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Three Dimensional Interface Problems for Elliptic Equations

Lung'an YING*

Abstract The author studies the structure of solutions to the interface problems for second order linear elliptic partial differential equations in three space dimension. The set of singular points consists of some singular lines and some isolated singular points. It is proved that near a singular line or a singular point, each weak solution can be decomposed into two parts, a singular part and a regular part. The singular parts are some finite sum of particular solutions to some simpler equations, and the regular parts are bounded in some norms, which are slightly weaker than that in the Sobolev space H^2.

Keywords Elliptic equation, Interface problem, Singular line, Singular point, Particular solution
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1 Introduction

The structure of solutions near singular points for the interface problems of two dimensional elliptic equations has been studied by a number of works (see [2, 5, 6, 9–14]). The results can be summarized into a decomposition theorem. Solution near a singular point can be decomposed into two parts, a singular part and a regular part. The singular part is a finite sum of particular solutions with the form of $r^\alpha\varphi(\theta)$, or $r^\alpha\log^m r\varphi(\theta)$, where r is the distance to the singular point, and θ is the polar angle.

The aim of this paper is to study three dimensional problems. The set of singular points, which is one dimensional, is more complicated than that of two dimensional problems, where the set of singular points is finite. Therefore we will study the structure of solutions near singular lines and near singular points, the intersections of singular lines. The results on two dimensional problems will be applied here.

We consider the equation

$$Lu=\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big)+\sum_{i=1}^{3}b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=f(x),\quad x\in\Omega,\tag{1.1}$$

where $\Omega\subset\mathbb{R}^3$ is a polyhedral domain. We assume that Ω is decomposed into a finite number of polyhedral sub-domains $\Omega^{(k)}$, such that $\cup\overline{\Omega^{(k)}}=\overline{\Omega}$, and we assume that $a_{ij}\in C^1\big(\overline{\Omega^{(k)}}\big)$, $b_i\in L^\infty(\Omega)$, $c\in L^\infty(\Omega)$. The matrix (a_{ij}) is not necessarily symmetric, but the condition of

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*School of Mathematical Sciences, Xiamen University, Xiamen 361005, Fujian, China; Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing 100871, China.
E-mail: yingla@math.pku.edu.cn

ellipticity,

$$\sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \geq \chi|\xi|^2, \quad \forall\, \xi_i, \xi_j \in \mathbb{R}, \tag{1.2}$$

should be satisfied, where $\chi > 0$ is a constant. For simplicity, we impose the Dirichlet boundary condition,

$$u|_{x\in\partial\Omega} = 0, \tag{1.3}$$

on (1.1), where $\partial\Omega$ is the boundary. If 0 is not an eigenvalue of the operator L, then the problem (1.1), (1.3) admits a weak solution $u \in H_0^1(\Omega)$ provided $f \in H^{-1}(\Omega)$ (see [3]).

The edges of the sub-domains $\Omega^{(k)}$ will be generally known as singular line segments, and the vertices of them will be known as singular points. In the following the terminology "singular line" is understood as the open singular line segments.

We will recall some results on the two dimensional problems in the next section. Then we will study the singular lines in Section 3. Then we will study the singular points in the last three sections. In what follows we assume that the singular lines and the singular points are situated in the interior of the domain Ω. For those singular lines and singular points on the boundary the argument is analogous. We denote throughout this paper that C is a generic constant and the notations of the Sobolev norms $\|\cdot\|_s$ and semi-norms $|\cdot|_s$ are applied.

2 Two Dimensional Problems

We recall the results in [11] with some generalization. First of all we consider a homogeneous equation with piecewise constant coefficients, depending on a parameter $x_3 \in \overline{I}$, $I = (\alpha, \beta)$,

$$L_0 u = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{ij}(x_3)\frac{\partial u}{\partial x_i}\Big) = 0, \quad x \in D(o, r_0), \tag{2.1}$$

where a_{ij} depend on x_3 continuously, $D(o, r_0)$ is a disk with center o and radius r_0. The domain $D(o, r_0)$ is divided into some sectors S_m, $m = 1, \cdots, m_0$, by some rays starting from the origin. a_{ij} are constants in each sector for a given x_3 and satisfy the same elliptic condition as (1.2). Let (r, φ) be the polar coordinates with the origin o. We take a parameter $\xi \in (0, 1)$ and define $\Gamma_0 = \{(r, \theta); r = r_0\}$, and $\Gamma_k = \{(r, \theta); r = \xi^k r_0\}$. Denote the set $\{(r, \theta); 0 \leq r < \xi^k r_0\}$ by $\xi^k\Omega$. Then we have

Lemma 2.1 *Let $u \in C(\overline{I}; H^1(\Omega))$ be a solution to* (2.1) *for all $x_3 \in \overline{I}$. Then $u = u_1 + u_2$,*

$$u_1 = \sum_{n=1}^{N} c_n(x_3) r^{\alpha_n(x_3)} \log^{m_n(x_3)} r \varphi_n^{(x_3)}(\theta), \tag{2.2}$$

where $\alpha_n(x_3) > 0$, $\varphi_n^{(x_3)}$ are continuous and piecewise sufficiently smooth, and

$$\|u_2\|_{2,\xi\Omega\cap S_m} \leq C\|u\|_1, \quad \forall\, x_3 \in \overline{I},\ 1 \leq m \leq m_0. \tag{2.3}$$

u_2 and each term in (2.2) *are the solutions to* (2.1)*, and the constants N, C are independent of x_3.*

Proof Let $g = u|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$. Then $g \to u|_{\Gamma_1}$ defines a compact operator X in $H = H^{1/2}$ (see [11]). The eigenvalues are arranged as $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Two spectrum sets $\{\lambda_1, \cdots, \lambda_N\}$, $\{\lambda_{N+1}, \cdots, 0\}$ are defined such that $|\lambda_N| \geq \xi$ and $|\lambda_{N+1}| < \xi$. N depends on x_3, so it is denoted by $N(x_3)$. The space H is decomposed into two subspaces, $H = H_1 \oplus H_2$, accordingly. λ_n depends on x_3 continuously. For a given $x_3^{(0)} \in \overline{I}$, one has $N_0 = N(x_3^{(0)})$. There is a neighborhood of $x_3^{(0)}$ such that $|\lambda_{N_0+1}| < \xi$ for all x_3, so $N(x_3) \leq N_0$. We pick up a finite number of those neighborhoods covering the interval $\overline{I}$. Then we get an upper bound of $N(x_3)$, denoted by N. The constant N in (2.2) is thus obtained.

We construct a closed curve $\mathcal{C}$, such that $\lambda_{N+1}, \cdots, 0$ are situated in the interior of it, and $\lambda_1, \cdots, \lambda_N$ in the exterior of it. Denote by $R(\lambda, X)$ the resolvents of the operators X. In a neighborhood of $x_3^{(0)}$, it is analytic near $\mathcal{C}$. The projection operator from H to H_2 is

$$P = -\frac{1}{2\pi i} \int_{\mathcal{C}} R(\lambda, X)\, d\lambda.$$

Therefore P is bounded in this neighborhood. Using the same argument, we can prove that P is bounded on $\overline{I}$.

The remaining part of the proof is the same as that in [11], and thus is omitted.

Secondly, let us consider the nonhomogeneous equation

$$L_0 u = \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{ij}(x_3)\frac{\partial u}{\partial x_i}\Big) = f, \quad x \in \Omega = D(o, r_0). \tag{2.4}$$

Lemma 2.2 *If $f \in L^2(\Omega)$, then there is a particular solution u to* (2.4) *such that*

$$\|u\|_{1,\xi\Omega} + \|r^\gamma D^2 u\|_{0,\xi\Omega\cap S_m} \leq C\|r^\gamma f\|_0, \tag{2.5}$$

where $\gamma \in (0, b)$, b is a positive constant, and C depends on γ, but is independent of x_3.

Proof Let ε be a positive constant to be determined. For a fixed x_3, we define the spectrum sets like the previous lemma but require that $|\lambda_N| \geq \xi + \varepsilon$ and $|\lambda_{N+1}| < \xi + \varepsilon$. Since

$$\lim_{k\to\infty} \|X_{H_2}^k\|^{\frac{1}{k}} = |\lambda_{N+1}|,$$

where $\|\cdot\|$ stands for the spectrum norm and X_{H_2} is the operator X restricted on H_2, we have

$$\|X_{H_2}^k\| < (|\lambda_{N+1}| + \varepsilon)^k \tag{2.6}$$

for sufficiently large k. Let us fix one k, denoted by K_0. By continuity, there is a neighborhood of x_3 such that (2.6) holds with $k = K_0$, and $|\lambda_N| \geq \xi + \frac{\varepsilon}{2}$, $|\lambda_{N+1}| < \xi + \varepsilon$. Using the same argument as the previous lemma, we know that K_0 is uniformly bounded on $\overline{I}$.

Let $\Omega_k = \{(r, \theta); \xi^k > r > \xi^{k+1}\}$. Following the proof of Lemma 8 in [11], we define

$$f_k = \begin{cases} f, & x \in \Omega_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = \sum f_k$. Let u_k be the solution to the equation (2.4) on the whole plane, where f is replaced by f_k. Because u_k satisfies the homogeneous equation (2.1) on $\xi^{k+1}\Omega$, we have the decomposition $u_k = u_k^{(1)} + u_k^{(2)}$ with $u_k^{(1)}|_{\Gamma_{k+1}} \in H_1$ and $u_k^{(2)}|_{\Gamma_{k+1}} \in H_2$. We extend $u_k^{(1)}$ analytically to Ω which is still denoted by $u_k^{(1)}$. Let $u = \sum_{k=1}^{\infty}(u_k - u_k^{(1)})$. On the sub-domain Ω_l,

$$u = \sum_{k\geq l-1} u_k + \sum_{k<l-1} u_k^{(2)} - \sum_{k\geq l-1} u_k^{(1)}. \tag{2.7}$$

The estimate of the first and the third terms is the same as that in [11]. We study the second term only. Define a mapping $T_k : x \to \xi^k x$. Let $\widetilde{u} = u_k \circ T_k$. Then $\widetilde{u}$ satisfies $L_0\widetilde{u} = \xi^{2k} f_k \circ T_k$. Let $l = k + 2 + \kappa K_0 + l'$, where κ is a nonnegative integer and $0 \leq l' < K_0$. We notice that X is a bounded operator. Then using an estimate in [11], for the solutions of the equation (2.4), we have

$$\begin{aligned} \|(u_k^{(2)} \circ T_k)|_{\Gamma_{l-k-1}}\|_H &= \|X^{l-k-2}(u_k^{(2)} \circ T_k)|_{\Gamma_1}\|_H \\ &\leq C(|\lambda_{N+1}| + \varepsilon)^{\kappa K_0} \|(u_k^{(2)} \circ T_k)|_{\Gamma_1}\|_H \\ &\leq C(|\lambda_{N+1}| + \varepsilon)^{\kappa K_0} \xi^k \|f_k\|_0. \end{aligned}$$

Then by scaling and interior estimation, we get

$$|u_k^{(2)}|_{2,\Omega_l \cap S_m} \leq C(|\lambda_{N+1}| + \varepsilon)^{\kappa K_0} \xi^{k-l+1} \|f_k\|_0.$$

Consequently, we have

$$\Big|\sum_{k<l-1} u_k^{(2)}\Big|_{2,\Omega_l\cap S_m} \leq C\sum_{k=1}^{l-2}(|\lambda_{N+1}| + \varepsilon)^{\kappa K_0}\xi^{k-l+1}\|f_k\|_0 \leq C\sum_{k=1}^{l-2}\Big(\frac{\xi + 2\varepsilon}{\xi}\Big)^{l-k}\|f_k\|_0.$$

Combining the estimates for these three terms, we have

$$|u|^2_{2,\Omega_l\cap S_m} \leq C\Big(\xi^{-bl}\|r^{\frac{b}{2}} f\|^2_{0,\xi^{l-1}\Omega} + \Big(\sum_{k=1}^{l-2}\Big(\frac{\xi+2\varepsilon}{\xi}\Big)^{l-k}\|f\|_{0,\Omega_k}\Big)^2\Big),$$

where b is a sufficiently small positive constant. We multiply the both sides by $\xi^{2\gamma l}$ and then sum the inequality with respect to l. We have the estimate for the second term:

$$\begin{aligned} &C\sum_{l=1}^{\infty}\xi^{2\gamma l}\Big(\sum_{k=1}^{l-2}\Big(\frac{\xi+2\varepsilon}{\xi}\Big)^{l-k}\|f\|_{0,\Omega_k}\Big)^2 \\ \leq\; &C\sum_{l=1}^{\infty}\xi^{2\gamma l}\sum_{k=1}^{l-2}\Big(\frac{\xi}{\xi+2\varepsilon}\Big)^{l-k}\sum_{k=1}^{l-2}\Big(\frac{\xi+2\varepsilon}{\xi}\Big)^{3(l-k)}\|f\|^2_{0,\Omega_k} \\ \leq\; &C\sum_{l=1}^{\infty}\xi^{2\gamma l}\sum_{k=1}^{l-2}\Big(\frac{\xi+2\varepsilon}{\xi}\Big)^{3(l-k)}\|f\|^2_{0,\Omega_k} \\ =\; &C\sum_{k=1}^{\infty}\|f\|^2_{0,\Omega_k}\sum_{l=k+2}^{\infty}\xi^{2\gamma l}\Big(\frac{\xi+2\varepsilon}{\xi}\Big)^{3(l-k)}. \end{aligned}$$

Let ε satisfy $0 < \varepsilon < \frac{1}{2}(\xi^{1-\frac{2\gamma}{3}} - \xi)$. Then the right hand side is bounded by $C\|r^\gamma f\|_0$.

The estimate of the first term is the same as that in [11].

We remark that because we intend to get an estimate independent of x_3, the result of Lemma 2.2 is different from that in [11].

We are now in a position to consider the general equations. Let $u \in H^1(\Omega)$ be a solution to the equation

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{ij}(x,x_3)\frac{\partial u}{\partial x_i}\Big) + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x) + \sum_{i=1}^{2}\frac{\partial f_i(x)}{\partial x_i}, \tag{2.8}$$

where $f \in L^2$, $f_i \in H^1(S_m)$, $m = 1, 2, \cdots, m_0$, and $a_{ij} \in C(\overline{I}; C^1(\overline{S_m}))$, $b_i \in L^\infty$, $c \in L^\infty$. The matrix (a_{ij}) satisfies the condition of ellipticity.

Theorem 2.1 *u can be decomposed into two parts, $u = v + w$, and v, w possess the following properties:*

$$v = \sum_{n=1}^{N} u_n = \sum_{n=1}^{N} c_n(x_3) r^{\alpha_n(x_3)} \log^{m_n(x_3)} r \varphi_n^{(x_3)}(\theta), \tag{2.9}$$

where $\alpha_n(x_3) > 0$. Each term u_n is a particular solution to the homogeneous equation

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{ij}(0,x_3)\frac{\partial u}{\partial x_i}\Big) = 0, \tag{2.10}$$

where $a_{ij}(0,x_3)$ is a piecewise constant function, the coefficients a_{ij} frozen at o. v and w satisfy the estimation

$$\|v\|_1 + \|r^\gamma D^2 w\|_0 \le C\Big(\|u\|_1 + \|r^\gamma f\|_0 + \sum_m \sum_i |f_i|_{1,S_m}\Big), \tag{2.11}$$

where the constant C is independent of x_3 and γ is given in Lemma 2.2.

Proof The proof follows the same lines as that in [11]. The weak form of (2.8) can be written as

$$\int_{\xi\Omega} \sum_{i,j=1}^{2} a_{ij}(0,x_3)\frac{\partial u}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\,dx = \int_{\xi\Omega} \sum_{i,j=1}^{2}\Big((a_{ij}(0,x_3) - a_{ij}(x,x_3))\frac{\partial u}{\partial x_i} + f_j\Big)\overline{\frac{\partial v}{\partial x_j}}\,dx$$
$$+ \int_{\xi\Omega}\Big(\sum_{i=1}^{2} b_i\frac{\partial u}{\partial x_i} + cu - f\Big)\overline{v}\,dx, \quad \forall\, v \in H_0^1(\xi\Omega).$$

Then a function z in H^1 is defined such that it belongs to H^2 in each sub-domain S_m and

$$\sum_{i,j=1}^{2} a_{ij}(0,x_3)\frac{\partial z}{\partial x_i} n_j = \sum_{i,j=1}^{2}(a_{ij}(0,x_3) - a_{ij}(x,x_3))\frac{\partial u}{\partial x_i} n_j + \sum_{j=1}^{2} f_j n_j,$$

on the interfaces, where (n_1, n_2) is the unit exterior normal vector on the boundary of each S_m. The weak form is thus reduced to

$$\int_{\xi\Omega} \sum_{i,j=1}^{2} a_{ij}(0,x_3)\frac{\partial(u-z)}{\partial x_i}\overline{\frac{\partial v}{\partial x_j}}\,dx = \int_{\xi\Omega}\Big\{\sum_{i,j=1}^{2}\sum_m\Big(-\frac{\partial}{\partial x_j}\Big((a_{ij}(0,x_3) - a_{ij}(x,x_3))\frac{\partial u}{\partial x_i}\Big)$$
$$-\frac{\partial f_j}{\partial x_j} + \frac{\partial}{\partial x_j}\Big(a_{ij}(0,x_3)\frac{\partial z}{\partial x_i}\Big)\Big) + \sum_{i=1}^{2} b_i\frac{\partial u}{\partial x_i} + cu - f\Big\}\overline{v}\,dx.$$

By virtue of Lemma 2.2 a function w is taken to satisfy the above nonhomogeneous equation. Then $u - z - w$ is a solution to the equation (2.10). Using the results of Lemma 2.1, we have the desired decomposition of $u - z - w$. Then we regard $z + w + u_2$ as the function w, and u_1 the function v in the theorem, and the proof is completed.

3 Singular Lines

Without losing generality, we assume that the singular line of the solution u to the equation (1.1) is $x_1 = x_2 = 0, x_3 \in \overline{I}$. We consider a cylinder $S = D(o, r_0) \times I$, where r_0 is small enough so that there is no other singular point in S. We study the regularity and decomposition of the solution u.

The solution u is heterogeneous in S. We introduce some notations. As usual D^2u stands for the Hessian matrix $\big(\frac{\partial^2 u}{\partial x_i \partial x_j}\big)_{i,j=1}^3$. For our convenience, its sub-matrix $\big(\frac{\partial^2 u}{\partial x_i \partial x_j}\big)_{i,j=1}^2$ is denoted by $\partial^2 u$. A function $f \in L^2(S)$ is regarded as a mapping $x_3 (\in I) \to L^2(D(o, r_0))$ and it is expressed by $f \in L^2(I; L^2(D(o, r_0)))$, or in a simpler form, $f \in L^2(L^2)$. The above spaces L^2 can be replaced by other Banach spaces.

Lemma 3.1 $\frac{\partial u}{\partial x_3} \in H^1(S')$, *where* $S' \subset\subset S$.

Proof Denote by τ_h the shift operator, $\tau_h u(x_1, x_2, x_3) = u(x_1, x_2, x_3 + h)$, and by Δ_h the difference operator $\Delta_h = \frac{1}{h}(\tau_h - E)$, where E is the identity operator. Then we have

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\Big(\Delta_h a_{ij}\frac{\partial u}{\partial x_i} + \tau_h a_{ij}\frac{\partial \Delta_h u}{\partial x_i}\Big) = \Delta_h\Big\{f - \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} - cu\Big\}.$$

The H^{-1} norm of the right hand side is bounded, and

$$\Big\| \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\Big(\Delta_h a_{ij}\frac{\partial u}{\partial x_i}\Big)\Big\|_{-1}$$

is also bounded. Using the standard argument (see [3]), we can get the H^1-norm estimate for $\frac{\partial u}{\partial x_3}$.

Theorem 3.1 *The solution* u *of* (1.1) *can be decomposed in* S *into two parts:* $u = v + w$.

$$v = \sum_{n=1}^{N} u_n = \sum_{n=1}^{N} c_n(x_3) r^{\alpha_n(x_3)} \log^{m_n(x_3)} r \varphi_n^{(x_3)}(\theta),$$

where (r, θ) *are the polar coordinates on the* (x_1, x_2) *plane at each point* $x_3 \in I$, $\alpha_n(x_3) > 0$, *and* N, $\alpha_n(x_3)$, $m_n(x_3)$ *and the functions* $\varphi_n^{(x_3)}$ *depend on* $a_{ij}(0, 0, x_3)$ *only. Moreover, each term* u_n *is a particular solution to the homogeneous equation*

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{ij}(0, 0, x_3)\frac{\partial u}{\partial x_i}\Big) = 0, \tag{3.1}$$

$$\|v\|_{L^2(H^1)} + \|r^\gamma \partial^2 w\|_{L^2(L^2)} \le C(\|u\|_1 + \|r^\gamma f\|_0). \tag{3.2}$$

Proof The equation (1.1) can be rewritten as

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big) + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\Big(a_{3j}(x)\frac{\partial u}{\partial x_3}\Big)$$
$$+ \sum_{i=1}^{3} a_{i3}(x)\frac{\partial^2 u}{\partial x_i \partial x_3} + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i} + b_3(x)\frac{\partial u}{\partial x_3} + c(x)u = f(x).$$

By Lemma 3.1 $a_{3j}\frac{\partial u}{\partial x_3}$ piecewise belongs to H^1, and $a_{i3}\frac{\partial^2 u}{\partial x_i \partial x_3}$ and $b_3\frac{\partial u}{\partial x_3}$ belong to L^2. We notice that for almost all x_3, the solution $u(\,\cdot\,,\,\cdot\,,x_3)$ belongs to $H^1(D(o,r_0))$. Then by the results of Theorem 2.1, the conclusion follows.

The number $1-\min\limits_{x_3}\operatorname{Re}\alpha_1(x_3)$ describes the singularity of the solution. We take $\eta > 1-\min\limits_{x_3}\operatorname{Re}\alpha_1(x_3)$, $\eta<1$. Then by Theorem 3.1 and Lemma 3.1 we have

Corollary 3.1 *The solution u to the equation* (1.1) *satisfies*

$$\|r^\eta D^2 u\|_0 \le C(|u|_1 + \|f\|_0)$$

near a singular line.

The above result does not imply the regularity of the functions v and w with respect to x_3. In fact, $m_n(x_3)$ and $\varphi_n^{(x_3)}$ are in general discontinuous with respect to x_3. As a particular case, if a_{ij} are independent of x_3, then the spectrum of X is independent of x_3, some results for the regularity can be proved. We take derivatives of (2.1) and (2.4) with respect to x_3, and then follow the proof of Lemma 2.1 and Lemma 2.2 to get

Lemma 3.2 *Under the conditions of Lemma* 2.1 *if $u\in C^1(\overline{I};H^1(\Omega))$ and a_{ij} is independent of x_3, then $u_1\in C^1(\overline{I};H^1(D(0,r_0)))$, and*

$$|c_n|_{C^1} + \|u_2\|_{C^1(H^2)} \le C\|u\|_{C^1(H^1)},$$

where c_n is given in (2.2).

Lemma 3.3 *There is a particular solution u to the nonhomogeneous equation*

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big) = f(x_1,x_2,x_3),$$

such that

$$\|u\|_{C^1(H^1)} + \|r^\gamma \partial^2 u\|_{C^1(L^2)} \le C\|r^\gamma f\|_{C^1(L^2)}.$$

Then we have the result for the regularity with respect to x_3.

Corollary 3.2 *If $b_i, c\in C^1(\overline{I};L^\infty)$ and $f\in C^1(\overline{I};L^2)$, then the functions v and w in Theorem* 3.1 *satisfy*

$$|c_n|_{C^1} + \|r^\gamma\partial^2 w\|_{C^1(L^2)} \le C(\|u\|_{C^1(\overline{I};H^1)} + \|r^\gamma f\|_{C^1(\overline{I};L^2)}).$$

Remark 3.1 It is easy to see that if higher order derivatives of b_i, c, f exist, the higher order derivatives of c_n and w exist too.

4 Singular Points-Homogeneous Equations

Without losing generality, let the domain be $\Omega = B(o,1)$, a ball with center o and radius 1. We assume that o is a singular point, and there are a finite number of singular line segments, $\{l_i\}$, starting from o, which are the intersections of interfaces. We consider the homogeneous equation with piecewise constant coefficients

$$\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial u}{\partial x_i}\Big) = 0 \tag{4.1}$$

in this section. Let the boundary of Ω, $\partial\Omega = \Gamma_0$. The same notations $\xi \in (0,1)$, Γ_k, Ω_k, S_m, $H = H^{\frac{1}{2}}(\Gamma_0)$, and $X : H \to H$ are defined as in the two dimensional case. The definitions of them are not repeated here.

Lemma 4.1 *X is a compact operator.*

Proof Let $g \in H$ be the boundary value. Then the equation (4.1) admits a unique solution $u \in H^1(\Omega)$. For a given bounded set $\{g_n\} \subset H$, there is a subsequence of the solutions $\{u_n\}$, which is still denoted by $\{u_n\}$, converging to u in $H^1(\Omega)$ weakly. We may assume that $\xi > \frac{1}{2}$. Then we consider the domain $\{x; \frac{1}{2} < \rho < 1\}$, where ρ is the distance to o. For each singular line in it, we construct a cylinder S surrounding it as in the previous section. By Corollary 3.2 and Remark 3.1, $u_n = v_n + w_n$ and w_n are uniformly bounded in the norm $\|r^\gamma \partial^2 \cdot\|_{H^s(L^2)}$, where s can be arbitrarily large. Thus there is a subsequence of w_n converging in H^1 strongly on each domain $S \cap S_m$. There are only a finite number of terms in v_n, so converging subsequences also exist. Then it is easy to see that there is a subsequence of u_n converging in any compact sub-domain of $\{x; \frac{1}{2} < \rho < 1\} \cap \overline{S}_m$ in the norm of H^1 strongly, the limit of which is still u. Restricted on Γ_1, the subsequence converges in the norm of $H^{\frac{1}{2}}$.

By the singularity of solutions near singular lines, we define a weight function on Ω. Let $t \in \Gamma_0$ be a point, $t_i \in \Gamma_0$ be the intersections of the singular lines l_i with Γ_0, and $t - t_i$ be the vector from t_i to t. By Corollary 3.1, there is an exponent η_i corresponding to t_i. Let $x \in \Omega$, and the angle between l_i with the ray ox be θ_i. Then we set

$$\psi_0(x) = \begin{cases} \prod \theta_i^{\eta_i}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The following semi-norm with a weight ψ is denoted by

$$|u|_{2,\psi,\Omega} = \Big(\sum_m \|\psi D^2 u\|^2_{0,S_m} \Big)^{\frac{1}{2}}.$$

By Corollary 3.1, if $u \in H^1(\Omega)$ is a solution to (4.1), then $|u|_{2,\psi_0,\Omega'} < \infty$ in any compact sub-domain Ω'. Applying Corollary 3.2 to v and w, we find that the derivatives along singular lines can be any order, so it also holds that $|v|_{2,\psi_0,\Omega'} < \infty$ and $|w|_{2,\psi_0,\Omega'} < \infty$ in any compact sub-domain Ω'.

Let (λ, g) be an eigenpair of the operator X. Different from the two dimensional case the eigenfunction g is singular, and the singular points are t_i. We study the property of g.

Lemma 4.2 *Near a singular point t_i the eigenfunction g can be decomposed into two parts: $g = g_v + g_w$. g_v is a finite summation of some particular functions,*

$$g_v = \sum_{n=1}^{N} c_n |t - t_i|^{\alpha_n} \log^{m_n} |t - t_i| \varphi_n\Big(\frac{t - t_i}{|t - t_i|}\Big), \tag{4.2}$$

where $\alpha_n > 0$, φ_n are continuous and piecewise sufficiently smooth, and $\|g_w\|_{\frac{3}{2}-\delta} < \infty$ in $S_m \bigcap \Gamma_0$, where $\delta > 0$.

Proof We consider the solution u of the equation (4.1) with boundary data g. Then $u|_{\Gamma_1} = \lambda g$. We apply Theorem 3.1 in the domain Ω_1 to get $u = v + w$. The traces of v and w on Γ_1 possess the desired property.

Corollary 4.1 *If the elementary divisor is quadratic for an eigenvalue λ of X, $h \in N((X - \lambda E)^2)$, and $(X - \lambda E)h = g$, then the conclusion in Lemma 4.2 keeps true for h.*

By the Riesz-Schauder Theorem, the spectrum of X consists of isolated eigenvalues and the point 0. The eigenvalues are arranged as $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Two spectrum sets $\{\lambda_1, \cdots, \lambda_N\}$, $\{\lambda_{N+1}, \cdots, 0\}$ are defined such that $|\lambda_N| \geq \xi^{\frac{1}{2}}$ and $|\lambda_{N+1}| < \xi^{\frac{1}{2}}$. The space H is decomposed into two subspaces, $H = H_1 \oplus H_2$, accordingly, such that the spectrum of X_{H_1} in H_1 is just $\{\lambda_1, \cdots, \lambda_N\}$ and the spectrum of X_{H_2} in H_2 is $\{\lambda_{N+1}, \cdots, 0\}$. For any $g \in H$, we have a unique decomposition $g = g_1 + g_2$, $g_1 \in H_1$, $g_2 \in H_2$. Let u_1, u_2 be the solutions corresponding to g_1, g_2 respectively.

Theorem 4.1 *u_1 is a finite summation in the form of*

$$u_1 = \sum_{n=1}^{N} c_n \rho^{\alpha_n} \log^{m_n} \rho \varphi_n(t),$$

where $\alpha_n > -\frac{1}{2}$, φ_n can be decomposed into two parts, as stated in Lemma 4.2, $|c_n| \leq C\|u\|_{1,\Omega}$, and $|u_2|_{2,\psi_0,\xi\Omega} \leq C|u|_{1,\Omega}$.

Proof The proof follows the same lines as for two dimensional problems. We only need to check the regularity of u_2. By Corollary 3.1 we have

$$\|\psi_0 D^2 u_2\|^2_{0,\Omega_k \bigcap S_m} = \xi^{-k+1} \|\psi_0 D^2 (u_2 \circ T_{k-1})\|^2_{0,\Omega_1 \bigcap S_m} \leq C\xi^{-k+1} \|u_2 \circ T_{k-1}\|_{1,\Omega\setminus\overline{\xi^3\Omega}}.$$

We consider the boundary value problem on $\Omega\setminus\overline{\xi^3\Omega}$ and obtain

$$\|u_2 \circ T_{k-1}\|_{1,\Omega\setminus\overline{\xi^3\Omega}} \leq C(\|X^{k-1} g_2\|^2_H + \|X^{k+2} g_2\|^2_H).$$

Let $\varepsilon > 0$ and $|\lambda_{N+1}| + \varepsilon < \xi^{\frac{1}{2}}$. Then for k large enough, we have

$$\|\psi_0 D^2 u_2\|^2_{0,\Omega_k \bigcap S_m} \leq C\xi^{-k+1}((|\lambda_{N+1}| + \varepsilon)^{2(k-1)} + (|\lambda_{N+1}| + \varepsilon)^{2(k+2)})\|g_2\|^2_H.$$

Therefore $\sum\limits_{k=1}^{\infty} \|\psi_0 D^2 u_2\|^2_{0,\Omega_k \bigcap S_m}$ converges, which proves the assertion.

5 Singular Points-Nonhomogeneous Equations

We consider the equation

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\Big(a_{ij} \frac{\partial u}{\partial x_i}\Big) = f \tag{5.1}$$

in this section. We make one hypothesis:

(H) The exponents α_n in (4.2) satisfy $\mathrm{Re}(\alpha_n) > \frac{1}{3}$ for all n.

Then we prove an existence result for the problems on the entire space $\mathbb{R}^3$. Let the closure of $C_0^\infty(\mathbb{R}^3)$ with respect the norm $|\cdot|_1$ be $Z^1(\mathbb{R}^3)$, which is a Hilbert space with this norm (see [7]).

Lemma 5.1 *We assume that* (H) *holds, $\psi_0 f \in L^2(\mathbb{R}^3)$ and $\mathrm{supp}\, f \in B(o, 1)$. Then the equation* (5.1) *admits a unique solution $u \in Z^1(\mathbb{R}^3)$.*

Proof Let $v \in Z^1(\mathbb{R}^3)$. It is proved in [7] that the norm

$$\Big(\int |\nabla v|^2\, dx + \int \frac{v^2}{\rho^2+1}\, dx\Big)^{\frac{1}{2}}$$

is equivalent to $|v|_1$ in $Z^1(\mathbb{R}^3)$. It is also equivalent to the norm $\|v\|_1$ if the domain is the unit ball. We show that the inner product in L^2, (f,v), defines a bounded operator on $Z^1(\mathbb{R}^3)$. In fact by the Hölder inequality

$$|(f,v)| = \Big|\int_{r<1} fv\, dx\Big| \le \Big(\int_{r<1} \psi_0^2 |f|^2\, dx\Big)^{\frac{1}{2}} \Big(\int_{r<1} |v|^6\, dx\Big)^{\frac{1}{6}} \Big(\int_{r<1} \psi_0^{-3}\, dx\Big)^{\frac{1}{3}}.$$

By the embedding theorem $H^1 \to L^6$, and by the hypothesis (H), we can take $\eta_i < \frac{2}{3}$, which implies $\int \psi_0^{-3}\, dx < \infty$. Therefore

$$|(f,v)| \le C\|\psi_0 f\|_0 |v|_1.$$

Then by the Lax-Milgram Theorem the existence and uniqueness follows.

We define another weight

$$\psi_1(x) = \begin{cases} \prod \theta_i, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

Theorem 5.1 *If* (H) *holds and* $\psi_0 f \in L^2$*, there is a particular solution* u *in* $\Omega = B(o,1)$ *to the equation* (5.1) *such that*

$$\|u\|_{1,\xi\Omega} + |u|_{2,\psi_2,\xi\Omega} \le C\|\psi_0 f\|_0,$$

where

$$\psi_2(x) = \frac{\psi_1(x)}{(|\log\rho|+1)^M},$$

the integer M *depends on the equation* (4.1)*. Moreover, there is another solution* u *such that*

$$\|u\|_{1,\xi\Omega} + |u|_{2,\psi_1,\xi\Omega} \le C\|\psi_0(|\log\rho|+1)^M f\|_0,$$

provided the right hand side is bounded.

Proof We prove a weighted norm estimate in Ω_1. Let u be a weak solution to (5.1). Let $\widetilde{\Omega} \subset \Omega_1 \bigcap S_m$ and the distance $\mathrm{dist}(\widetilde{\Omega}, l_i) \ge d > 0$, $\forall\, i$, and let $\Omega' = \{x \in S_m; \mathrm{dist}\,(x,\widetilde{\Omega}) < \frac{d}{2}\}$. Then by the interior estimate

$$|u|_{2,\widetilde{\Omega}} \le C(d^{-1}|u|_{1,\Omega'} + \|f\|_{0,\Omega'}).$$

Let d be small enough, then $\Omega' \subset \Omega \setminus \overline{\xi^3\Omega}$. Consequently,

$$|u|_{2,\psi_1,\Omega_1} \le C(|u|_{1,\Omega\setminus\overline{\xi^3\Omega}} + \|\psi_1 f\|_{0,\Omega\setminus\overline{\xi^3\Omega}}). \tag{5.2}$$

Analogous to Lemma 2.2, we define

$$f_k = \begin{cases} f, & x \in \Omega_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then applying Lemma 5.1, we construct the solution u_k to (5.1) with the right hand side f_k. The solution u is given by (2.7). Applying (5.2), we see that the estimate for u is the same as in [11] except that $|u|_{2,\Omega_1\bigcap S_m}$ is replaced by $\|\psi_1 D^2 u\|_{0,\Omega_1\bigcap S_m}$.

6 Singular Points-General Linear Equations

We consider the equation (1.1) in this section. We consider one singular point and denote $\Omega = B(o,1)$.

Theorem 6.1 *Under the hypothesis* (H) *let* $u \in H^1(\Omega)$ *be a solution to the equation* (1.1). *Then* u *can be decomposed into* $u = v + w$. v *is a solution to the equation* (4.1), *where the coefficients are* $a_{ij}(x)$ *frozen at* o, *and* v *is a finite summation in the form of*

$$v = \sum_{n=1}^{N} c_n \rho^{\alpha_n} \log^{m_n} \rho \varphi_n(t),$$

where $\alpha_n > -\frac{1}{2}$, *and*

$$|c_n| + \|w\|_{1,\xi\Omega} + |w|_{2,\psi_2,\xi\Omega} \le C(\|u\|_{1,\Omega} + \|f\|_{0,\Omega}).$$

Moreover, we have

$$|c_n| + \|w\|_{1,\xi\Omega} + |w|_{2,\psi_1,\xi\Omega} \le C(\|u\|_{1,\Omega} + \|(|\log \rho| + 1)^M f\|_{0,\Omega}),$$

provided the right hand side is bounded.

Proof Analogous to the proof in [11], we obtain

$$|u|_{2,\psi_3,\xi\Omega} \le C(\|u\|_{1,\Omega} + \|\rho f\|_{0,\Omega}),$$

where $\psi_3(x) = \rho\psi_0(x)$. The weak form for the boundary value problem of (1.1) can be written as

$$\int_{\xi\Omega} \sum_{i,j=1}^{3} a_{ij}(0) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}}\, dx = \int_{\xi\Omega} \sum_{i,j=1}^{3} \Big((a_{ij}(0) - a_{ij}(x)) \frac{\partial u}{\partial x_i} \Big) \overline{\frac{\partial v}{\partial x_j}}\, dx$$
$$+ \int_{\xi\Omega} \Big(\sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu - f \Big) \overline{v}\, dx, \quad \forall\, v \in H_0^1(\xi\Omega).$$

We consider the open sets, $G_{kl} = \{x \in \Omega;\ \xi^{k-1} > \rho > \xi^{k+1},\ \frac{\varepsilon}{2^{l-1}} > |t - t_i| > \frac{\varepsilon}{2^{l+1}}\}$, $k = 1, 2, \cdots$, $l = 0, 1, \cdots$, $\varepsilon > 0$, and $G_{00} = \{x \in \Omega;\ |t - t_i| > \varepsilon,\ \forall\, i\}$. They cover the domain Ω. Let $\{\chi_{kl}\}$ be a partition of unity. Restricted to one sub-domain S_m, by the inverse trace theorem (see [8]), there exist $\widetilde{z}_{mkl} \in H^2(G_{kl} \cap S_m)$ such that

$$\widetilde{z}_{mkl} = 0, \quad \frac{\partial \widetilde{z}_{mkl}}{\partial n} = \xi^{k-1} \Big\{ \chi_{kl} \frac{(a_{ij}(0) - a_{ij}) \frac{\partial u}{\partial n} n_j}{n^T A n} \Big\} \circ T_{k-1}$$

on ∂S_m, and

$$\|\widetilde{z}_{mkl}\|_2 \le C \Big\| \xi^{k-1} \Big\{ \chi_{kl} \frac{(a_{ij}(0) - a_{ij}) \frac{\partial u}{\partial n} n_j}{n^T A n} \Big\} \circ T_{k-1} \Big\|_1,$$

where $n = (n_1, n_2, n_3)$ is the unit exterior normal vector on ∂S_m, and the matrix $A = (a_{ij}(0))$. Let $z_{mkl} = \widetilde{z}_{mkl} \circ T_{1-k}$ and $z = \sum\limits_{mkl} z_{mkl}$. Then

$$\frac{\partial z}{\partial n} = \frac{(a_{ij}(0) - a_{ij}) \frac{\partial u}{\partial n} n_j}{n^T A n}$$

on the boundary provided that ε is small enough. Moreover, z satisfies

$$|z|_{2,\psi_0,\xi\Omega} \le C(\|u\|_{1,\Omega} + \|\rho f\|_{0,\Omega}).$$

Integrating by parts we get

$$\begin{aligned}
&\int_{\xi\Omega} \sum_{i,j=1}^{3} a_{ij}(0) \frac{\partial(u-z)}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}}\, dx \\
&= \sum_m \int_{\xi\Omega \bigcap S_m} \Big\{ - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\Big((a_{ij}(0) - a_{ij}(x))\frac{\partial u}{\partial x_i}\Big) + \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\Big(a_{ij}(0)\frac{\partial z}{\partial x_i}\Big)\Big\}\overline{v}\, dx \\
&\quad + \int_{\xi\Omega} \Big(\sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu - f\Big)\overline{v}\, dx, \quad \forall\, v \in H_0^1(\xi\Omega).
\end{aligned}$$

Then applying the results of Theorem 4.1 and Theorem 5.1, we complete the proof. For details, see [11].

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