+\mathcal{E}(p))$, where $\chi_{R_1}(\cdot)=\chi(\cdot/R_1)$. After easy computations (involving the L^∞ bound for the electric field E_R) we find (5.14), (5.15) which implies that

$$\lim_{R\to+\infty}(1+\mathcal{E}(p))f_R=(1+\mathcal{E}(p))f,\qquad\text{weakly in } L^1(]0,T[\times\Omega\times\mathbb{R}^N_p),$$

$$\lim_{R\to+\infty}(v(p)\cdot n(x))(1+\mathcal{E}(p))f_R=(v(p)\cdot n(x))(1+\mathcal{E}(p))f,\quad\text{weakly in } L^1(]0,T[\times\Sigma^+).$$

The passing to the limit for $R\to+\infty$ in (5.11), (5.13) follows now easily by using the above weak convergence. Observe also that by passing to the L^∞ weak $*$ limit $f=\lim_{R\to+\infty}f_R$ we have

$$\lim_{R_1\to+\infty}\operatorname*{ess\,sup}_{0<t<T}\int_\Omega\int_{\mathbb{R}^N_p}(1+\mathcal{E}(p))f\cdot\mathbf{1}_{\{|p|>R_1\}}\,dxdp=0.\tag{5.16}$$

Another direct consequence of Theorem 5.1 is the propagation of the moments.

Proposition 5.2 *Under the hypotheses* (i)–(v) *of Theorem* 5.1 *with* $1 \leq N \leq 3$ *in the classical case and* $1 \leq N \leq 2$ *in the relativistic case denote by* $(f, E = -\nabla_x \Phi_s - \nabla_x \Phi_0)$ *the solution constructed previously. Suppose also that for some* m *such that* $m > 2$ *in the classical case and* $m > 1$ *in the relativistic case the initial-boundary conditions verify*

$$\int_\Omega \int_{\mathbb{R}_p^N} |p|^m \cdot f_0(x,p)\, dxdp + \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))| \cdot |p|^m \cdot g(t,x,p)\, dtd\sigma dp < +\infty. \tag{5.17}$$

Then we have

$$\Big\| \int_\Omega \int_{\mathbb{R}_p^N} |p|^m \cdot f(\cdot,x,p)\, dxdp \Big\|_{L^\infty(]0,T[)} + \int_0^T \int_{\Sigma^+} (v(p) \cdot n(x)) \cdot |p|^m \cdot \gamma^+ f(t,x,p)\, dtd\sigma dp < +\infty. \tag{5.18}$$

Proof It is sufficient to prove (5.18) for smooth solutions. The conclusion follows easily by observing that for $r = m, m-1, \cdots$ we have

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \int_{\mathbb{R}_p^N} |p|^r \cdot f_\alpha(t,x,p)\, dxdp + \int_\Sigma (v(p) \cdot n(x)) \cdot |p|^r \cdot \gamma f_\alpha\, d\sigma dp \\
&= \int_\Omega \int_{\mathbb{R}_p^N} q \cdot f_\alpha(t,x,p) \cdot r \cdot |p|^{r-2} (E_\alpha(t,x) \cdot p)\, dxdp \\
&\leq |q| \cdot r \cdot \|E_\alpha\|_{L^\infty} \int_\Omega \int_{\mathbb{R}_p^N} |p|^{r-1} \cdot f_\alpha(t,x,p)\, dxdp.
\end{aligned}$$

Proposition 5.3 *Under the hypotheses* (i)–(v) *of Theorem* 5.1 *with* $1 \leq N \leq 3$ *in the classical case and* $1 \leq N \leq 2$ *in the relativistic case we suppose also that for some* $m > 0$ *we have*

$$\widetilde{M}_m := \int_{\mathbb{R}_p^N} |p|^m \cdot F_0(|p|)\, dp + \int_{\mathbb{R}_p^N} |p|^m \cdot G(|p|)\, dp < +\infty. \tag{5.19}$$

Then we have

$$\Big\| \int_{\mathbb{R}_p^N} |p|^m \cdot f(\,\cdot\,,\,\cdot\,,p)\, dp \Big\|_{L^\infty(]0,T[\times\Omega)} + \Big\| \int_{\mathbb{R}_p^N} |p|^m \cdot \gamma f(\,\cdot\,,\,\cdot\,,p)\, dp \Big\|_{L^\infty(]0,T[\times\partial\Omega)} < +\infty. \tag{5.20}$$

Proof Write

$$\int_{\mathbb{R}_p^N} |p|^m \cdot f(t,x,p)\, dp = \int_{|p| \leq 4D} \{\cdots\, dp\} + \int_{|p| > 4D} \{\cdots\, dp\}$$

and continue as it was done for the cases $m = 0$, $m = 1$.

5.1 The time periodic case

We end this paper by considering permanent regimes. We assume that the boundary data g, φ_0 are T periodic and under natural hypotheses we construct weak solutions for the Vlasov-Poisson system with bounded electric field. We consider the classical case. First of all let us deduce bounds for the total mass and energy by performing formal computations (for more details see [27]). We assume that the boundary conditions verify

$$0 \le g \in L^\infty(\mathbb{R}_t \times \Sigma^-), \quad \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))|(1 + \mathcal{E}(p))g(t,x,p)\; dt d\sigma dp < +\infty,$$
$$\varphi_0 \in L^2(]0,T[; H^1(\partial\Omega)), \quad \nabla_x \Phi_0 \in L^\infty(\mathbb{R}_t \times \Omega),$$

where Φ_0 is the exterior potential $(-\Delta_x \Phi_0 = 0,\ (t,x) \in \mathbb{R}_t \times \Omega,\ \Phi_0 = \varphi_0,\ (t,x) \in \mathbb{R}_t \times \partial\Omega)$. Consider $(f, \Phi = \Phi_s + \Phi_0)$ a T periodic smooth solution with compact support in momentum. The conservations of the mass and kinetic energy give

$$\frac{d}{dt}\int_\Omega \int_{\mathbb{R}_p^N} f(t,x,p)\; dxdp + \int_\Sigma (v(p) \cdot n(x))\gamma f(t,x,p)\; d\sigma dp = 0, \quad t \in \mathbb{R}_t, \tag{5.21}$$

$$\frac{d}{dt}\int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}(p) f(t,x,p)\; dxdp + \int_\Sigma (v(p) \cdot n(x))\mathcal{E}(p)\gamma f(t,x,p)\; d\sigma dp$$
$$= \int_\Omega \int_{\mathbb{R}_p^N} q(E(t,x) \cdot v(p)) f\; dxdp = -\int_\Omega j(t,x) \cdot (\nabla_x \Phi_s + \nabla_x \Phi_0)\; dx, \quad t \in \mathbb{R}_t. \tag{5.22}$$

After multiplying the Vlasov equation by $q\Phi_s$ and by using the Poisson equation we find as before

$$\frac{d}{dt}\Big\{ \int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}(p) f(t,x,p)\; dxdp + \frac{1}{2}\int_\Omega \rho(t,x)\Phi_s(t,x)\; dx \Big\}$$
$$+ \int_\Sigma (v(p) \cdot n(x))\mathcal{E}(p)\gamma f(t,x,p)\; d\sigma dp$$
$$= -\int_\Omega j(t,x) \cdot \nabla_x \Phi_0\; dx. \tag{5.23}$$

After integration on $]0,T[$ we deduce that

$$\int_0^T \int_{\Sigma^+} (v(p) \cdot n(x))\gamma^+ f(t,x,p)\; dt d\sigma dp = \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))| g(t,x,p)\; dt d\sigma dp, \tag{5.24}$$

$$\int_0^T \int_{\Sigma^+} (v(p) \cdot n(x))\mathcal{E}(p)\gamma^+ f\; dt d\sigma dp = \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))|\mathcal{E}(p) g\; dt d\sigma dp$$
$$- \int_0^T \int_\Omega j(t,x) \cdot \nabla_x \Phi_0\; dt dx. \tag{5.25}$$

We multiply the Vlasov equation by $(p \cdot x)$ and we suppose that $\partial\Omega$ is strictly star-shaped with respect to $0 \in \Omega$, i.e., $\exists\, r > 0$ such that $r \le (n(x) \cdot x),\ \forall\, x \in \partial\Omega$. We obtain

$$\frac{d}{dt}\int_\Omega \int_{\mathbb{R}_p^N} (p \cdot x) f\; dxdp + \int_\Sigma (v(p) \cdot n(x))\ (p \cdot x)\gamma f\; d\sigma dp$$
$$= \int_\Omega \int_{\mathbb{R}_p^N} (v(p) \cdot p) f\; dxdp + \int_\Omega \int_{\mathbb{R}_p^N} q(E \cdot x) f\; dxdp$$
$$= \int_\Omega \int_{\mathbb{R}_p^N} (v(p) \cdot p) f\; dxdp + \int_\Omega \rho(E \cdot x)\; dx. \tag{5.26}$$

We use the identity

$$E_i \text{div } E = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(E_i E_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|E|^2, \quad \forall 1 \le i \le N, \tag{5.27}$$

if $\frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i}$, $\forall 1 \le i, j \le N$. After integration by parts and by using the decomposition $E = (E \cdot n)n + E_\tau$, $(t,x) \in \mathbb{R}_t \times \partial\Omega$ we find

$$\begin{aligned}\int_\Omega (E \cdot x)\text{div } E \, dx = & \int_\Omega \sum_{i=1}^{N} x_i \Big\{ \sum_{j=1}^{N} \frac{\partial}{\partial x_j}(E_i E_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|E|^2 \Big\} \, dx \\ = & \Big(\frac{N}{2} - 1\Big) \int_\Omega |E(t,x)|^2 \, dx + \frac{1}{2}\int_{\partial\Omega} (n(x) \cdot x)(E \cdot n)^2 \, d\sigma \\ & + \int_{\partial\Omega} (E_\tau \cdot x) \cdot (E \cdot n(x)) \, d\sigma - \frac{1}{2}\int_{\partial\Omega} (n(x) \cdot x) \cdot |E_\tau|^2 \, d\sigma. \end{aligned} \tag{5.28}$$

By using (5.26), (5.28) we deduce that

$$\begin{aligned}&\int_0^T \int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}(p) f dt dx dp + \varepsilon_0 \Big(\frac{N}{2} - 1\Big) \int_0^T \int_\Omega |E|^2 \, dt dx + \frac{\varepsilon_0 r}{2} \int_0^T \int_{\partial\Omega} (E \cdot n(x))^2 \, dt d\sigma \\ &\le \int_0^T \int_\Sigma (v(p) \cdot n(x))(p \cdot x)\gamma f \, dt d\sigma dp + \frac{\varepsilon_0}{2} \int_0^T \int_{\partial\Omega} (n(x) \cdot x) \cdot |E_\tau|^2 \, dt d\sigma \\ &- \varepsilon_0 \int_0^T \int_{\partial\Omega} (E_\tau \cdot x) \cdot (E \cdot n(x)) \, dt d\sigma. \end{aligned} \tag{5.29}$$

Observe that $\|E_\tau\|_{L^2(]0,T[\times\partial\Omega)} \le C \cdot \|\varphi_0\|_{L^2(]0,T[;H^1(\partial\Omega))}$ and from (5.24), (5.25) note that

$$\begin{aligned}\Big| \int_0^T \int_\Sigma (v(p) \cdot n(x))(p \cdot x)\gamma f \, dt d\sigma dp \Big| &\le C \cdot \int_0^T \int_\Sigma |(v(p) \cdot n(x))|(1 + \mathcal{E}(p))\gamma f \, dt d\sigma dp \\ &\le C \cdot \int_0^T \int_{\Sigma^-} |(v(p) \cdot n(x))|(1 + \mathcal{E}(p)) g \, dt d\sigma dp \\ &\quad + C \cdot \|\nabla_x \Phi_0\|_{L^\infty} \cdot \int_0^T \int_\Omega |j(t,x)| \, dt dx. \end{aligned} \tag{5.30}$$

By using interpolation inequalities and (5.29), (5.30) we obtain bounds for

$$\begin{aligned}&\int_0^T \int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}(p) f dt dx dp + \frac{\varepsilon_0}{2} \int_0^T \int_\Omega |E|^2 \, dt dx \\ &+ \int_0^T \int_{\Sigma^+} (v(p) \cdot n(x))(1 + \mathcal{E}(p))\gamma^+ f \, dt d\sigma dp + \frac{\varepsilon_0}{2} \int_0^T \int_{\partial\Omega} (E \cdot n)^2 \, dt d\sigma \le C \end{aligned}$$

for the case $N > 2$. In the case $N = 2$ we obtain bounds only for

$$\begin{aligned} W = &\int_0^T \int_\Omega \int_{\mathbb{R}_p^N} \mathcal{E}(p) f dt dx dp + \frac{\varepsilon_0}{2} \int_0^T \int_{\partial\Omega} (E \cdot n)^2 \, dt d\sigma \\ &+ \int_0^T \int_{\Sigma^+} (v(p) \cdot n(x))(1 + \mathcal{E}(p))\gamma^+ f \, dt d\sigma dp. \end{aligned}$$

By interpolation inequalities we have $\|\rho\|_{L^2(]0,T[\times\Omega)} \le C$ and therefore

$$\int_0^T\int_\Omega |\nabla_x \Phi_s|^2 \, dtdx \le C \cdot \int_0^T\int_\Omega \rho^2 \, dtdx \le C.$$

In fact the total energy is uniformly bounded in time. Indeed, since $\int_0^T \{K(t) + V_s(t)\}\, dt \le C$, there is t_0 such that $K(t_0)+V_s(t_0) \le \frac{C}{T}$ and we can propagate the total energy for $t \in [t_0, t_0+T]$. Suppose also that there is non-increasing $G : [0, +\infty[\to [0, +\infty[$ such that

$$g(t, x, p) \le G(|p|), \quad \forall\, (t, x, p) \in \mathbb{R}_t \times \Sigma^-,$$
$$\widetilde{M}^- := \int_{\mathbb{R}_p^N} G(|p|)\, dp < +\infty.$$

By using the method presented at Subsection 4.2 we deduce a bound for the L^∞ norm of the electric field and the charge density in the cases $N \in \{2, 3\}$. The one dimensional case was studied in [26]. In this case, we write

$$\|E_s(t)\|_{L^\infty} \le C \cdot \|\rho(t)\|_{L^1} \le C \cdot \|\rho_1(t)\|_{L^1} + C \cdot \|\rho_2(t)\|_{L^1}, \tag{5.31}$$

where $\rho_1(t, x) = q \cdot \int_{|p|\le 4D} f(t, x, p)\, dp$ and $\rho_2(t, x) = q \cdot \int_{|p|>4D} f(t, x, p)\, dp$. For the first charge density we have

$$\|\rho_1(t)\|_{L^1} \le C \cdot \|\rho_1(t)\|_{L^\infty} \le C \cdot \|f\|_{L^\infty} \cdot D \le C \cdot \|E\|_{L^\infty}^{\frac{1}{2}}, \tag{5.32}$$

and for the second charge density we have as usual

$$\|\rho_2(t)\|_{L^1} \le C \cdot \|\rho_2(t)\|_{L^\infty} \le C \cdot \int_{\mathbb{R}_p} G(|p|)\, dp. \tag{5.33}$$

From (5.31)–(5.33) we obtain a bound for the L^∞ norm of E and ρ.

A direct consequence of the L^∞ bound for the electric field is the existence of weak solution for the time periodic Vlasov-Poisson system with particle distribution compactly supported in momentum, when the boundary condition has compact support in momentum, i.e., $\exists R > 0$ such that $g = g \cdot \mathbf{1}_{\{|p|\le R\}}$ (cf. Theorem 2.4).

6 Appendix

We give here the proof of momentum change lemmas for the classical and relativistic cases.

6.1 The classical case

We will need the following easy lemma.

Lemma 6.1 *Consider the quadratic function* $F : \mathbb{R} \to \mathbb{R}$ *given by* $F(s) = \frac{1}{2}a(s - s_1)^2 - b(s - s_1) + c$, *with* $a, b, c > 0$, $\Delta = b^2 - 2ac > 0$ *and* $s_1 \le s_2$ *such that* $F(s) \ge 0$, $\forall\, s_1 \le s \le s_2$. *Then we have* $s_2 - s_1 \le \frac{b-\sqrt{\Delta}}{a} \le \frac{2c}{b}$.

Proof Without loss of generality, we can suppose that $s_1 = 0$. The equation $F(s) = 0$ has two positive real roots $r_{1,2} = \frac{b\mp\sqrt{\Delta}}{a}$, $0 < r_1 < r_2$. Since $a > 0$ we have $F(s) < 0$, $\forall\, r_1 < s < r_2$.

Suppose that $s_2 > r_1$ and consider $s_0 \in [0, s_2] \cap]r_1, r_2[\neq \emptyset$. Thus, since $0 \le s_0 \le s_2$ by the hypothesis we have $F(s_0) \ge 0$. On the other hand, since $r_1 < s_0 < r_2$ we have $F(s_0) < 0$. Therefore $s_2 > r_1$ is not possible and we get that $s_2 \le r_1 = \frac{b-\sqrt{\Delta}}{a} \le \frac{2c}{b}$.

Remark 6.1 If $a = 0$, we still have the inequalities $s_2 - s_1 \le \frac{c}{b} < \frac{2c}{b}$.

Corollary 6.1 *Consider the function* $F_1 : \mathbb{R} \to \mathbb{R}$ *given by* $F_1(s) = \frac{1}{2}a(s-t)^2 - b|s-t| + c$ *with* $a \ge 0, b, c > 0$, $\Delta = b^2 - 2ac > 0$ *and* $s_1 \le t \le s_2$ *such that* $F_1(s) \ge 0$, $\forall s_1 \le s \le s_2$. *Then we have* $\max\{t - s_1, s_2 - t\} \le \frac{2c}{b}$ *and* $s_2 - s_1 \le \frac{4c}{b}$.

Proof Consider $F(r) = \frac{1}{2}ar^2 - br + c$. Observe that $F(r) \ge 0$, $\forall\, 0 \le r \le \max\{t - s_1, s_2 - t\}$. The conclusion follows by applying Lemma 6.1.

Proof of Lemma 2.1 (1) Let us consider for $s_{\text{in}} \le s \le s_{\text{out}}$

$$M(s) = \frac{q}{m}\begin{pmatrix} 0 & B_3(s, X(s)) & -B_2(s, X(s)) \\ -B_3(s, X(s)) & 0 & B_1(s, X(s)) \\ B_2(s, X(s)) & -B_1(s, X(s)) & 0 \end{pmatrix}.$$

We have

$$\|M(s)\| = \sup_{p \in \mathbb{R}^3_p} \frac{|M(s) \cdot p|}{|p|} = \frac{|q|}{m} \sup_{p \in \mathbb{R}^3_p} \frac{|p \wedge B(s)|}{|p|} \le \frac{|q|}{m}\|B\|_\infty, \quad \forall s_{\text{in}} \le s \le s_{\text{out}}.$$

Denote by $R(s;t)$ the resolvent for $\frac{\partial R}{\partial s}(s;t) = M(s)R(s;t)$ with $R(s = t; t) = I$. Since $M(s)$ is antisymmetric we have $\|R(s;t)\| = 1$, $\forall s_{\text{in}} \le s \le s_{\text{out}}$ (in fact $R(s;t)$ is orthogonal) and therefore we have

$$\|R(s;t) - I\| \le |s-t| \cdot \|M(\cdot)\|_\infty \le |s-t| \cdot \frac{|q|}{m} \cdot \|B\|_\infty.$$

By (2.7) we have $P(s) = R(s;t)P(t) + q\int_t^s R(s;\tau)E(\tau, X(\tau))d\tau$, $\forall s_{\text{in}} \le s \le s_{\text{out}}$, and therefore we obtain

$$|P(s) - P(t)| \le |s-t| \cdot \frac{|q|}{m} \cdot \|B\|_\infty \cdot |P(t)| + |q| \cdot |s-t| \cdot \|E\|_\infty. \tag{6.1}$$

We use now the equation $\frac{dX}{ds} = \frac{P(s)}{m}$ and (6.1) to obtain

$$\begin{aligned} \text{diam}(\Omega) &\ge \Big|\Big(X(s) - X(t), \frac{P(t)}{|P(t)|}\Big)\Big| = \Big|\int_t^s \Big(\frac{P(\tau)}{m}, \frac{P(t)}{|P(t)|}\Big)d\tau\Big| \\ &\ge \Big|\int_t^s \frac{|P(t)|}{m}d\tau\Big| - \Big|\int_t^s \Big(\frac{P(\tau) - P(t)}{m}, \frac{P(t)}{|P(t)|}\Big)d\tau\Big| \\ &\ge \frac{1}{m}|s-t| \cdot |P(t)| - \frac{1}{2m} \cdot |s-t|^2\Big(\frac{|q|}{m} \cdot \|B\|_\infty \cdot |P(t)| + |q| \cdot \|E\|_\infty\Big). \end{aligned} \tag{6.2}$$

Denote by $F_1 : \mathbb{R} \to \mathbb{R}$ the function given by

$$F_1(s) = \frac{1}{2}|s-t|^2\Big(\frac{|q|}{m} \cdot \|B\|_\infty \cdot |P(t)| + |q| \cdot \|E\|_\infty\Big) - |s-t| \cdot |P(t)| + m \cdot \text{diam}(\Omega).$$

The discriminant is

$$\begin{aligned} \Delta &= |P(t)|^2 - 2 \cdot \Big(\frac{|q|}{m} \cdot \|B\|_\infty \cdot |P(t)| + |q| \cdot \|E\|_\infty\Big) \cdot m \cdot \text{diam}(\Omega) \\ &= (|P(t)| - |q| \cdot \|B\|_\infty \cdot \text{diam}(\Omega))^2 \\ &\quad - (|q|^2 \cdot \|B\|_\infty^2 \cdot \text{diam}(\Omega)^2 + 2|q| \cdot \|E\|_\infty \cdot m \cdot \text{diam}(\Omega)) > 0, \end{aligned}$$

since $|P(t)| > D_{\text{cla}}$. By (6.2) we have that $F_1(s) \geq 0,\ \forall\, s_{\text{in}} \leq s \leq s_{\text{out}}$ and thus by applying Corollary 6.1 we deduce that $\max\{t - s_{\text{in}}, s_{\text{out}} - t\} \leq 2 \cdot m \cdot \frac{\text{diam}(\Omega)}{|P(t)|}$ and $s_{\text{out}} - s_{\text{in}} \leq 4 \cdot m \cdot \frac{\text{diam}(\Omega)}{|P(t)|} \leq 4 \cdot m \cdot \frac{\text{diam}(\Omega)}{D_{\text{cla}}}$. Using one more time (6.1), we deduce that for all $s_{\text{in}} \leq s \leq s_{\text{out}}$

$$\begin{aligned}
|P(s) - P(t)| &\leq |s - t| \Big(\frac{|q|}{m} \cdot \|B\|_\infty \cdot |P(t)| + |q| \cdot \|E\|_\infty \Big) \\
&\leq \frac{2 \cdot m \cdot \text{diam}(\Omega)}{|P(t)|} \Big(\frac{|q|}{m} \cdot \|B\|_\infty \cdot |P(t)| + |q| \cdot \|E\|_\infty \Big) \\
&\leq 2|q| \cdot \|B\|_\infty \cdot \text{diam}(\Omega) + \frac{2}{D_{\text{cla}}} \cdot |q| \cdot \|E\|_\infty \cdot m \cdot \text{diam}(\Omega) \\
&\leq D_{\text{cla}}.
\end{aligned}$$

We deduce that $|P(s_1) - P(s_2)| \leq 2D_{\text{cla}},\ \forall\, s_{\text{in}} \leq s_1 \leq s_2 \leq s_{\text{out}}$.

(2) If $|P(s_1)| \leq D_{\text{cla}}$ and $|P(s_2)| \leq D_{\text{cla}}$ we have $|P(s_1) - P(s_2)| \leq 2D_{\text{cla}}$. If $|P(s_1)| > D_{\text{cla}}$, by applying the previous point for $t = s_1$ we deduce that $|P(s_2) - P(s_1)| \leq D_{\text{cla}} \leq 2D_{\text{cla}},\ \forall\, s_2$. If $|P(s_2)| > D_{\text{cla}}$ we apply the previous point with $t = s_2$.

6.2 The relativistic case

Let us establish some preliminary properties concerning the function $v(p),\ p \in \mathbb{R}^N_p$.

Lemma 6.2 *Consider* $v : \mathbb{R}^N_p \to \mathbb{R}^N$ *given by* $v(p) = \frac{p}{m} \cdot \Big(1 + \frac{|p|^2}{(mc_0)^2}\Big)^{-1/2}$. *Then we have*

(1) $|v(p)| \leq c_0,\ \forall\, p \in \mathbb{R}^N_p$;

(2) $(v(p_1) - v(p_2), p_1 - p_2) > 0,\ \forall\, p_1 \neq p_2$;

(3) $|v(p_1) - v(p_2)|^2 \leq \dfrac{N}{m^2}|p_1 - p_2|^2 \displaystyle\int_0^1 \Big(1 + \frac{|tp_1 + (1-t)p_2|^2}{(mc_0)^2}\Big)^{-1} dt, \quad \forall\, p_1, p_2 \in \mathbb{R}^N_p$;

(4) $|v(p_1) - v(p_2)| \leq \dfrac{2\sqrt{N}}{m} \cdot |p_1 - p_2| \cdot \Big(1 + \dfrac{|p_1|^2}{(mc_0)^2}\Big)^{-1/2}, \quad$ *if* $\ |p_1 - p_2| \leq \dfrac{|p_1|}{2},\ \forall\, p_1, p_2 \in \mathbb{R}^N_p$.

Proof (1) is obvious. For the point (2) consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(u) = mc_0^2\big(1 + \frac{u^2}{(m^2c_0^2)^{1/2}} - 1\big)$ and check that φ is strictly convex on $\mathbb{R}$ and strictly increasing on $[0, +\infty[$. We deduce that $\mathcal{E}(p)$ is strictly convex on $\mathbb{R}^N_p$. Indeed, for $\lambda \in]0, 1[$ we have

$$\begin{aligned}
\mathcal{E}(\lambda p_1 + (1-\lambda)p_2) &= \varphi(|\lambda p_1 + (1-\lambda)p_2|) \\
&\leq \varphi(\lambda|p_1| + (1-\lambda)|p_2|) \leq \lambda\varphi(|p_1|) + (1-\lambda)\varphi(|p_2|) \\
&\leq \lambda\mathcal{E}(p_1) + (1-\lambda)\mathcal{E}(p_2),
\end{aligned}$$

with equality iff $|\lambda p_1 + (1-\lambda)p_2| = \lambda|p_1| + (1-\lambda)|p_2|$ and $|p_1| = |p_2|$, which means iff $p_1 = p_2$. Therefore we have for $p_1 \neq p_2$ that

$$(\nabla_p \mathcal{E}(p_1) - \nabla_p \mathcal{E}(p_2), p_1 - p_2) > 0 \quad \text{or} \quad (v(p_1) - v(p_2), p_1 - p_2) > 0.$$

The point (3) follows by direct computation by writing

$$v(p_1) - v(p_2) = \int_0^1 \nabla_p v(tp_1 + (1-t)p_2) \cdot (p_1 - p_2) dt.$$

For (4) we write

$$|tp_1 + (1-t)p_2| \geq |p_1| - (1-t)|p_1 - p_2| \geq |p_1| - |p_1 - p_2| \geq \frac{|p_1|}{2}, \quad \forall\, t \in [0,1],$$

and the conclusion follows by (3).

Proof of Lemma 2.2 We have $P(s) = P(t) + q\int_t^s E(\tau, X(\tau))d\tau$ and we deduce that

$$|P(s) - P(t)| \leq |q| \cdot |s-t| \cdot \|E\|_\infty \leq \frac{|P(t)|}{2}, \quad s_{\text{in}} \leq s \leq s_{\text{out}},\ |s-t| \leq \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}.$$

Note that if $\|E\|_\infty = 0$ the above inequality holds $\forall\, s \in [s_{\text{in}}, s_{\text{out}}]$. By Lemma 6.2 we have

$$\begin{aligned} |v(P(s)) - v(P(t))| &\leq \frac{2\sqrt{N}}{m} \cdot |P(s) - P(t)| \cdot \Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{-1/2} \\ &\leq \frac{2\sqrt{N}}{m} \cdot |q| \cdot \|E\|_\infty \cdot |s-t| \cdot \Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{-1/2}, \quad \forall\, r_1 \leq s \leq r_2, \end{aligned} \tag{6.3}$$

where

$$r_1 = \max\Big\{s_{\text{in}}, t - \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}\Big\}, \quad r_2 = \min\Big\{s_{\text{out}}, t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}\Big\},$$

if $\|E\|_\infty > 0$ and $r_1 = s_{\text{in}},\ r_2 = s_{\text{out}}$ if $\|E\|_\infty = 0$. By using the equation $\frac{dX}{ds} = v(P(s))$ and (6.3) we find for $r_1 \leq s \leq r_2$ that

$$\begin{aligned} \text{diam}(\Omega) &\geq \Big|\Big(X(s) - X(t), \frac{v(P(t))}{|v(P(t))|}\Big)\Big| = \Big|\int_t^s \Big(v(P(\tau)), \frac{v(P(t))}{|v(P(t))|}\Big)d\tau\Big| \\ &\geq \Big|\int_t^s \Big(v(P(t)), \frac{v(P(t))}{|v(P(t))|}\Big)d\tau\Big| - \Big|\int_t^s \Big(v(P(\tau)) - v(P(t)), \frac{v(P(t))}{|v(P(t))|}\Big)d\tau\Big| \\ &\geq |s-t| \cdot |v(P(t))| - \Big|\int_t^s |v(P(\tau)) - v(P(t))| d\tau\Big| \\ &\geq |s-t| \cdot |v(P(t))| - \frac{\sqrt{N} \cdot |q| \cdot \|E\|_\infty}{m}|s-t|^2\Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{-1/2}. \end{aligned}$$

We consider also the function

$$F_1(s) = \frac{1}{2}|s-t|^2 \cdot 2\sqrt{N} \cdot |q| \cdot \|E\|_\infty \Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{-1/2} - |s-t| \cdot |P(t)| \cdot \Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{-1/2} + m \cdot \text{diam}(\Omega).$$

By the above computations we have $F_1(s) \geq 0,\ \forall\, r_1 \leq s \leq r_2$. Moreover, the condition $\Delta > 0$ is equivalent to $\alpha^2 > \beta\sqrt{1+\alpha^2}$ where $\alpha = \frac{|P(t)|}{mc_0}$. The previous inequality can be written also as $\big(\alpha^2 - \frac{\beta^2}{2}\big)^2 > \beta^2 + \frac{\beta^4}{4}$ and thus $\Delta > 0$ if $\alpha^2 > \beta + \beta^2 > \frac{\beta^2}{2} + \sqrt{\beta^2 + \frac{\beta^4}{4}}$. But $\alpha = \frac{|P(t)|}{mc_0} > (\beta + \beta^2)^{1/2}$ is satisfied by hypothesis. By Corollary 6.1 we deduce that

$$\max\{t - r_1, r_2 - t\} \leq \frac{2m \cdot \text{diam}(\Omega)}{|P(t)|}\Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{1/2}. \tag{6.4}$$

Suppose that $t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty} < s_{\text{out}}$, or $r_2 = t + \frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty}$. We have by (6.4) that

$$\frac{|P(t)|}{2 \cdot |q| \cdot \|E\|_\infty} \leq \frac{2m \cdot \text{diam}(\Omega)}{|P(t)|}\Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{1/2},$$

which is equivalent to $\frac{\alpha^2}{\sqrt{1+\alpha^2}} \leq \frac{\beta}{\sqrt{N}}$ with the previous notations. Since $N \geq 1$ we would deduce that $\frac{\alpha^2}{\sqrt{1+\alpha^2}} \leq \beta$ or $\Delta \leq 0$ but we have proved that $\Delta > 0$. Finally we deduce that $s_{\text{out}} \leq t + \frac{|P(t)|}{2\cdot|q|\cdot\|E\|_\infty}$ and similarly we have $t - \frac{|P(t)|}{2\cdot|q|\cdot\|E\|_\infty} \leq s_{\text{in}}$. It follows that

$$r_1 = s_{\text{in}}, \quad r_2 = s_{\text{out}}, \quad \max\{t - s_{\text{in}}, s_{\text{out}} - t\} \leq \frac{2\text{diam}(\Omega)}{|v(P(t))|}, \quad s_{\text{out}} - s_{\text{in}} \leq \frac{4\text{diam}(\Omega)}{|v(P(t))|}.$$

We check easily that if $|P(t)| > D_{\text{rel}}^{\text{ele}}$, then

$$|v(P(t))| = c_0 \frac{|P(t)|}{mc_0}\Big(1 + \frac{|P(t)|^2}{m^2c_0^2}\Big)^{-1/2} > c_0 \frac{\sqrt{\beta(1+\beta)}}{\sqrt{1+\beta(1+\beta)}}$$

and thus we obtain that

$$\max\{t - s_{\text{in}}, s_{\text{out}} - t\} < \frac{2\text{diam}(\Omega)}{c_0} \cdot \frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}}.$$

Finally we find for $s_{\text{in}} \leq s \leq s_{\text{out}}$

$$\begin{aligned} |P(s) - P(t)| &\leq |q| \cdot \|E\|_\infty \cdot |s-t| < \frac{2|q| \cdot \|E\|_\infty \cdot \text{diam}(\Omega)}{c_0} \cdot \frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}} \\ &= \frac{\beta mc_0}{2\sqrt{N}} \cdot \frac{\sqrt{1+\beta(1+\beta)}}{\sqrt{\beta(1+\beta)}} < mc_0\sqrt{\beta(1+\beta)} = D_{\text{rel}}^{\text{ele}}. \end{aligned} \tag{6.5}$$

(2) If $\max\{|P(s_1)|, |P(s_2)|\} \leq D_{\text{rel}}^{\text{ele}}$, then we have $|P(s_1) - P(s_2)| \leq 2D_{\text{rel}}^{\text{ele}}$. If $|P(s_1)| > D_{\text{rel}}^{\text{ele}}$, by the point (1) with $t = s_1$ we deduce that $|P(s_2) - P(s_1)| \leq D_{\text{rel}}^{\text{ele}} \leq 2D_{\text{rel}}^{\text{ele}}$ and the same if $|P(s_2)| > D_{\text{rel}}^{\text{ele}}$ by taking $t = s_2$.

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