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Kähler Manifolds with Almost Non-negative Ricci Curvature

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Abstract Compact Kähler manifolds with semi-positive Ricci curvature have been investigated by various authors. From Peternell's work, if M is a compact Kähler n-manifold with semi-positive Ricci curvature and finite fundamental group, then the universal cover has a decomposition $\widetilde{M} \cong X_1 \times \cdots \times X_m$, where X_j is a Calabi-Yau manifold, or a hyperKähler manifold, or X_j satisfies $H^0(X_j, \Omega^p) = 0$. The purpose of this paper is to generalize this theorem to almost non-negative Ricci curvature Kähler manifolds by using the Gromov-Hausdorff convergence. Let M be a compact complex n-manifold with non-vanishing Euler number. If for any $\epsilon > 0$, there exists a Kähler structure (J_ϵ, g_ϵ) on M such that the volume $\mathrm{Vol}_{g_\epsilon}(M) < V$, the sectional curvature $|K(g_\epsilon)| < \Lambda^2$, and the Ricci-tensor $\mathrm{Ric}(g_\epsilon) > -\epsilon g_\epsilon$, where V and Λ are two constants independent of ϵ. Then the fundamental group of M is finite, and M is diffeomorphic to a complex manifold X such that the universal covering of X has a decomposition, $\widetilde{X} \cong X_1 \times \cdots \times X_s$, where X_i is a Calabi-Yau manifold, or a hyperKähler manifold, or X_i satisfies $H^0(X_i, \Omega^p) = \{0\}$, $p > 0$.

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1 Introduction

In [6], the uniformization theorem of Mok for nonnegative bisectional curvature Kähler manifolds is generalized to almost nonnegative bisectional curvature Kähler manifolds. On the other hand, compact Kähler manifolds with semi-positive Ricci curvature have been investigated in [5, 14]. If M is a compact Kähler n-manifold with semi-positive Ricci curvature and finite fundamental group, then the universal cover has a decomposition $\widetilde{M} \cong X_1 \times \cdots \times X_m$, where X_j is a Calabi-Yau manifold, or a hyperKähler manifold, or X_j satisfies $H^0(X_j, \Omega^p) = 0$, by [14, Theorem 5.13]. In this paper, we generalize this theorem to Kähler manifolds with almost non-negative Ricci curvature and bounded sectional curvature. We call (J, ω, g) a Kähler structure on a manifold M, if J is a complex structure on M, g is a Kähler metric compatible with J, and ω is the Kähler form associated to g.

Theorem 1.1 *Let M be a compact complex n-manifold with non-vanishing Euler number, $\chi(M) \neq 0$. If for any $\epsilon > 0$, there exists a Kähler structure (J_ϵ, g_ϵ) on M such that*

(1) $\mathrm{Vol}_{g_\epsilon}(M) < V$, *where* $\mathrm{Vol}_{g_\epsilon}(M)$ *is the volume of* (M, g_ϵ), *and V is a constant independent of ϵ.*

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(2) $|K(g_\epsilon)| < \Lambda^2$, *where* $K(g_\epsilon)$ *is the sectional curvature, and* Λ *is a constant independent of* ϵ.

(3) $\mathrm{Ric}(g_\epsilon) > -\epsilon g_\epsilon$, *where* $\mathrm{Ric}(g_\epsilon)$ *is the Ricci-tensor of* g_ϵ.

Then,

(1) *The fundamental group of* M, $\pi_1(M)$, *is finite.*

(2) M *is diffeomorphic to a complex manifold* X *such that the universal covering of* X *has a decomposition,* $\widetilde{X} \cong X_1 \times \cdots \times X_s$, *where* X_i *is a Calabi-Yau manifold, or a hyperKähler manifold, or* X_i *satisfies* $H^0(X_i, \Omega^p) = \{0\}$, $p > 0$.

(3) *The holomorphic Euler number of* M *satisfies that*

$$\chi(M, \mathcal{O}) \leq \frac{2^{n-1}}{|\pi_1(M)|}.$$

The condition (2) in the above theorem seems too strong. If we remove this condition, and fix a complex structure, then a weaker conclusion also can be obtained. Let (M, J, ω, g) be a compact Kähler n-manifold. We call the first Chern $c_1(M)$ class of M numerically effective, if for every $\epsilon > 0$ there is a smooth hermitian metric h_ϵ on the anti-canonical bundle $-K_M = \wedge_J^{0,n} T^*M$ such that the curvature satisfies $\sqrt{-1}\,\Theta_{h_\epsilon}(-K_M) \geq -\epsilon\omega$ (see [5, 13]). This implies that, for any $\epsilon > 0$, there exists a Kähler metric g_ϵ with Kähler form ω_ϵ such that the Ricci form satisfies $\mathrm{Ric}(g_\epsilon) \geq -\epsilon\omega_\epsilon$, and $\omega_\epsilon \in [\omega]$ by arguments in the proof of Theorem 1.1 in [5]. A $(1,1)$-class $[\alpha]$ is integrable if it contains a closed positive current T of the form $T = \alpha + \sqrt{-1}\,\partial\overline{\partial}\varphi$ with $\int_M e^{-\varphi}\omega^n < \infty$. For example, if $[\alpha]$ contains a closed positive current T whose Lelong numbers are small enough, i.e. $\max \nu(T, x) < 2$, then $[\alpha]$ is integrable (see [13, §2]).

Proposition 1.1 *Let* (M, J, ω, g) *be a compact Kähler* n*-manifold with the first Chern* $c_1(M)$ *class numerically effective and integrable. Then* $h^{p,0}(M) \leq \binom{n}{p}$. *Furthermore, if* $h^{n,0}(M) \neq 0$, *then* $c_1(M) = 0$.

We organize this paper as follows: In Section 2, we discuss the convergence of (p, q)-forms; in Section 3, we prove Theorem 1.1 and Proposition 1.1 respectively.

2 Convergence of Harmonic (p, q)-Forms

Let (M, J) be a compact complex manifold satisfying the hypothesis in Theorem 1.1. Then we have a sequence of Kähler structures (J_k, g_k) such that

(1) $\mathrm{Vol}_k(M) < V$, where $\mathrm{Vol}_k(M)$ is the volume of (M, g_k).

(2) $|K(g_k)| < \Lambda^2$, where $K(g_k)$ is the sectional curvature, and Λ is a constant independent of k.

(3) $\mathrm{Ric}_k > -\frac{1}{k} g_k$, where Ric_k is the Ricci-tensor of (M, g_k).

Lemma 2.1 *There are constants* D *and* v, *independent of* k, *such that* $\mathrm{Vol}_k(M) > v$ *and* $\mathrm{diam}_k(M) < D$, *where* $\mathrm{diam}_k(M)$ *is the diameter of* (M, g_k).

Proof First, we claim that there is a sequence $\{x^k\} \subset M$ such that, for any k, the injectivity radius of g_k at x^k satisfies $i_k(x^k) > \iota$, where ι is a constant independent of k. If not, there is a k such that the injectivity radius of (M, g_k) is less than ι, $i_k(M, g_k) < \iota$, where ι is the critical injectivity radius in the sense of [4]. Then by [4], there exists an F-structure of positive rank on M as $|K(g_k)| < \Lambda^2$. It is contradict to $\chi(M) \neq 0$ (see [4]).

Then, since $|K(g_k)| < \Lambda^2$, we have

$$\mathrm{Vol}_k(B_{x^k}(1)) > \mathrm{Vol}_k(B_{x^k}(\iota)) > v,$$

where v is a constant independent of k. Then by [13, Lemma 2.3], the diameter of (M, g_k) is bounded by a constant D from above, $\mathrm{diam}_k(M) < D$.

Finally, $\mathrm{Vol}_k(M) > \mathrm{Vol}_k(B_{x^k}(\iota)) > v$.

By the lemma, the Kähler manifolds (M, J_k, g_k) satisfy that $\mathrm{diam}_k(M) < D$, $\mathrm{Vol}_k(M) > v$, and $|K(g_k)| < \Lambda^2$. Then, by Gromov's compactness theorem (see [15, Theorem 4.1]), there exist a subsequence of $\{k\}$, denoted also by $\{k\}$, and a $C^{1,\alpha}$-Kähler manifold (X, J_∞, g_∞), $0 < \alpha < 1$, such that, for k large, there exist diffeomorphisms $f_k : X \to M$ satisfying:

(1) $\widehat{g}_k = f_k^* g_k$ converges to g_∞ in the $C^{1,\alpha}$ sense.

(2) $\widehat{J}_k = (f_k^{-1})_* \cdot J_k \cdot (f_k)_*$ converges to J_∞ in the $C^{1,\alpha}$ sense.

In fact, there is a finite family of harmonic balls $\{B_m(r)\}$ corresponding to $\widehat{g}_N$, $N \gg 1$, such that X is covered by $\{B_m(\frac{r}{2})\}$. For each $B_m(r)$, there is a harmonic coordinate, $\{(x^1, \cdots, x^{2n})\}$, $2n = \dim M$, such that $\widehat{g}_k = \sum\limits_{i,j} \widehat{g}_{k,ij} dx^i dx^j$ satisfying $\mu^{-1}\mathrm{Id} < (\widehat{g}_{k,ij}) < \mu\,\mathrm{Id}$, and $\widehat{g}_{k,ij} \to \widehat{g}_{\infty,ij}$ in $C^{1,\alpha}(B_m(r))$. Furthermore, $\widehat{g}_\infty = \sum\limits_{i,j} \widehat{g}_{\infty,ij} dx^i dx^j$ is a $C^{1,\alpha}$-Kähler metric, that is, $g_\infty(J_\infty\cdot, J_\infty\cdot) = g_\infty(\,\cdot\,,\,\cdot\,)$ and $\nabla^\infty J_\infty = 0$ where ∇^∞ is the Levi-Civita connection of g_∞ (see [15, 6]).

Denote the harmonic (p,q)-forms space of (M, J_k, g_k) by $H_k^{p,q}(M)$, and the complex Laplacian operator by $\triangle_{\overline{\partial}_k} = (\overline{\partial}_k + \overline{\partial}_k^*)^2$. Let β^k be a harmonic (p,q)-form of (M, J_k, g_k), $\triangle_{\overline{\partial}_k}\beta^k = 0$, such that

$$\frac{1}{\mathrm{Vol}_k(M)} \int_M |\beta^k|_k^2 \, d\,\mathrm{Vol}_k = 1, \tag{2.1}$$

where $d\,\mathrm{Vol}_k$ is the volume form. Let $\widehat{\beta}^k = f_k^* \beta^k$. It is obvious that $\widehat{\beta}^k$ is a harmonic (p,q)-form of $(X, \widehat{J}_k, \widehat{g}_k)$.

Lemma 2.2 *A subsequence of $\widehat{\beta}^k$ converges to a $C^{1,\alpha}$-form β^∞ in the $C^{1,\alpha}$ sense. Furthermore, β^∞ is a harmonic (p,q)-form of (X, g_∞, J_∞).*

Proof (I) On any $B_m(r)$, $\widehat{\beta}^k = \sum\limits_I \widehat{\beta}_I^k dx^I$, where $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_{p+q}}$, and $(x^1, \cdots, x^{2n})$ is the harmonic coordinate associated to $\widehat{g}_N$. By Weitzenböck formula, we have

$$0 = \triangle_{\overline{\partial}_k} \widehat{\beta}^k = \nabla^* \nabla \widehat{\beta}^k + R^k \widehat{\beta}^k, \tag{2.2}$$

where the operators R^k come from the curvature operators of $\widehat{g}_k$. In the coordinate, it can be written as

$$\sum_{ij} \widehat{g}_k^{ij} \frac{\partial^2 \widehat{\beta}_I^k}{\partial x^i \partial x^j} + \sum_{i,L} \widehat{b}_{k,I}^{i,L} \frac{\partial \widehat{\beta}_L^k}{\partial x^i} + \sum_L \widehat{c}_{k,I}^L \widehat{\beta}_L^k = 0. \tag{2.3}$$

First, we obtain the uniform $C^{0,\alpha}$ bounds of $\widehat{g}_k^{ij}$, $\widehat{b}_{k,I}^{i,L}$ and a uniform positive bound from below for the minimal eigenvalue of $\widehat{g}_k^{ij}$, $k \gg N$. Second, $\widehat{c}_{k,I}^{L}$ are uniformly L^∞-bounded by the hypothesis $|K(g_k)| < \Lambda^2$, and $\widehat{g}_k$ are Kähler metrics on X. By the elliptic estimate (see [1]), for any s, we obtain

$$\|\widehat{\beta}_I^k\|_{L^{2,s}(B(\frac{r}{2}))} \leq C \sum_L \|\widehat{\beta}_L^k\|_{L^s(B(r))}, \tag{2.4}$$

where C is a constant depending only on the bounds of the coefficients.

On the other hand,

$$\sum_L \|\widehat{\beta}_L^k\|_{L^s(B(r))} \leq C' \sum_L \sup_{B(r)} |\widehat{\beta}_L^k| \leq C'' \sup_{B(r)} |\widehat{\beta}^k|_k \leq C''' \sup_X |\widehat{\beta}^k|_k.$$

By [10, Lemma 8], we have

$$\triangle_d |\widehat{\beta}^k|_k \leq K |\widehat{\beta}^k|_k, \tag{2.5}$$

where $\triangle_d = (d + d^*)^2$, and K is a constant depending only on the bound Λ, $p+q$ and n. Then [9, Proposition 3.2] implies that

$$\sup_X |\widehat{\beta}^k|_k \leq K' \Big(\frac{1}{\mathrm{Vol}_k(M)} \int_M |\widehat{\beta}^k|_k^2 \Big)^{\frac{1}{2}}, \tag{2.6}$$

where K' is a constant depending only on V, Λ and D. Thus

$$\|\widehat{\beta}_I^k\|_{L^{2,s}(B(\frac{r}{2}))} \leq C, \tag{2.7}$$

where C is a constant independent of k. By the Sobolev embedding theorem, there is a compact embedding, $L^{2,s} \hookrightarrow C^{1,1-\frac{2n}{s}}$. Thus, a subsequence of $\widehat{\beta}^k$ converges to a $C^{1,\alpha}$ $(p+q)$-form β^∞ in the $C^{1,\alpha}$ sense, $\alpha < 1 - \frac{2n}{s}$.

(II) Let $\Pi_k^{p,q}$ be the projection operators to the (p,q)-forms corresponding to the complex structures $\widehat{J}_k$. Since

$$\Pi_k^{p,q} = \otimes^p \frac{1 - \sqrt{-1}\,\widehat{J}_k}{2} \otimes \otimes^q \frac{1 + \sqrt{-1}\,\widehat{J}_k}{2}, \tag{2.8}$$

and $\widehat{J}_k$ converges to J_∞ in the $C^{1,\alpha}$ sense, we obtain that $\Pi_k^{p,q}$ converges to $\Pi_\infty^{p,q}$ in the $C^{1,\alpha}$ sense, where $\Pi_\infty^{p,q}$ is the projection operator to the (p,q)-forms corresponding to the complex structure $\widehat{J}_\infty$. Then

$$0 = (\mathrm{Id} - \Pi_k^{p,q})\widehat{\beta}^k \to (\mathrm{Id} - \Pi_\infty^{p,q})\beta^\infty \tag{2.9}$$

in the $C^{1,\alpha}$ sense. Thus β^∞ is a (p,q)-form corresponding to J_∞.

(III) Notice that $d = \sum\limits_i dx^i \wedge \frac{\partial}{\partial x^i}$, and $d_k^* = -\sum\limits_{ij} \widehat{g}_k^{ij} \iota(dx^i) \frac{\partial}{\partial x^j} + \sum\limits_i b_k^i \iota(dx^i)$, where b_k^i involves only the 1-order derivations of $\widehat{g}_k^{ij}$ and $\widehat{g}_{k,ij}$. Thus

$$0 = (d + d_k^*)\widehat{\beta}^k \to (d + d_\infty^*)\beta^\infty \tag{2.10}$$

in the $C^{0,\alpha}$ sense. Since g_∞ is Kählerian,

$$\int_X |(d+d^*_\infty)\beta^\infty|^2 d\,\mathrm{Vol}_\infty = 2\int_X |(\overline{\partial}_\infty + \overline{\partial}^*_\infty)\beta^\infty|^2 d\,\mathrm{Vol}_\infty.$$

Then we obtain

$$(\overline{\partial}_\infty + \overline{\partial}^*_\infty)\beta^\infty = 0. \tag{2.11}$$

Thus, β^∞ is a harmonic (p,q)-form.

Lemma 2.3 *For any $\beta^\infty \in H^{p,q}_\infty(X)$, there exists a sequence $\beta^k \in H^{p,q}_k(X)$ such that β^k converges to β^∞ in the $C^{1,\alpha}$ sense.*

Proof For fixed p and q, since $\dim H^{p,q}_k(M) \le b_{p+q}(M)$, we assume $l \equiv \dim H^{p,q}_k(M)$ by passing to a subsequence. Suppose that $\{\beta^k_1, \cdots, \beta^k_l\}$ is an orthonormal basis of the harmonic space $H^{p,q}_k(M)$ such that

$$\frac{1}{\mathrm{Vol}_k(M)}\int_M \langle \beta^k_i, \beta^k_j\rangle_k d\,\mathrm{Vol}_k = \frac{1}{\mathrm{Vol}_k(X)}\int_X \langle \widehat{\beta}^k_i, \widehat{\beta}^k_j\rangle_k d\,\mathrm{Vol}_k = \delta_{ij}. \tag{2.12}$$

By Lemma 2.2, we can assume that $\widehat{\beta}^k_i$ converges to a $C^{1,\alpha}$ form β^∞_i, for each i.

$$\begin{aligned}
&\Big|\frac{1}{\mathrm{Vol}_\infty(X)}\int_X \langle \beta^\infty_i, \beta^\infty_j\rangle_\infty d\,\mathrm{Vol}_\infty - \delta_{ij}\Big| \\
&= \Big|\frac{1}{\mathrm{Vol}_\infty(X)}\int_X \langle \beta^\infty_i, \beta^\infty_j\rangle_\infty d\,\mathrm{Vol}_\infty - \frac{1}{\mathrm{Vol}_k(X)}\int_X \langle \widehat{\beta}^k_i, \widehat{\beta}^k_j\rangle_k d\,\mathrm{Vol}_k\Big| \\
&= \Big|\int_X \Big(\sum_{L,H} g_\infty^{l_1h_1}\cdots g_\infty^{l_{p+q}h_{p+q}}\beta^\infty_{i,L}\overline{\beta^\infty_{j,H}}\frac{\sqrt{\det g_\infty}}{\mathrm{Vol}_\infty(X)} \\
&\quad - \sum_{L,H}\widehat{g}_k^{l_1h_1}\cdots \widehat{g}_k^{l_{p+q}h_{p+q}}\widehat{\beta}^k_{i,L}\overline{\widehat{\beta}^k_{j,H}}\frac{\sqrt{\det \widehat{g}_k}}{\mathrm{Vol}_k(X)}\Big)dx^1\cdots dx^{2n}\Big| \\
&\to 0,
\end{aligned}$$

when $k\to\infty$. Thus

$$\frac{1}{\mathrm{Vol}_\infty(X)}\int_X \langle \beta^\infty_i, \beta^\infty_j\rangle_\infty d\,\mathrm{Vol}_\infty = \delta_{ij}. \tag{2.13}$$

By Lemma 2.2, we obtain $H^{p,q}_\infty(X) \supset \mathrm{Span}\{\beta^\infty_1, \cdots, \beta^\infty_l\}$. So $h^{p,q}(X) = \dim H^{p,q}_\infty(X) \ge l$. Then we have

$$b_s(X) = \sum_{p+q=s} h^{p,q}(X) \ge \sum_{p+q=s} h^{p,q}(M) = b_s(M),$$

where b_s is the Betti-number. Thus

$$H^{p,q}_\infty(X) = \mathrm{Span}\{\beta^\infty_1, \cdots, \beta^\infty_l\}.$$

Lemma 2.4 *All harmonic $(p,0)$-forms of (X, J_∞, g_∞) are parallel, that is, $\nabla^\infty\beta^\infty = 0$ for any $\beta^\infty \in H^{p,0}_\infty(X)$.*

Proof By Lemma 2.3, for any $\beta^\infty \in H_\infty^{p,0}(X)$, there exists a sequence $\beta^k \in H_k^{p,0}(X)$ such that β^k converges to β^∞ in the $C^{1,\alpha}$ sense. By the Weitzenböck formula (see [17]), we have

$$-\Delta_k|\beta^k|_k^2 = |\nabla^k\beta^k|_k^2 + |\overline{\nabla}^k\beta^k|_k^2 + \sum_I \Big(\sum_{i\in I}\mathrm{Ric}_k(e_i,e_i)\Big)|\beta_I^k|^2, \tag{2.14}$$

where $\{e_i, \widehat{J}_k e_i\}$ is an orthnormal basis of T_xX corresponding to $\widehat{g}_k$, and $\beta^k = \sum\limits_I \beta_I^k\xi^I$, $\xi^I = (e_{i_1}^* + \sqrt{-1}\,\widehat{J}_k e_{i_1}^*)\wedge\cdots\wedge(e_{i_p}^* + \sqrt{-1}\,\widehat{J}_k e_{i_p}^*)$.

$$\begin{aligned} 0 &= \int_M \Big(|\nabla^k\beta^k|^2 + |\overline{\nabla}^k\beta^k|^2 + \sum_I\Big(\sum_{i\in I}\mathrm{Ric}_k(e_i,e_i)\Big)|\beta_I^k|^2\Big)d\,\mathrm{Vol}_k \\ &\geq \int_M \Big(|\nabla^k\beta^k|^2 + |\overline{\nabla}^k\beta^k|^2 - \frac{C}{k}\int_M|\beta^k|^2\Big)d\,\mathrm{Vol}_k \\ &\to \int_X (|\nabla^\infty\beta^\infty|^2 + |\overline{\nabla}^\infty\beta^\infty|^2)d\,\mathrm{Vol}_\infty \geq 0, \end{aligned}$$

when $k\to\infty$. Thus, we obtain $\nabla^\infty\beta^\infty \equiv 0$, where ∇^∞ is the Levi-Civita connection of g_∞.

3 Proofs of Theorem 1.1 and Proposition 1.1

Now, it is the place to prove Theorem 1.1 and Proposition 1.1.

Proof of Theorem 1.1 (I) Let $\overline{M}$ be any finite covering of M, $\pi:\overline{M}\to M$. Give $\overline{M}$ the metric $\overline{g}_k = \pi^*g_k$. By the same arguments as in Section 2, there exist diffeomorphisms from $\overline{M}$ to $\overline{X}$, each of which is a finite covering of X, and $\overline{g}_k$ converges to a $C^{1,\alpha}$ Kähler metric $\overline{g}_\infty$. Furthermore, all lemmas in Section 2 are valid for the present case.

If $h^{1,0}(\overline{X}) \neq 0$, let $\beta^\infty \in H_\infty^{1,0}(\overline{X})$, $\beta^\infty \neq 0$. By Lemma 2.4, β^∞ is parallel, $\nabla^\infty\beta^\infty \equiv 0$. Thus $|\beta^\infty| \equiv$constant$\neq 0$. It contradicts $\chi(\overline{X}) = d\chi(X) \neq 0$. So $h^{1,0}(\overline{X}) = 0$.

On the other hand, by the hypothesis $\mathrm{diam}_k(M) < D$, $\mathrm{Vol}_k(M) > v$, $|K(g_k)| < \Lambda^2$, $\mathrm{Ric}_k > -\frac{1}{k}g_k$, and the final remark in [7], the fundamental group of M, $\pi_1(M)$, has polynormial growth. By [14, Proposition 5.8], we obtain that $\pi_1(M)$ is finite.

(II) Let $\widetilde{M}$, $\widetilde{X}$ be the universal covering of M, X respectively. All lemmas in Section 2 are valid for $\widetilde{M}$ and $\widetilde{X}$. Let $\widetilde{g}_\infty$ be the $C^{1,\alpha}$-Kähler metric on $\widetilde{X}$ obtained by Section 2. Since de Rham decomposition Theorem is valid for $C^{1,\alpha}$-Kähler metric (see [6]), we obtain

$$(\widetilde{X},\widetilde{g}_\infty) \cong (X_1,\widetilde{g}_{\infty,1})\times\cdots\times(X_s,\widetilde{g}_{\infty,s}), \tag{3.1}$$

where $(X_i,\widetilde{g}_{\infty,i})$ are $C^{1,\alpha}$-Kähler manifolds with irreducible holonomy groups H_i. If β_i is a harmonic $(p,0)$-form of $(X_i,\widetilde{g}_{\infty,i})$, then $P_i^*\beta_i$ is a harmonic $(p,0)$-form of $(\widetilde{X},\widetilde{g}_\infty)$, where P_i is the projection from $(\widetilde{X},\widetilde{g}_\infty)$ to $(X_i,\widetilde{g}_{\infty,i})$. By Lemma 2.4, $P_i^*\beta_i$ is parallel. So β_i is parallel corresponding to $\widetilde{g}_{\infty,i}$. Thus, any element of $H_\infty^{p,0}(X_i)$ is parallel.

By the Berger Theorem (see [3]), there are only three possibilities for H_i, namely $\mathrm{U}(n_i)$, $\mathrm{SU}(n_i)$ and $\mathrm{Sp}(\frac{n_i}{2})$, in case n_i is even. Since Calabi-Yau manifolds are characterised by $H_i = \mathrm{SU}(n_i)$ and hyperKähler manifolds by $H_i = \mathrm{Sp}(\frac{n_i}{2})$, we only have to exclude that $H_i = \mathrm{U}(n_i)$. As there is no $\mathrm{U}(n_i)$-invariant linear subspace of $T_x^{*,(p,0)}X_i$, $p>0$, we have $H_\infty^{p,0}(X_i) = \{0\}$.

(III) Finally, note that $h^{p,0}(\widetilde{X}) \le \binom{n}{p}$, and $\chi(M,\mathcal{O}) = \chi(X,\mathcal{O})$. Then we have $\chi(M,\mathcal{O}) = \chi(X,\mathcal{O}) = \frac{\chi(\widetilde{X},\mathcal{O})}{|\pi_1(M)|} \le \frac{2^{n-1}}{|\pi_1(M)|}$.

Proof of Proposition 1.1 (I) By [13, Theorem 2], there is a family of Kähler metrics $\{g_k\}$ such that $\mathrm{Ric}_k \cdot \mathrm{diam}_k^2 > -\frac{\mathrm{diam}_k^2}{k} g_k$, and $\frac{\mathrm{diam}_k^2}{k} \to 0$ when $k \to \infty$. For any holomorphic $(p,0)$ form $\beta \in H^{p,0}(M)$, by the Weitzenböck formula (see [17]), we have

$$-\Delta_k |\beta|_k^2 = |\nabla^k \beta|_k^2 + \sum_I \Big(\sum_{i \in I} \mathrm{Ric}_k(e_i, e_i)\Big) |\beta_I|^2, \tag{3.2}$$

where $\{e_i, J_k e_i\}$ is an orthnormal basis of T_xM corresponding to g_k, and $\beta = \sum\limits_I \beta_I \zeta^I$, $\zeta^I = (e_{i_1}^* + \sqrt{-1} J_k e_{i_1}^*) \wedge \cdots \wedge (e_{i_p}^* + \sqrt{-1} J_k e_{i_p}^*)$. By $|\nabla^k \beta|_k^2 \ge |d|\beta|_k|^2$, we have $\Delta_k |\beta|_k \le \frac{pC_n}{k \cdot \mathrm{diam}^2(g_k)} |\beta|_k$. So [8, Theorem 3.3] is valid for the present case. Thus we obtain

$$\dim H^{p,0}(M) \le \binom{n}{p} \xi^2 \Big(\frac{\mathrm{diam}_k}{\sqrt{k}}\Big),$$

where $\xi\big(\frac{\mathrm{diam}_k}{\sqrt{k}}\big)$ is a function such that $\xi\big(\frac{\mathrm{diam}_k}{\sqrt{k}}\big) \to 1$ when $k \to \infty$. Letting k_0 be large enough, we have

$$h^{(p,0)}(M) = \dim H^{p,0}(M) \le \binom{n}{p}. \tag{3.3}$$

(II) The following argument is similar to the proof of Proposition 2.1 in [12]. Let $K_M = \wedge_J^{n,0} T^*M$ be the canonical line bundle of (M,J). Since the first Chern class $c_1(M)$ of M is numerically effective, there is a family of hermitian metric $\{h_k'\}$ on $-K_M$ such that the curvature forms satisfy $\sqrt{-1}\,\Theta_k(-K_M) \ge -\frac{1}{k}\omega$ (see [5]). Then we obtain a family of hermitian metric $\{h_k\}$ on K_M whose curvature forms satisfy

$$\sqrt{-1}\,\Theta_k(K_M) \le \frac{1}{k}\omega. \tag{3.4}$$

If $h^{n,0}(M) \ne 0$, there exists a non-zero holomorphic section s of K_M, $s \in H^0(M, \mathcal{O}(K_M))$. For any $x \in M$, we choose complex normal coordinates $(z_1, \cdots, z_n)$ such that $\omega = \sqrt{-1} \sum g_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta$ satisfies $g_{\alpha\overline{\beta}}|_x = \delta_{\alpha\overline{\beta}}$ and $dg_{\alpha\overline{\beta}}|_x = 0$. For any k, we choose a holomorphic basis e^k of K_M in a neighborhood of x such that $s = s_k e^k$, $\|s\|_{h_k}^2 = h_k s_k \overline{s_k}$, $h_k(x) = 1$, $dh_k(x) = 0$ and $\sqrt{-1}\,\Theta_k(K_M) = -\sqrt{-1}\,\partial\overline{\partial}\log h_k$. At x, for any $\epsilon > 0$, we have

$$\partial\overline{\partial}\,\log(\|s\|_{h_k}^2 + \epsilon^2) = \frac{\partial\overline{\partial}\,\|s\|_{h_k}^2}{\|s\|_{h_k}^2 + \epsilon^2} - \frac{\partial\|s\|_{h_k}^2 \wedge \overline{\partial}\,\|s\|_{h_k}^2}{(\|s\|_{h_k}^2 + \epsilon^2)^2},$$

$$\partial\overline{\partial}\,\|s\|_{h_k}^2 = \partial s_k \wedge \overline{\partial s_k} - \|s\|_{h_k}^2 \Theta_k(K_M), \quad \text{and} \quad \partial\|s\|_{h_k}^2 \wedge \overline{\partial}\,\|s\|_{h_k}^2 = \|s\|_{h_k}^2 \partial s_k \wedge \overline{\partial s_k}.$$

Hence, at x,

$$\begin{aligned}\triangle\log(\|s\|_{h_k}^2 + \epsilon^2)\omega^n &= \sum g^{\alpha\overline{\beta}} \frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \log(\|s\|_{h_k}^2 + \epsilon^2)\omega^n \\ &= -\frac{\|s\|_{h_k}^2}{\|s\|_{h_k}^2 + \epsilon^2} \sqrt{-1}\,\Theta_k(K_M) \wedge \omega^{n-1} \\ &\quad + \frac{\epsilon^2}{(\|s\|_{h_k}^2 + \epsilon^2)^2} \sqrt{-1}\,\partial s_k \wedge \overline{\partial s_k} \wedge \omega^{n-1} \\ &\ge -\frac{1}{k}\omega^n. \end{aligned} \tag{3.5}$$

Thus $\triangle\log(\|s\|^2_{h_k}+\epsilon^2)\omega^n \geq -\frac{1}{k}\omega^n$ on M. By Fatou Lemma, we have

$$\int_M \triangle\log\|s\|^2_{h_k}\omega^n \leq \lim_{\epsilon\to 0}\int_M \triangle\log(\|s\|^2_{h_k}+\epsilon^2)\omega^n = 0.$$

Note that the Poincare-Lelong equation shows that

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|s\|^2_{h_k} = -\frac{\sqrt{-1}}{2\pi}\Theta_k(K_M) + [Z_s],$$

where $Z_s = s^{-1}(0)$ is the zero divisor of s (see [12]). Hence

$$\mathrm{Vol}_\omega(Z_s) \leq \frac{1}{k}\mathrm{Vol}_\omega(M) \to 0, \tag{3.6}$$

when $k\to\infty$. This implies that Z_s is empty. Thus K_M is a trivial bundle, and $c_1(M) = 0$.

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