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On Hardy's Theorem on SU(1, 1)***

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Abstract The classical Hardy theorem asserts that f and its Fourier transform $\widehat{f}$ can not both be very rapidly decreasing. This theorem was generalized on Lie groups and also for the Fourier-Jacobi transform. However, on $\mathrm{SU}(1,1)$ there are infinitely many "good" functions in the sense that f and its spherical Fourier transform $\widetilde{f}$ both have good decay. In this paper, we shall characterize such functions on $\mathrm{SU}(1,1)$.

Keywords Heat kernel, Jacobi transform, Plancherel formula
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1 Introduction

The classical Hardy theorem (see [2]) asserts that f and its Fourier transform $\widetilde{f}$ can not both be "very rapidly decreasing". More precisely, suppose that a measurable function f on $\mathbb{R}$ and its Fourier transform $\widetilde{f}$ on $\mathbb{R}$ satisfy

$$|f(x)| \le Ae^{-ax^2} \quad \text{and} \quad |\widetilde{f}(\lambda)| \le Be^{-b\lambda^2} \tag{1.1}$$

for some positive constants A, B, a and b. If $ab > \frac{1}{4}$, then $f = 0$, and if $ab = \frac{1}{4}$, then f is a constant multiple of e^{-ax^2}. Recently, an analogue of Hardy's theorem was established on Lie groups by various people, where the heat kernel on Lie groups plays an essential role in controlling the decay of f and, in the case of $ab = \frac{1}{4}$, in expressing a unique function up to a constant multiplication. We refer to [9] and the references therein for more information. Moreover, Hardy's theorem was generalized for the Fourier-Jacobi transform (see [1, 3]) and, as an application, Andersen pointed out that Hardy's theorem on $\mathrm{SU}(1,1)$ does not hold unless the K-type of f is fixed: Let $G = \mathrm{SU}(1,1)$, and for $g \in G$ let $g = k_\phi a_x k_\psi$, $0 \le x$, $0 \le \phi, \psi \le 4\pi$, denote the Cartan decomposition of g. Let h_t denote the heat kernel on G and for integrable functions f on G let $\widetilde{f}_{n,m}$, $n, m \in \frac{1}{2}\mathbb{Z}$, the spherical Fourier transform of f corresponding to the K-type (n, m) (see (2.10) below). We suppose that a measurable function f on G and its spherical Fourier transform $\widetilde{f}_{n,m}$ on $\mathbb{R}$ satisfy

$$|f_{n,m}(g)| \le Ah_{1/(4a)}(g) \quad \text{and} \quad |\widetilde{f}_{n,m}(\lambda)| \le Be^{-b\lambda^2} \quad \text{for all } n, m \in \frac{1}{2}\mathbb{Z} \tag{1.2}$$

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for some positive constants A, B, a and b. Then, $f = 0$ if $ab > \frac{1}{4}$. However, there are infinitely many linearly independent functions on G satisfying the above condition if $ab = \frac{1}{4}$ (see Corollary 4.1).

In this paper, we restrict our attention to functions on G with K-types (n, m), $n, m = 0, 1, 2, \cdots$, and we show that the condition (1.2) under $ab = \frac{1}{4}$ determines a function on G uniquely in the following sense: In the classical case the condition (1.1) under $ab = \frac{1}{4}$ guarantees the limit

$$\lim_{x \to \infty} e^{ax^2} f(x) = c$$

and then f is uniquely determined as $f(x) = ce^{-ax^2}$. On $\mathrm{SU}(1,1)$, similarly, the condition (1.2) under $ab = \frac{1}{4}$ guarantees the limit

$$\lim_{x \to \infty} h_{1/(4a)}(x)^{-1} f(k_\phi a_x) = F(\phi)$$

and then f is uniquely determined by using the Fourier coefficients of F. Here $F \in H^2(\mathbb{T})$ is real analytic. Moreover, the L^2-norm of F on $\mathbb{T}$ coincides with the L^2-norm of the principal part of f on G and the Fourier coefficients $\{d_n; n = 0, 1, 2, \cdots\}$ of F satisfy

$$\sum_{n=0}^{\infty} |d_n|^2 \Big(1 + \sum_{k=0}^{n-1} k e^{2b(2k+1)^2}\Big) < \infty.$$

In Theorem 5.1 we shall give a characterization of F.

2 Notations

Let $G = \mathrm{SU}(1,1)$ and A, K the subgroups of G of the matrices

$$a_x = \begin{pmatrix} \cosh\frac{x}{2} & \sinh\frac{x}{2} \\ \sinh\frac{x}{2} & \cosh\frac{x}{2} \end{pmatrix}, \quad x \in \mathbb{R} \quad \text{and} \quad k_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \quad 0 \le \phi \le 4\pi$$

respectively. According to the Cartan decomposition of G, each $g \in G$ can be written uniquely as $g = k_\phi a_x k_\psi$ where $0 \le x$, $0 \le \phi, \psi \le 4\pi$. Let $\pi_{j,\lambda}$ ($j = 0, \frac{1}{2}, \lambda \in \mathbb{R}$) denote the principal series representation of G. Then the (vector-valued) spherical Fourier transform $\pi_{j,\lambda}(f)$ of f on G is defined as $\pi_{j,\lambda}(f) = \int_G f(f)\pi_{j,\lambda}(g)dg$, where dg a Haar measure on G. In the following, we shall consider functions f on G satisfying

$$f(a_x) = f(a_{-x}), \quad x \in \mathbb{R}$$

and we identify f with an even function on $\mathbb{R}$, which is denoted by the same symbol f. Under this restriction, we may suppose that $\pi_{j,\lambda}(f)$ is supported on $j = 0$ and $\lambda > 0$ and the K-types (m, n) of f is supported on $m, n \in \mathbb{Z}$ (cf. [6] and [8, §8]).

Before introducing the explicit form of the spherical Fourier transform of f on G, we shall recall the theory of the Jacobi transform on $\mathbb{R}_+$ (see [4, 5]). Let $\alpha, \beta, \lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ and consider the differential equation

$$(L_{\alpha,\beta} + \lambda^2 + \rho^2) f(x) = 0, \tag{2.1}$$

where $\rho = \alpha + \beta + 1$ and

$$L_{\alpha,\beta} = \frac{d^2}{dx^2} + ((2\alpha+1)\coth x + (2\beta+1)\tanh x)\frac{d}{dx}.$$

Then, for $\alpha \notin -\mathbb{N}$, the Jacobi function of the first kind with order (α, β)

$$\phi_\lambda^{\alpha,\beta}(x) = F\Big(\frac{\rho+i\lambda}{2}, \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\Big) \tag{2.2}$$

is a unique solution of (2.1) satisfying $\phi_\lambda^{\alpha,\beta}(0) = 1$ and $\frac{d\phi_\lambda^{\alpha,\beta}}{dx}(0) = 0$. For $\lambda \notin -i\mathbb{N}$, the Jacobi function of the second kind with order (α, β)

$$\Phi_\lambda^{\alpha,\beta}(t) = (e^t - e^{-t})^{i\lambda-\rho} F\Big(\frac{\rho-2\alpha-i\lambda}{2}, \frac{\rho-i\lambda}{2}; 1-i\lambda; -\sinh^{-2} t\Big) \tag{2.3}$$

is another solution of (2.1). Then $\Gamma(\alpha+1)^{-1}\phi_\lambda^{\alpha,\beta}$ is entire of α, β, and for $\lambda \notin i\mathbb{Z}$, we have the identity

$$\frac{\sqrt{\pi}}{\Gamma(\alpha+1)}\phi_\lambda^{\alpha,\beta}(t) = \frac{1}{2}(C_{\alpha,\beta}(\lambda)\Phi_\lambda^{\alpha,\beta}(t) + C_{\alpha,\beta}(-\lambda)\Phi_{-\lambda}^{\alpha,\beta}(t)), \tag{2.4}$$

where $C_{\alpha,\beta}(\lambda)$ is the C-function given by

$$C_{\alpha,\beta}(\lambda) = \frac{2^\rho \Gamma(\frac{i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})}{\Gamma(\frac{\rho+i\lambda}{2})\Gamma(\frac{\rho-2\beta+i\lambda}{2})} \tag{2.5}$$

(see [4, (2.5), (2.6)]). For convenience we assume $\alpha > -1$ and $\beta \in \mathbb{R}$ in the following. Then $C_{\alpha,\beta}(-\lambda)^{-1}$ has only simple poles for $\Im\lambda \geq 0$ which lie in the finite set $D_{\alpha,\beta} = \{i(|\beta| - \alpha - 1 - 2m); m = 0, 1, 2, \cdots, |\beta| - \alpha - 1 - 2m > 0\}$. We denote the residue of $(C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda))^{-1}$ at $\gamma \in D_{\alpha,\beta}$ by

$$d_{\alpha,\beta}(\gamma) = -i\mathrm{Res}_{\lambda=\gamma}(C_{\alpha,\beta}(\lambda)C_{\alpha,\beta}(-\lambda))^{-1}.$$

Let f be a compactly supported C^∞ even function on $\mathbb{R}$. We define the Jacobi transform $\widehat{f}_{\alpha,\beta}(\lambda)$ by

$$\widehat{f}_{\alpha,\beta}(\lambda) = \frac{\sqrt{2}}{\Gamma(\alpha+1)}\int_0^\infty f(x)\phi_\lambda^{\alpha,\beta}(x)\Delta_{\alpha,\beta}(x)dx, \tag{2.6}$$

where $\Delta_{\alpha,\beta}(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$ (see [4, (3.2)] and [5, (2.12)]). Then the inversion formula and the Plancherel formula are respectively given as follows:

$$f(x) = \frac{\sqrt{2}}{\Gamma(\alpha+1)}\Big(\int_0^\infty \widehat{f}_{\alpha,\beta}(\lambda)\phi_\lambda^{\alpha,\beta}(x)|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda + \sum_{\gamma\in D_{\alpha,\beta}} \widehat{f}_{\alpha,\beta}(\gamma)\phi_\gamma^{\alpha,\beta}(x)d_{\alpha,\beta}(\gamma)\Big), \tag{2.7}$$

$$\int_0^\infty |f(x)|^2\Delta_{\alpha,\beta}(x)dx = \int_0^\infty |\widehat{f}_{\alpha,\beta}(\lambda)|^2|C_{\alpha,\beta}(\lambda)|^{-2}d\lambda + \sum_{\gamma\in D_{\alpha,\beta}} |\widehat{f}_{\alpha,\beta}(\gamma)|^2 d_{\alpha,\beta}(\gamma) \tag{2.8}$$

(see [4, Theorem 4.2, (5.1)] and [5, Theorems 2.3 and 2.4]).

Let $h_t^{\alpha,\beta}$ denote the heat kernel for the Jacobi transform, that is, an even function on $\mathbb{R}$ satisfying

$$(h_t^{\alpha,\beta})^\wedge_{\alpha,\beta}(\lambda) = e^{-t(\lambda^2+\rho^2)}, \quad t, \lambda \in \mathbb{R}. \tag{2.9}$$

We return to harmonic analysis on SU(1,1). Let $n, m \in \mathbb{Z}$ and $\psi_\lambda^{n,m}(g)$ $(\lambda \in \mathbb{R},\ g \in G)$ denote the matrix coefficient of $\pi_{0,\lambda}(g)$ with K-type (n, m). Let f be a compactly supported C^∞ function on G. We define the scalar-valued spherical Fourier transform $\widetilde{f}_{n,m}(\lambda)$ by

$$\widetilde{f}_{n,m}(\lambda) = \int_0^\infty f(x)\psi_\lambda^{(n,m)}(x)\Delta_{0,0}(x)dx. \tag{2.10}$$

We recall that the explicit form of $\psi_\lambda^{n,m}(g)$ is given by using the Jacobi function (2.2) (cf. [5, (4.17)] and [6, (3.4.10)]): for $g = k_\phi a_x k_\psi \in G$,

$$\psi_\lambda^{n,m}(g) = (\cosh x)^{n+m}(\sinh x)^{|n-m|}Q_{n,m}(\lambda)\phi_\lambda^{|n-m|,n+m}(x)e^{in\phi}e^{im\psi},$$

where $Q_{n,m}(\lambda)$ can be expressed by binomial coefficient as

$$Q_{n,m}(\lambda) = \begin{pmatrix} -\frac{1}{2} - \frac{i\lambda}{2} \mp m \\ |n-m| \end{pmatrix}, \tag{2.11}$$

and $\mp m$ is equal to $-m$ if $m \geq n$ and m if $m \leq n$. Hence from (2.6) and (2.9) it follows that

$$\begin{aligned}\widetilde{f}_{n,m}(\lambda) = {}& 2^{-2(n+m)-2|n-m|}Q_{n,m}(\lambda) \\ & \times \Big(f(x)(\sinh x)^{-|n-m|}(\cosh x)^{-(n+m)}\Big)^\wedge_{|n-m|,n+m}(\lambda).\end{aligned} \tag{2.12}$$

We shall consider the case of $n = m$. Let F be a compactly supported C^∞ even function on $\mathbb{R}$. We put

$$f(g) = F(x)(\cosh x)^{2n}e^{in(\phi+\psi)}, \quad g = k_\phi a_x k_\psi \in G. \tag{2.13}$$

Then letting $\alpha = 0, \beta = 2n$ in (2.8) and (2.11), we see that

$$\int_0^\infty |f(x)|^2\Delta_{0,0}(x)dx = \int_0^\infty |\widetilde{f}_{n,n}(\lambda)|^2|C_{0,0}(\lambda)|^{-2}d\lambda + \sum_{k=0}^{|n|-1}\Big(k+\frac{1}{2}\Big)|\widetilde{f}_{n,n}((2k+1)i)|^2 \tag{2.14}$$

(see [6, (4.21)] and [8, Theorem 8.2]). This is nothing but the Plancherel formula for central compactly supported C^∞ functions on G. We denote by f_P and ${}^\circ f$ respectively the principal part and discrete part of f on G;

$$f = f_P + {}^\circ f.$$

Then (2.14) corresponds to the relation $\|f\|^2_{L^2(G)} = \|f_P\|^2_{L^2(G)} + \|{}^\circ f\|^2_{L^2(G)}$.

3 Asymptotic Behavior of Heat Kernels

When $\alpha \geq \beta \geq -\frac{1}{2}$, the asymptotic behavior of $h_t^{\alpha,\beta}(x)$ is well-known (see [1] and [3, Theorem 3.1]). In particular,

$$h_t^{0,0}(x) \sim t^{-1}e^{-\rho^2 t}e^{-\rho x}e^{-x^2/(4t)}(1+t+x)^{-1/2}(1+x). \tag{3.1}$$

In this section we shall treat the case of $\alpha, \beta = 0, 1, 2, \cdots$, and we shall investigate a leading term of $h_t^{\alpha,\beta}(x)$ when $x \to \infty$. In the following, we fix $t > 0$ and we denote $a = \frac{1}{4t}$ for simplicity.

For an even function f on $\mathbb{R}$ let $W^\sigma_\mu(f)$, $\mu \in \mathbb{C}$, $\sigma > 0$, denote the Weyl type fractional integral of f, which is defined by

$$W^\sigma_\mu(f)(y) = \Gamma(\mu)^{-1} \int_y^\infty f(x)(\cosh\sigma x - \cosh\sigma y)^{\mu-1} d(\cosh\sigma x) \tag{3.2}$$

for $\Re\mu > 0$ and is extended to an entire function in μ (see [4, (3,10), (3.11)]). Then it is known that

$$\widehat{f}_{\alpha,\beta}(\lambda) = \mathcal{F}(2^{3\alpha+3/2} W^1_{\alpha-\beta} \circ W^2_{\beta+1/2}(f)),$$

where $\mathcal{F}$ denotes the Euclidean Fourier transform (see [4, (3.7), (3.12)]). Therefore, letting $\alpha = \beta = 0$, it follows from (2.9) that

$$W^2_{1/2}(h^{0,0}_t)(x) = \frac{1}{2^2\sqrt{t}} e^{-t} e^{-ax^2} \tag{3.3}$$

and moreover, letting $\alpha = m,\ \beta = n$, $2^{3m+3/2} e^{t(m+n+1)^2} W^1_{m-n} \circ W^2_{n+1/2}(h^{m,n}_t)$ does not depend on m, n. Hence, it follows that

$$h^{m,n}_t = 2^{-3m} e^{-t((m+n+1)^2-1)} W^2_{-1/2-n} \circ W^1_{n-m} \circ W^2_{1/2}(h^{0,0}_t). \tag{3.4}$$

Lemma 3.1 *For* $n = 0, 1, 2, \cdots$,

$$W^2_{-n} \circ W^1_n(f)(x) = \sum_{l=0}^{n-1} c^n_l (\cosh x)^{-(n+l)} W^1_l(f)(x), \tag{3.5}$$

where $4c^n_l = c^{n-1}_l - (n+l-2)c^{n-1}_{l-1}$. *In particular,* $c^n_0 = 2^{-2n}$, $|c^n_{n-1}| = 2^{-2n}(2n-3)!!$, $c^n_l > 0$ *if* l *is even and* $c^n_l < 0$ *if* l *is odd, and*

$$|c^n_l| \le \frac{(2n-3)!!}{2^{2n}(n-1-l)!}, \quad 0 < l \le n-1.$$

Proof Since

$$W^2_{-1} = \frac{1}{2\sinh 2x}\frac{d}{dx} = \frac{1}{4\cosh x} W^1_{-1},$$

(3.5) and the recursive relation $4c^n_l = c^{n-1}_l - (n+l-2)c^{n-1}_{l-1}$ follows from the induction on n. In particular, $4c^n_0 = c^{n-1}_l$ and $4c^n_{n-1} = -(2n-3)c^{n-1}_{n-2}$, and thus, $c^n_0 = 2^{-2n} > 0$ and $|c^n_{n-1}| = 2^{-2n}(2n-3)!!$. The sign of general c^n_l follows from the recursive relation. Since $4^{n-1}(2n-3)|c^{n-1}_{l-1}| \le 4^n|c^n_l|$, it follows that $4|c^n_l| \le |c^{n-1}_l| + \frac{4(n+l-2)}{2n-3}|c^n_l|$ and thus, $4|c^n_l| \le \frac{2n-3}{n-l-1}|c^{n-1}_l|$. This means that

$$|c^n_l| \le \frac{(2n-3)!!}{2^{2(n-l-1)}(n-l-1)!(2l-1)!!}|c^{l+1}_l| = \frac{(2n-3)!!}{2^{2n}(n-1-l)!}.$$

Lemma 3.2. *Let* $p, q \ge 0$ *and suppose* $q = 0$ *if* $p = 0$. *Then there exists a positive constant* c *such that for all* $l = 0, 1, 2, \cdots,$ *and* $x \ge \max(1, \frac{1}{a})$,

$$W^1_l(e^{-ax^2-px}x^q) \le c2^{-l}(2a)^l e^{l^2/(4a)} e^{-ax^2-(p-l)x} x^{q-l}.$$

Here, if $l \geq 1$*, then* $(2a)^l x^{-l}$ *can be replaced by* $\Gamma(l)^{-1}(2a)^{-1}x^{-1}$*. Moreover, if* $2^{-l}e^{l^2/(4a)}$ *is replaced by* 2^{-2l}*, then the lower bound follows.*

Proof The case of $l = 0$ is obvious, so we may suppose that $l \geq 1$. Since $2\sinh x \leq e^x$, $s + x \leq (1+s)(1+x) \leq 2x(1+s)$ for $x \geq 1,\ s \geq 0$, and $e^{-px}(1+s)^q \leq c$, it follows from (3.2) that

$$\begin{aligned}
&\Gamma(l)W_l^1(e^{-ax^2-px}x^q)\\
&= \int_x^\infty e^{-as^2-ps}s^q(\cosh s - \cosh x)^{l-1}\sinh s ds\\
&= e^{-ax^2-px}\int_0^\infty e^{-as^2-ps-2axs}(x+s)^q\Big(2\sinh\Big(\frac{s}{2}+x\Big)\sinh\Big(\frac{s}{2}\Big)\Big)^{l-1}\sinh(s+x)ds\\
&\leq ce^{-ax^2-(p-l)x}x^q\int_0^\infty e^{-as^2+(l/2+1/2)s-2axs}\Big(\sinh\Big(\frac{s}{2}\Big)\Big)^{l-1}ds\\
&\leq c2^{-l}e^{-ax^2-(p-l)x}x^q\int_0^\infty e^{-as^2+ls-2axs}ds\\
&= c2^{-l}e^{l^2/(4a)}e^{-ax^2-(p-l)x}x^q(2ax)^{-1}, \qquad (3.6)
\end{aligned}$$

where we used the fact that $-as^2 + ls = -a(s - \frac{l}{2a})^2 + \frac{l^2}{4a}$. Since $\sinh x \leq xe^x$, the integral in (3.6) is also estimated as

$$\leq c2^{-l}e^{l^2/(4a)}c\int_0^\infty e^{-2axs}s^{l-1}ds = c2^{-l}ce^{l^2/(4a)}\Gamma(l)(2ax)^{-l}.$$

Therefore, we can deduce the first estimate.

We note that $\sinh x \geq \frac{xe^x}{2}$ for $0 \leq x \leq \frac{1}{2}$. Since $0 \leq \frac{1}{2ax} \leq \frac{1}{2}$ and $s + x \geq x$ for $s, x \geq 0$, it follows that

$$\begin{aligned}
\Gamma(l)W_l^1(e^{-ax^2-px}x^q) &\geq c2^{-2l}e^{-ax^2-(p-l)x}x^q\int_0^{1/(2bx)} e^{-as^2-(p-l)s-2axs}s^{l-1}ds\\
&\geq c2^{-2l}e^{-ax^2-(p-l)x}(2ax)^{-l}\int_0^1 e^{-s}s^{l-1}ds.
\end{aligned}$$

Since $\Gamma(l)^{-1}\int_0^1 e^{-s}s^{l-1}ds$ is bounded below, the lower estimate follows.

Lemma 3.3 *Let* $p, q \geq 0$ *and suppose* $q = 0$ *if* $p = 0$*. Then there exist a positive constants* c *such that for all* $l, n = 0, 1, 2, \cdots,$ *and* $x \geq \max(1, \frac{1}{a})$*,*

$$\begin{aligned}
&|W_{-1/2}^2((\cosh x)^{n+l}W_l^1(e^{-ax^2-px}x^q))|\\
&\leq c2^n(2a)^{-l}e^{l^2/(4a)}e^{-ax^2-(n+1+p)x}x^{q-l}((n+l)x^{-1/2} + x^{1/2}).
\end{aligned}$$

Here, if $l \geq 1$*, then* $(2a)^{-l}x^{-l}$ *can be replaced by* $\Gamma(l)^{-1}(2a)^{-1}x^{-1}$*. Moreover, if* $2^n e^{l^2/(4a)}$ *is replaced by* 2^{n-l}*, then the lower bound follows.*

Proof Since $W_{-1/2}^2 = W_{1/2}^2 \circ W_{-1}^2$, it follows that

$$\begin{aligned}
&W_{-1/2}^2 \circ ((\cosh x)^{-(n+l)}W_l^1)\\
&= \frac{1}{4}W_{1/2}^2 \circ (-(n+l)(\cosh x)^{-(n+l+2)}W_l^1 + (\cosh x)^{-(n+l+1)}W_{l-1}^1). \qquad (3.7)
\end{aligned}$$

Therefore, we need to estimate

$$W^2_{1/2}((\cosh x)^{-(n+l+2)}W^1_l(e^{-ax^2-px}x^q)), \quad l=-1,0,1,2,\cdots.$$

Substituting the estimate obtained in Lemma 3.2, we see that for $l \geq 0$,

$$\begin{aligned} &W^2_{1/2}((\cosh x)^{-(n+l+2)}W^1_l(e^{-ax^2-px}x^q)) \\ &\leq cc_0\int_x^\infty e^{-as^2-(n+2+p)s}s^{q-l}(\cosh 2s-\cosh 2x)^{-1/2}2\sinh 2s ds \\ &\leq cc_0 e^{-ax^2-(n+p)x}\int_0^\infty e^{-as^2-(n+p+2ax)s}(s+x)^{q-l}(\cosh 2(s+x)-\cosh 2x)^{-1/2}ds, \end{aligned}$$

where $c_0 = c2^n(2a)^l e^{l^2/(4a)}$. We note that, if $l \geq q$, then $(s+x)^{q-l} \leq x^{q-l}$, and if $l \leq q$, then $(s+x)^{q-l} \leq (2x(1+s))^{q-l}$ and $e^{-ps}(1+s)^{q-l} \leq e^{-ps}(1+s)^q \leq c$. Therefore, applying [3, (3.1)] to $(\cosh 2(s+x)-\cosh 2x)^{-1/2}$, we have that the above formula could be estimated as

$$\begin{aligned} &\leq cc_0 e^{-ax^2-(n+1+p)x}x^{q-l}\int_0^\infty e^{-ax^2-(n+1+2ax)s}\Big(\frac{1+2(x+s)}{s(x+s)}\Big)^{1/2}ds \\ &\leq cc_0 e^{-ax^2-(n+1+p)x}x^{q-l}\Big(\frac{1}{x}+1\Big)^{1/2}\int_0^\infty e^{-2axs}\frac{1}{\sqrt{s}}ds \\ &\leq cc_0 e^{-ax^2-(n+1+p)x}x^{q-l-1/2}. \end{aligned} \tag{3.8}$$

When $l=-1$, we note that $|W^1_{-1}(e^{-ax^2-px}x^q)| \leq c(1+x)^{q+1}e^{-ax^2-px}(\sinh x)^{-1}$. Hence, (3.8) is replaced by

$$\begin{aligned} &\leq c2^n e^{-ax^2-(n+p)x}\int_0^\infty e^{-as^2-(n+p+2ax)s}(s+x)^{q+1}(\cosh(s+x)-\cosh x)^{-1/2}ds \\ &\leq c2^n e^{-ax^2-(n+1+p)x}x^q\int_0^\infty e^{-2axs}\frac{1}{\sqrt{s}}((x+s)(1+2(x+s))^{1/2}ds. \end{aligned}$$

The last integral is dominated by $x^{1/2}$. Substituting these estimates to (3.7), we can deduce the desired upper estimate. Other desired estimates follow from Lemma 3.2 and the arguments used in [3, Theorem 3.1].

When $p=q=0$ and $l=0$, we have the following refinement.

Lemma 3.4 *For all* $n=0,1,2,\cdots$,

$$W^2_{-1/2}((\cosh x)^{-n}e^{-ax^2}) = c_0(\cosh x)^{-n}h_t^{0,0}(x) + O(2^n n e^{-ax^2-(n+1)x}x^{-1/2}),$$

where $c_0 = 2^2\sqrt{t}\,e^t$. *Here* $f=O(g)$ *means that* $\left|\frac{f(x)}{g(x)}\right| \leq C$ *when* $x \to \infty$. *If* C *depends on some parameters* γ, *then we use the symbol* $f = O_{(\gamma)}(g)$.

Proof Since $e^{-ax^2} = c_0 W^2_{1/2}(h_t^{0,0})$ (see (3.3)), the case of $n=0$ is obvious and moreover, for $n \geq 1$, it follows that

$$\begin{aligned} &c_0^{-1}W^2_{-1/2}((\cosh x)^{-n}e^{-ax^2}) \\ &= -\int_x^\infty \frac{d}{d\cosh 2s}(((\cosh x)^{-n}-(\cosh s)^{-n})W^2_{1/2}(h_t^{0,0})(s))(\cosh 2s-\cosh 2x)^{-1/2}d\cosh 2s \\ &\quad + (\cosh x)^{-n}h_t^{0,0}(x) \\ &= -\int_x^\infty((\cosh x)^{-n}-(\cosh s)^{-n})e^{-as^2}(\cosh 2s-\cosh 2x)^{-3/2}2\sinh 2s ds + (\cosh x)^{-n}h_t^{0,0}(x). \end{aligned}$$

We note that for $0 \le x \le s$,

$$(\cosh x)^{-n} - (\cosh s)^{-n} \le \frac{n(\cosh s - \cosh x)}{\cosh x (\cosh x)^n}.$$

Therefore, the similar argument in the proof of Lemma 3.3 (or [3, Theorem 3.1]) yields that the last integral is dominated by $2^n n e^{-ax^2-(n+1)x} x^{-1/2}$.

Now we shall obtain the asymptotic behavior of $h_t^{m,n}(x)$ as $x \to \infty$. It follows from (3.3), (3.4) and (3.5) that

$$h_t^{m,n} = c_0^{-1} 2^{-3m} e^{-t((m+n+1)^2-1)} \sum_{l=0}^{n-1} c_l^n W_{-1/2}^2((\cosh x)^{-(n+l)} W_{l-m}^1(e^{-ax^2})). \tag{3.9}$$

Since

$$W_{-m}^1(e^{-ax^2}) \sim_{(m)} (2ax)^m e^{-ax^2-mx},$$

here $f \sim g$ means that there exist positive constants c_1, c_2 such that $c_1 f(x) \le g(x) \le c_2 f(x)$. If c_1, c_2 depend on some parameters γ, then we use the symbol $f \sim_{(\gamma)} g$. Lemmas 3.4 implies that, when $x \to \infty$, the term corresponding to $l = 0$ contributes to the asymptotic behavior of $h_t^{m,n}(x)$:

Proposition 3.1 *We fix $t > 0$ and $m, n = 0, 1, 2, \cdots$. Then for $x, ax \ge 1$,*

$$h_t^{m,n}(x) \sim_{(t,m,n)} e^{-\rho^2 t} e^{-\rho x} e^{-x^2/(4t)} (1+x)^{m+1/2}. \tag{3.10}$$

Next we shall consider the behavior of $(\cosh x)^n h_t^{0,n}(x)$. Let $\epsilon > 0$ and we suppose that

$$x \ge \frac{1}{2} \log\left(\frac{1}{2^\epsilon - 1}\right) = x(\epsilon),$$

that is, $\cosh x \le 2^{-1+\epsilon} e^x$ if $x \ge x(\epsilon)$ and $x \to \infty$ if $\epsilon \to 0$. Then it follows from (3.9), Lemmas 3.3 and 3.4 that

$$(\cosh x)^n h_t^{0,n}(x) = 2^{n\epsilon} e^{-t((n+1)^2-1)} \Big(c_0^n h_t^{0,0}(x) + O\Big(\sum_{l=1}^{n-1} |c_l^n| e^{l^2/(4a)} \Gamma(l)^{-1} e^{-ax^2-x} n x^{-1/2} \Big) \Big).$$

We note that $e^{l^2/(4a)} \le e^{(n-1)^2/(4a)}$ and

$$\sum_{l=1}^{n-1} |c_l^n| \Gamma(l)^{-1} \le \sum_{l=1}^{n-1} \frac{(n-1)!}{2^{n+2}(n-l-1)!\Gamma(l)} = 2^{-4}(n-1).$$

Hence, it follows that

$$\begin{aligned}(\cosh x)^n h_t^{0,n}(x) &= 2^{n\epsilon} e^{-t((n+1)^2-1)} (c_0^n h_t^{0,0}(x) + O(e^{(n-1)^2/(4a)} e^{-ax^2-x} n^2 x^{-1/2})) \\ &= 2^{n\epsilon} 2^{-2n} e^{-t((n+1)^2-1)} h_t^{0,0}(x)(1 + O(n^2 2^{2n} e^{(n-1)^2/(4a)} x^{-1})).\end{aligned} \tag{3.11}$$

Letting $x \to \infty$, we have the following

Proposition 3.2 *We fix $t > 0$ and $n = 0, 1, 2, \cdots$. Then*

$$\lim_{x \to \infty} \frac{(\cosh x)^n h_t^{0,n}(x)}{h_t^{0,0}(x)} = 2^{-2n} e^{-t((n+1)^2-1)}. \tag{3.12}$$

4 Hardy's Theorem

We keep the notations in the previous section. We recall the proof of Hardy's theorem for the Jacobi transform of (α,β), $\alpha \geq \beta \geq -\frac{1}{2}$ (see [1, 3]). Then it is easy to see that Hardy's theorem for the Jacobi transform of (m, n), $m, n = 0, 1, 2, \cdots$, also holds:

Theorem 4.1 *Let $m, n = 0, 1, 2, \cdots$, and f be a measurable function on $\mathbb{R}_+$ satisfying*

(i) $g(x) = O_{(m,n)}(h^{m,n}_{1/(4a)}(x))$,

(ii) $\widehat{g}_{m,n}(\lambda) = O_{(m,n)}(e^{-b\lambda^2})$.

If $ab > \frac{1}{4}$, then $g = 0$, and if $ab = \frac{1}{4}$, then g is a constant multiple of $h^{m,n}_b(x)$.

Applying this theorem, we have Hardy's theorem on SU(1, 1) for a fixed K-type (see the example in [7]).

Theorem 4.2 *Let f be a measurable function on G of K-type (n, m), $n, m = 0, 1, 2, \cdots$, satisfying*

(i) $f(x) = O_{(n,m)}(h^{|n-m|,n+m}_{1/(4a)}(x)(\sinh x)^{|n-m|}(\cosh x)^{n+m})$,

(ii) $\widetilde{f}_{n,m}(\lambda) = O_{(n,m)}(Q_{n,m}(\lambda)e^{-b\lambda^2})$.

If $ab > \frac{1}{4}$, then $f = 0$, and if $ab = \frac{1}{4}$, then f is a constant multiple of $h^{|n-m|,n+m}_b(x)$ $\cdot(\sinh x)^{|n-m|}(\cosh x)^{n+m}$.

Proof Let $g(x) = f(x)(\sinh x)^{-|n-m|}(\cosh x)^{-(n+m)}$. Then

$$g(x) = O_{(n,m)}(h^{|n-m|,n+m}_{1/(4a)}(x)),$$

$$\widehat{g}_{|n-m|,n+m}(\lambda) = 2^{2(n+m)+2|n-m|}\widetilde{f}_{n,m}(\lambda)Q^{-1}_{n,m}(\lambda) = O_{(n,m)}(e^{-b\lambda^2})$$

(see (2.12)). Then Theorem 4.1 implies that, if $ab > \frac{1}{4}$, then $g = 0$, and if $ab = \frac{1}{4}$, then g is a constant multiple of $h^{|n-m|,n+m}_b(x)$ and thus, f is the desired form.

Let $L^2_{0+}(G)$ denote the subspace of $L^2(G)$ consisting of all f of the form

$$f = \sum_{n,m=0}^{\infty} f_{n,m},$$

where $f_{n,m}$ is of K-type (n, m).

Corollary 4.1 *Let f be in $L^2_{0+}(G)$ and satisfy for all $n, m = 0, 1, 2, \cdots$,*

(i) $f_{n,m}(x) = O_{(n,m)}(h^{0,0}_{1/(4a)}(x))$,

(ii) $\widetilde{f}_{n,m}(\lambda) = O_{(n,m)}(e^{-b\lambda^2})$.

If $ab > \frac{1}{4}$, then $f = 0$, and if $ab = \frac{1}{4}$, then f is of the form

$$f(g) = \sum_{n=0}^{\infty} a_n h^{0,2n}_b(x)(\cosh x)^{2n}e^{in(\phi+\psi)}, \tag{4.1}$$

where $g = k_\phi a_x k_\psi$ and $a_n \in \mathbb{C}$.

Proof Proposition 3.1 implies that

$$h_t^{|n-m|,n+m}(x)(\sinh x)^{|m-n|}(\cosh x)^{m+n} \sim_{(m,n)} h_t^{0,0}(x)(1+x)^{|m-n|} \quad \text{for } x, ax \geq 1. \tag{4.2}$$

Hence $f_{n,m}(x) = O_{(n,m)}(h_{1/(4a)}^{|n-m|,n+m}(x)\ (\sinh x)^{|n-m|}(\cosh x)^{n+m}(1+x)^{-|n-m|})$ and $\widetilde{f}_{n,m}(\lambda) = O_{(n,m)}\ (e^{-b\lambda^2}) = O_{(n,m)}(Q_{n,m}(\lambda)e^{-b\lambda^2})$ (see (2.11)). Hence Theorem 4.2 implies that, if $ab > \frac{1}{4}$, then $f_{n,m} = 0$, for all $n, m \in \mathbb{Z}$, and thus $f = 0$. If $ab = \frac{1}{4}$, then $f_{n,m}$ is a constant multiple of $h_b^{|n-m|,n+m}(x)(\sinh x)^{|n-m|}(\cosh x)^{n+m}$. Since $f_{n,m}(x) = O_{(n,m)}(h_{1/(4a)}^{0,0}(x))$, it follows that $|n-m| = 0$ (see (4.2)). Thereby, f must be of the desired form.

5 Main Theorem

We retain the notations and suppose that

$$f(g) = \sum_{n=0}^{\infty} a_n h_b^{0,2n}(x)(\cosh x)^{2n} e^{in(\phi+\psi)} \in L_{0+}^2(G).$$

We recall that

$$\widetilde{f}_{n,n}(\lambda) = a_n 2^{-4n} e^{-b((2n+1)^2+\lambda^2)} \tag{5.1}$$

(see (2.9) and (2.12)). Then letting $t = b = \frac{1}{4a}$ in (2.14), we obtain the L^2-norm of f on G as follows.

$$\begin{aligned}
\int_G |f(g)|^2 dg &= \sum_{n=0}^{\infty} |a_n|^2 \int_0^{\infty} |h_b^{0,2n}(x)(\cosh x)^{2n}|^2 \Delta_{0,0}(x)dx \\
&= \sum_{n=0}^{\infty} |a_n|^2 2^{-8n} e^{-2b(2n+1)^2} \\
&\quad \times \Big(\int_0^{\infty} e^{-2b\lambda^2} |C_{0,0}(\lambda)|^{-2} d\lambda + \sum_{k=0}^{n-1} \Big(k + \frac{1}{2}\Big) e^{2b(2k+1)^2} \Big).
\end{aligned} \tag{5.2}$$

We define the partial sum f_N, $N = 0, 1, 2, \cdots$, of f as

$$f_N(g) = \sum_{n=0}^{N} a_n h_b^{0,2n}(x)(\cosh x)^{2n} e^{in(\phi+\psi)}.$$

Then Proposition 3.2 implies that

$$\lim_{N\to\infty} \lim_{x\to\infty} h_b^{0,0}(x)^{-1} f_N(k_\phi a_x) = \sum_{n=0}^{\infty} a_n 2^{-4n} e^{-b((2n+1)^2-1)} e^{in\phi} = \sum_{n=0}^{\infty} d_n e^{in\phi} = F(\phi), \tag{5.3}$$

where $d_n = 2^{-4n} e^{-b((2n+1)^2-1)} a_n$. Obviously, (5.2) implies that

$$\|F\|_{L^2(\mathbb{T})} = c\|f_P\|_{L^2(G)} \quad \text{and} \quad \sum_{n=0}^{\infty} |d_n|^2 \Big(1 + \sum_{k=0}^{n-1} k e^{2b(2k+1)^2}\Big) \sim \|f\|_{L^2(G)}^2. \tag{5.4}$$

Since $\sum\limits_{n=0}^{\infty} |d_n|^2 < \infty$ and $\sum\limits_{n=1}^{\infty} |d_n|^2 (n-1) e^{2b(2n-1)^2} < \infty$, there exists a positive constant C such that $|d_n|(1+n)^{1/2} e^{b(2n-1)^2} \leq C$ for all $n = 0, 1, 2, \cdots$. This means that F is real analytic and

$$|a_n| 2^{-4n} (1+n)^{1/2} e^{-b(2n+1)^2} e^{b(2n-1)^2} \leq C \quad \text{for all } n = 0, 1, 2, \cdots.$$

Hence (5.1) implies

$$\widetilde{f}_{n,n}(\lambda) = O((1+n)^{-1/2} e^{-b(2n-1)^2} e^{-b\lambda^2}).$$

We introduce a subspace $A_b^2(\mathbb{T})$ of $H^2(\mathbb{T})$ as follows:

$$A_b^2(\mathbb{T}) = \Big\{ F(\phi) = \sum_{n=0}^{\infty} d_n e^{in\phi} \in H^2(\mathbb{T});\ \|F\|^2_{A_b^2(\mathbb{T})} = \sum_{n=0}^{\infty} |d_n|^2 \Big(1 + \sum_{k=0}^{n-1} k e^{2b(2k+1)^2}\Big) < \infty \Big\}.$$

For $F(\phi) = \sum_{n=0}^{\infty} d_n e^{in\phi} \in A_b^2(\mathbb{T})$, we define a function f on G as (4.1) with $a_n = 2^{4n} e^{b((2n+1)^2-1)} d_n$. Then (5.2) and (5.4) imply $\|f\|_{L^2(G)} \le c\|F\|^2_{A_b^2(\mathbb{T})}$. Clearly, $|f_{n.n}(x)| = |a_n| h_b^{0,2n}(x)(\cosh x)^{2n} = O_{(n)}(h_b^{0,0}(x))$ and (5.1) implies $\widetilde{f}_{n,n}(\lambda) = O_{(n)}(e^{-b\lambda^2})$. Moreover,

$$\lim_{N\to\infty} \lim_{x\to\infty} h_b^{0,0}(x)^{-1} f_N(k_\phi a_x) = F(\phi).$$

Finally, we have the following theorem.

Theorem 5.1 *Let $ab = \frac{1}{4}$. Let f be in $L^2_{0+}(G)$ and satisfy for all $n, m = 0, 1, 2, \cdots$,*

(i) $f_{n,m}(x) = O_{(n,m)}(h^{0,0}_{1/(4a)}(x))$,

(ii) $\widetilde{f}_{n,m}(\lambda) = O_{(n,m)}(e^{-b\lambda^2})$.

Then, as an L^2-function on $\mathbb{T}$,

(iii) $\lim_{N\to\infty} \lim_{x\to\infty} h_b^{0,0}(x)^{-1} f_N(k_\phi a_x) = F(\phi)$

exists and $F \in A_b^2(\mathbb{T})$. Here $\|F\|_{L^2(\mathbb{T})} = c\|f_P\|_{L^2(G)}$ and $\|F\|_{A_b^2(\mathbb{T})} \sim \|f\|_{L^2(G)}$. Let $F(\phi) = \sum_{n=0}^{\infty} d_n e^{in\phi}$ denote the Fourier series of F. Then f is uniquely determined as a central function

$$f(g) = \sum_{n=0}^{\infty} d_n 2^{4n} e^{b((2n+1)^2-1)} h_b^{0,2n}(x)(\cosh x)^{2n} e^{in(\phi+\psi)},$$

where $g = k_\phi a_x k_\psi$, and each $\widetilde{f}_{n,n}(\lambda)$ satisfies

$$\widetilde{f}_{n,n}(\lambda) = O((1+n)^{-1/2} e^{-b(2n-1)^2} e^{-b\lambda^2}).$$

Conversely, if $F \in A_b^2(\mathbb{T})$, then there exists a function $f \in L^2_{0+}(G)$ such that f satisfies (i), (ii) *and* (iii).

Remark 5.1 (1) We note that, if $f \in L^2_{0+}(G)$ is of the form in (4.1) and F is given by Theorem 5.1(iii), then $|f(g) - h_b^{0,0}(x)F(\phi)|$, $g = k_\phi a_x k_\psi$, is dominated as

$$\Big(\sum_{n=0}^{\infty} |a_n| 2^{n\epsilon} e^{-t((n+1)^2-1)} n^2 e^{(n-1)^2/(4a)}\Big) h_b^{0,0}(x) x^{-1}$$

(see (3.11)). Therefore, if this sum is finite, then we can replace Theorem 5.1(iii) by

(iii)′ $\lim_{x\to\infty} h_b^{0,0}(x)^{-1} f(k_\phi a_x) = F(\phi)$,

and deduce that $f \in L^1(G)$ and $f(x) = O(h_b^{0,0}(x))$.

(2) In Corollary 4.1 we can replace the condition (i) by

(i)′ $f(x) = O(h^{0,0}_{1/(4a)}(x))$,

because (i)′ implies (i). Also, in Theorem 5.1, it is true if we ignore the last statement of the existence of f for $F \in A^2_b(\mathbb{T})$. As remarked in (1), in order to construct $f \in L^2_{0+}(G)$ from $F \in A^2_b(\mathbb{T})$, which satisfies (i)′, (ii) and (iii), it is necessary to control the series in (1).

(3) In Corollary 4.1 and Theorem 5.1, if we replace the condition (i) by

(i)$'_P$ $(f_P)_{m,n}(x) = O_{(n,m)}(h^{0,0}_{1/(4a)}(x))$,

then $a_n = 0$ for $n \neq 0$, that is, f is K-biinvariant. Actually, since $\widetilde{f}_{m,n} = (f_P)^{\sim}_{m,n}$, (i)$'_P$ and (ii) imply that f_P is of the form in (4.1). Since f_P has no discrete part, (5.2) implies that a_n must be 0 if $n \neq 0$.

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