Chin. Ann. Math.
28B(4), 2007, 453–462
DOI: 10.1007/s11401-005-0300-z

Chinese Annals of Mathematics, Series B

Delay Induced Hopf Bifurcation of Small-World Networks

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Abstract In this paper, the stability and the Hopf bifurcation of small-world networks with time delay are studied. By analyzing the change of delay, we obtain several sufficient conditions on stable and unstable properties. When the delay passes a critical value, a Hopf bifurcation may appear. Furthermore, the direction and the stability of bifurcating periodic solutions are investigated by the normal form theory and the center manifold reduction. At last, by numerical simulations, we further illustrate the effectiveness of theorems in this paper.

Keywords Small-world networks, Time delay, Hopf bifurcation
2000 MR Subject Classification 34K15

1 Introduction

In 1998, Watts and Strogatz [1] proposed a model of small-world network. This kind of network is different from regular network and random network. It is a special model with a high degree of local clustering as well as a small average distance. Because of its promising potential of applications in biological, social and man-made systems, more and more attentions are paid to the research of small-world network recently (see [1–5]). In 1999, Newman and Watts [2] studied the behavior of the small-world model, which is concentrated on its scaling properties. Due to the existence of time delay and the nonlinear interaction in spreading, in 2001, Yang [3] considered a new model of small-world network which extended that in [2]. Moreover, by numerical simulations and analytic analysis, chaos was investigated in different type of small-world networks. In 2004, Li and Chen [9] considered the local stability and the Hopf bifurcation in small-world delayed networks, and nonlinear interaction strength was regarded as a parameter.

In order to further show the effect of delay on small-world networks, we will regard the delay as a parameter and investigate the stability and the Hopf bifurcation. This paper is organized as follows. In Section 2, we study the existence of the Hopf bifurcation and the local stability of the positive equilibrium for one dimensional and high dimensional small-world networks, respectively. In Section 3, by using the normal form theory and the center manifold reduction (see [6]), we derive the formulas which determine the direction, stability and period

Manuscript received July 15, 2005. Published online July 9, 2007.
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of the bifurcating periodic solutions. In Section 4, we give a numerical example to show the effectiveness of theorems in this paper. In Section 5, we conclude the paper.

2 Existence of Hopf Bifurcation and Local Stability of Positive Equilibrium

Consider one dimensional small-world networks with delay (see [3]) as follows

$$\frac{dV(t)}{dt} = \xi + V(t-\tau) - \mu\xi V^2(t-\tau), \tag{2.1}$$

where $V(t)$ is the total influenced volume, ξ is the Newman-Watts length scale, μ is a measure of nonlinear interaction, and τ is the time delay.

The initial condition of equation (2.1) is given by

$$V(s) = \phi(s), \quad s \in [-\tau, 0].$$

It easily follows that $V^* = \frac{1+\sqrt{1+4\mu\xi^2}}{2\mu\xi}$ is a positive equilibrium of equation (2.1). Let $u(t) = V(t) - V^*$. Then equation (2.1) can be written as

$$\frac{du(t)}{dt} = -\sqrt{1+4\mu\xi^2}\, u(t-\tau) - \mu\xi u^2(t-\tau). \tag{2.2}$$

The characteristic equation of linear part of equation (2.2) is given by

$$\lambda + \sqrt{1+4\mu\xi^2}\, e^{-\lambda\tau} = 0. \tag{2.3}$$

Let $\lambda = \pm i\omega$ $(\omega > 0)$ be a pair of purely imaginary roots of equation (2.3). Then

$$\begin{cases} \sqrt{1+4\mu\xi^2}\, \cos(\omega\tau) = 0, \\ \omega - \sqrt{1+4\mu\xi^2}\, \sin(\omega\tau) = 0. \end{cases}$$

Then

$$\begin{cases} \omega\tau = \dfrac{(2n+1)\pi}{2}, \\ \dfrac{(2n+1)\pi}{2\tau} - \sqrt{1+4\mu\xi^2}\,(-1)^n = 0, \end{cases}$$

where $n = 0, 1, 2, \cdots$.

Thus by solving the above equations, we have

$$\begin{cases} \tau_n = \dfrac{(2n+1)\pi}{2\sqrt{1+4\mu\xi^2}}, \\ \omega_n = \dfrac{(2n+1)\pi}{2\tau_n}, \quad n = 0, 2, 4, \cdots . \end{cases} \tag{2.4}$$

That is, if $\tau_n = \frac{(2n+1)\pi}{2\sqrt{1+4\mu\xi^2}}$ $(n = 0, 2, 4, \cdots)$, then equation (2.3) has a pair of purely imaginary roots $\pm i\sqrt{1+4\mu\xi^2}$ $(n = 0, 2, 4, \cdots)$.

Next, we consider the case that equation (2.3) has roots with positive real parts. Assume that $\alpha + i\omega$ is a root of equation (2.3) with $\tau_n = \frac{(2n+1)\pi}{2\sqrt{1+4\mu\xi^2}}$ $(n = 0, 2, 4, \cdots)$, where $\alpha,\ \omega > 0$.

From equation (2.3), we have

$$\begin{cases}\alpha + \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau_n}\cos(\omega\tau_n) = 0,\\ \omega = \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau_n}\sin(\omega\tau_n),\end{cases}$$

which implies that

$$\Big(m+\frac{1}{2}\Big)\pi < \omega\tau_n < (m+1)\pi, \quad m,n = 0,2,4,\cdots, \qquad (2.5)$$
$$\omega < \sqrt{1+4\mu\xi^2}.$$

Then, from the first equation of (2.4), we have

$$\frac{2m+1}{2n+1}\sqrt{1+4\mu\xi^2} < \omega < \frac{2m+2}{2n+1}\sqrt{1+4\mu\xi^2}, \quad m,n = 0,2,4,\cdots.$$

So, equation (2.3) may have roots with positive real parts except for $n=0$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of equation (2.3) satisfying

$$\alpha(\tau_n) = 0, \quad \omega(\tau_n) = \sqrt{1+4\mu\xi^2}.$$

By equation (2.3), we have

$$\begin{cases}\alpha + \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau}\cos(\omega\tau) = 0,\\ \omega - \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau}\sin(\omega\tau) = 0.\end{cases} \qquad (2.6)$$

Taking the derivative of α and ω with respect to τ in (2.6), we have

$$\begin{cases}\dfrac{d\alpha}{d\tau} + \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau}\Big(-\dfrac{d\alpha}{d\tau}\tau - \alpha\Big)\cos(\omega\tau) - \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau}\sin(\omega\tau)\Big(\dfrac{d\omega}{d\tau}\tau + \omega\Big) = 0,\\ \dfrac{d\omega}{d\tau} - \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau}\Big(-\dfrac{d\alpha}{d\tau}\tau - \alpha\Big)\sin(\omega\tau) - \sqrt{1+4\mu\xi^2}\, e^{-\alpha\tau}\cos(\omega\tau)\Big(\dfrac{d\omega}{d\tau}\tau + \omega\Big) = 0,\end{cases} \qquad (2.7)$$

which means

$$\frac{d\alpha}{d\tau}(\tau_n, \alpha(\tau_n), \omega(\tau_n)) = \frac{4+16\mu\xi^2}{4+(2n+1)^2\pi^2} > 0. \qquad (2.8)$$

When $\tau = 0$, the root of equation (2.3) has negative real parts. By the above analysis and [11, Lemma 2.2], we obtain the following lemma.

Lemma 2.1 (i) *If* $\tau \in [0, \tau_0)$, *all the roots of equation* (2.3) *have strictly negative real parts.*

(ii) *If* $\tau = \tau_0$, *equation* (2.3) *has a pair of purely imaginary roots* $\pm i\omega_0$ *and all other roots have strictly negative real parts.*

(iii) *If* $\tau > \tau_0$, *equation* (2.3) *has at least one pair of roots with positive real parts.*

Applying (2.8), Lemma 2.1 and [12, Chapter 11, Theorem 1.1], we have the following theorem.

Theorem 2.1 (i) *If* $\tau \in [0, \tau_0)$, *the positive equilibrium* V^* *of equation* (2.1) *is asymptotically stable.*

(ii) *If* $\tau > \tau_0$, *the positive equilibrium* V^* *of equation* (2.1) *is unstable.*

(iii) τ_n $(n = 0,2,4,\cdots)$ *are Hopf bifurcation values for equation* (2.1).

Next, we consider the high dimensional small-world networks with delay (see [3]) as follows:

$$\frac{d^d V(t)}{dt^d} = \xi^d + V(t-\tau) - \mu\xi^d V^2(t-\tau), \tag{2.9}$$

where $V(t),\ \xi,\ \mu$ and τ are the same as those of equation (2.1), and d is the dimension of the network.

The initial condition of equation (2.9) is given by

$$V(s) = \phi(s), \quad s \in [-\tau, 0].$$

It easily follows that $V^* = \frac{1+\sqrt{1+4\mu\xi^{2d}}}{2\mu\xi^d}$ is a positive equilibrium of equation (2.9). Let $u(t) = V(t) - V^*$. Then equation (2.9) can be written as

$$\frac{d^d u(t)}{dt^d} = -\sqrt{1+4\mu\xi^{2d}}\, u(t-\tau) - \mu\xi^d u^2(t-\tau). \tag{2.10}$$

The characteristic equation of linear part of equation (2.10) is given by

$$\lambda^d + \sqrt{1+4\mu\xi^{2d}}\, e^{-\lambda\tau} = 0. \tag{2.11}$$

When $\tau = 0$, equation (2.11) has at least a pair of roots with positive real parts for $d \geq 3$ and $d \in N$. Similarly to the analysis in one dimensional case, we obtain the following remarks.

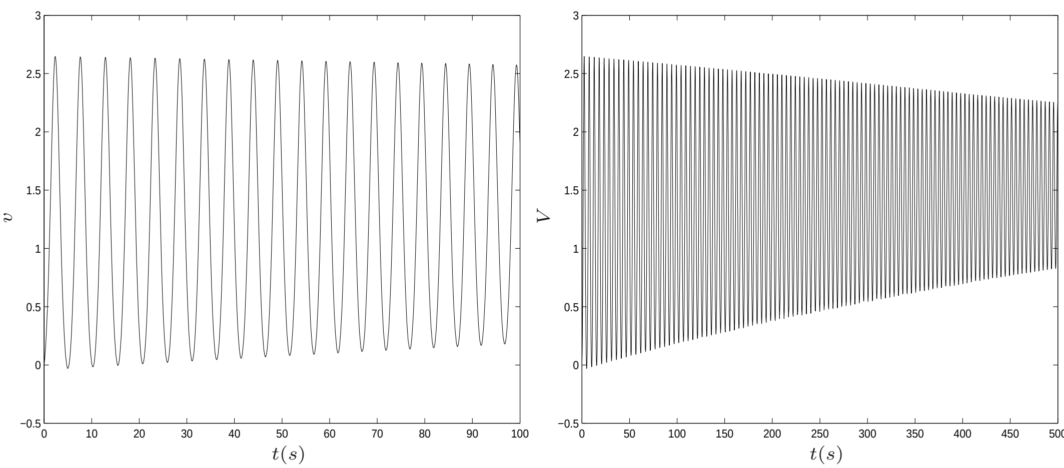

Figure 1 Waveform plots for system (2.9) with $d = 2,\ \tau = 0,\ \xi = 1,\ \mu = 1;\ V(0) = 0.001,\ V'(0) = 0.3$, for left figure $t = 100s$, for right figure $t = 500s$.

Remark 2.1 (i) For $d = 2$, if $\tau \in (0, +\infty)$, equation (2.11) has at least a pair of roots with positive real parts.

For $d \geq 3$ and $d \in N$, if $\tau \in [0, +\infty)$, equation (2.11) has at least a pair of roots with positive real parts.

(ii) If $d = 4k-1,\ k = 1, 2, \cdots$, equation (2.11) has a pair of purely imaginary roots $\pm i\omega_0$ at $\tau = \tau_n$, where $\tau_n = \frac{(2n+1)\pi}{2\sqrt[8k-2]{1+4\mu\xi^{8k-2}}},\ n = 1, 3, \cdots,\ \omega_0 = \sqrt[8k-2]{1+4\mu\xi^{8k-2}}$.

If $d = 4k-3,\ k = 1, 2, \cdots$, equation (2.11) has a pair of purely imaginary roots $\pm i\omega_0$ at $\tau = \tau_n$, where $\tau_n = \frac{(2n+1)\pi}{2\sqrt[8k-6]{1+4\mu\xi^{8k-6}}},\ n = 0, 2, 4, \cdots,\ \omega_0 = \sqrt[8k-6]{1+4\mu\xi^{8k-6}}$.

If $d=4k,\ k=1,2,\cdots$, equation (2.11) has a pair of purely imaginary roots $\pm i\omega_0$ at $\tau=\tau_n$, where $\tau_n=\frac{n\pi}{\sqrt[2d]{1+4\mu\xi^{2d}}},\ n=1,3,\cdots,\ \omega_0=\sqrt[2d]{1+4\mu\xi^{2d}}$.

If $d=4k-2,\ k=1,2,\cdots$, equation (2.11) has a pair of purely imaginary roots $\pm i\omega_0$ at $\tau=\tau_n$, where $\tau_n=\frac{n\pi}{\sqrt[2d]{1+4\mu\xi^{2d}}},\ n=0,2,4,\cdots,\ \omega_0=\sqrt[2d]{1+4\mu\xi^{2d}}$.

Remark 2.2 (i) For $d=2$, if $\tau\in(0,+\infty)$, the positive equilibrium V^* of equation (2.9) is unstable; for $d=2$ and $\tau=0$, the zero solution is a center of the corresponding linear equations of equation (2.10). By a numerical simulation, we know that the zero solution of equation (2.10) is a stable focus (see Figure 1). For $d\geq 3$ and $d\in N$, if $\tau\in[0,+\infty)$, the positive equilibrium V^* of equation (2.9) is unstable.

(ii) For $d=4k-1,\ k=1,2,\cdots,\ \tau_n$ are Hopf bifurcation values for equation (2.9), where $\tau_n=\frac{(2n+1)\pi}{2\sqrt[8k-2]{1+4\mu\xi^{8k-2}}},\ n=1,3,\cdots$.

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3 Stability and Direction of Bifurcating Periodic Solutions

For convenience, let $t=s\tau,\ u(s\tau)=X(s),\ \tau=\tau_0+\gamma,\ \gamma\in R$. Then equation (2.2) can be written as

$$\frac{dX(s)}{ds}=-(\tau_0+\gamma)[\sqrt{1+4\mu\xi^2}\,X(s-1)+\mu\xi X^2(s-1)]. \tag{3.1}$$

For $\varphi(\theta)\in C[-1,0]$, define a family of operators

$$L_\gamma\varphi=\int_{-1}^{0}d\eta(\theta,\gamma)\varphi(\theta),\quad F(\gamma,\varphi)=-(\tau_0+\gamma)\mu\xi\varphi^2(-1),$$

where $d\eta(\theta,\gamma)=-(\tau_0+\gamma)\sqrt{1+4\mu\xi^2}\,\delta(\theta+1)$ and δ is the Dirac function.

For $\varphi\in C^1[-1,0]$, define

$$A(\gamma)\varphi=\begin{cases}\dfrac{d\varphi(\theta)}{d\theta}, & \theta\in[-1,0),\\ \displaystyle\int_{-1}^{0}d\eta(s,\gamma)\varphi(s), & \theta=0,\end{cases}\qquad R(\gamma)\varphi=\begin{cases}0, & \theta\in[-1,0),\\ F(\gamma,\varphi), & \theta=0.\end{cases}$$

Then equation (3.1) can be written as

$$\dot{X}_t=A(\gamma)X_t+R(\gamma)X_t, \tag{3.2}$$

where $X_t(\theta)=X(t+\theta)$ for $\theta\in[-1,0]$.

For $\psi\in C^1[0,1]$, the adjoint operator A^* of A is defined as

$$A^*(\gamma)\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \displaystyle\int_{-1}^{0}d\eta(t,\gamma)\psi(-t), & s=0.\end{cases}$$

For $\varphi \in C([-1,0],R)$ and $\psi \in C([0,1],R)$, we define the bilinear form by

$$\langle \psi, \varphi \rangle = \overline{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \overline{\psi}(s-\theta)d\eta(\theta)\varphi(s)ds,$$

where $\eta(\theta) = \eta(\theta, 0)$.

From the discussion in Section 2, we know that $\pm i\tau_0\omega_0$ are eigenvalues of $A(0)$ and other eigenvalues have strictly negative real parts. Furthermore, they are eigenvalues of $A^*(0)$, too.

We easily obtain that

$$q(\theta) = e^{i\omega_0\tau_0\theta}, \quad \theta \in [-1,0]$$

is the eigenvector of $A(0)$ with respect to $i\omega_0\tau_0$, and

$$q^*(\theta) = Be^{i\omega_0\tau_0 s}, \quad s \in [0,1]$$

is the eigenvector of $A^*(0)$ with respect to $-i\omega_0\tau_0$, where

$$B = \frac{1}{1 - \tau_0\sqrt{1+4\mu\xi^2}\, e^{i\omega_0\tau_0}}.$$

Moreover,

$$\langle q^*, q\rangle = 1, \quad \langle q^*, \overline{q}\rangle = 0.$$

Let $z(t) = \langle q^*, X_t\rangle,\ w(t,\theta) = X_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\}$.

On the center manifold C_0 (refer to [6]), we have

$$w(t,\theta) = w(z(t), \overline{z}(t), \theta),$$

where $w(z,\overline{z},\theta) = w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} + w_{02}(\theta)\frac{\overline{z}^2}{2} + w_{30}\frac{z^3}{6} + \cdots$, z and $\overline{z}$ are the local coordinates for the center manifold C_0 in the directions of q^* and $\overline{q^*}$ respectively.

For the solution $X_t \in C_0$ of equation (3.1), since $\gamma = 0$, we have

$$\dot{z}(t) = i\tau_0\omega_0 z(t) + \langle q^*(\theta), F(0, w + 2\mathrm{Re}\{z(t)q(\theta)\})\rangle = i\tau_0\omega_0 z(t) + \overline{q^*}(0)F_0(z,\overline{z}), \tag{3.3}$$

where $F_0(z,\overline{z}) = F(0, w(z,\overline{z},\theta) + 2\mathrm{Re}\{z(t)q(\theta)\})$. Rewrite (3.3) as

$$\dot{z}(t) = i\tau_0\omega_0 z(t) + g(z,\overline{z}),$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{3.4}$$

By (3.2) and (3.3), we have

$$\dot{w} = \dot{X}_t - \dot{z}q - \dot{\overline{z}}\,\overline{q} = \begin{cases} Aw - 2\mathrm{Re}\{q^*(0)F_0 q(\theta)\}, & \theta \in [-1,0) \\ Aw - 2\mathrm{Re}\{q^*(0)F_0 q(\theta)\} + F_0, & \theta = 0 \end{cases} := Aw + H(z,\overline{z},\theta),$$

where $H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}\frac{\overline{z}^2}{2} + \cdots$.

Expanding the above series and comparing the coefficients, we obtain

$$\begin{cases} (A - 2i\omega_0\tau_0)w_{20}(\theta) = -H_{20}(\theta), \\ Aw_{11}(\theta) = -H_{11}(\theta). \end{cases} \tag{3.5}$$

$$\begin{aligned}
F_0(z,\overline{z}) &= -\tau_0\mu\xi X_t^2(t-1) = -\mu\xi\tau_0[w(z,\overline{z},-1)+ze^{-i\omega_0\tau_0}+\overline{z}e^{i\omega_0\tau_0}]^2\\
&= -\mu\xi\tau_0[z^2e^{-2i\omega_0\tau_0}+\overline{z}^2e^{2i\omega_0\tau_0}+2z\overline{z}+2w(z,\overline{z},-1)ze^{-i\omega_0\tau_0}\\
&\quad +2w(z,\overline{z},-1)\overline{z}e^{i\omega_0\tau_0}+\cdots]\\
&= -\mu\xi\tau_0[z^2e^{-2i\omega_0\tau_0}+\overline{z}^2e^{2i\omega_0\tau_0}+2z\overline{z}+2w_{11}(-1)e^{-i\omega_0\tau_0}z^2\overline{z}+w_{02}(-1)e^{i\omega_0\tau_0}\overline{z}^2z\\
&\quad +w_{20}(-1)e^{i\omega_0\tau_0}z^2\overline{z}+2w_{11}(-1)e^{i\omega_0\tau_0}z\overline{z}^2+\cdots]\\
&= -\mu\xi\tau_0e^{-2i\omega_0\tau_0}z^2-2\mu\xi\tau_0z\overline{z}-\mu\xi\tau_0e^{2i\omega_0\tau_0}\overline{z}^2-\mu\xi\tau_0[2e^{-i\omega_0\tau_0}w_{11}(-1)\\
&\quad +e^{i\omega_0\tau_0}w_{20}(-1)]z^2\overline{z}+\cdots\\
&= \mu\xi\tau_0z^2-2\mu\xi\tau_0z\overline{z}+\mu\xi\tau_0\overline{z}^2-\mu\xi\tau_0i[-2w_{11}(-1)+w_{20}(-1)]z^2\overline{z}+\cdots.
\end{aligned}$$

So

$$g(z,\overline{z}) = \overline{q^*}(0)F_0(z,\overline{z}) = \overline{B}\{\mu\xi\tau_0z^2-2\mu\xi\tau_0z\overline{z}+\mu\xi\tau_0\overline{z}^2-\mu\xi\tau_0i[-2w_{11}(-1)+w_{20}(-1)]z^2\overline{z}+\cdots\}.$$

Comparing the above coefficients with those in (3.4), we have

$$\begin{aligned}
g_{20} &= 2\mu\xi\tau_0\overline{B}, \quad g_{11} = -2\mu\xi\tau_0\overline{B},\\
g_{02} &= 2\mu\xi\tau_0\overline{B}, \quad g_{21} = -2\mu\xi\tau_0\overline{B}i[-2w_{11}(-1)+w_{20}(-1)].
\end{aligned}$$

Next, we will compute $w_{11}(-1)$ and $w_{20}(-1)$.

Since $H(z,\overline{z})|_{\theta=0} = -2\mathrm{Re}\{\overline{q^*}(0)F_0q(0)\}+F_0$. We have

$$\begin{aligned}
H_{20}(0) &= -g_{20}q(0)-\overline{g}_{02}\overline{q}(0)+2\mu\xi\tau_0,\\
H_{11}(0) &= -g_{11}q(0)-\overline{g}_{11}\overline{q}(0)-2\mu\xi\tau_0.
\end{aligned}$$

By (3.5),

$$-\tau_0\sqrt{1+4\mu\xi^2}\,w_{20}(-1) = 2i\omega_0\tau_0w_{20}(0)-H_{20}(0), \tag{3.6}$$

$$-\tau_0\sqrt{1+4\mu\xi^2}\,w_{11}(-1) = -H_{11}(0). \tag{3.7}$$

It follows easily that

$$w_{11}(-1) = \frac{H_{11}(0)}{\tau_0\sqrt{1+4\mu\xi^2}}.$$

Next, we need only to computer $w_{20}(-1)$.

By (3.5) and the definition of A, we have

$$\dot{w}_{20}(\theta) = 2i\omega_0\tau_0w_{20}(\theta)-g_{20}q(0)e^{i\omega_0\tau_0\theta}-\overline{g}_{02}\overline{q}(0)e^{-i\omega_0\tau_0\theta}.$$

Solving for w_{20}, we get

$$w_{20}(\theta) = \frac{g_{20}}{i\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta}-\frac{\overline{g}_{20}}{3i\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta}+E_1e^{2i\omega_0\tau_0\theta}. \tag{3.8}$$

Substituting (3.8) into (3.6), we have

$$E_1 = \frac{-\sqrt{1+4\mu\xi^2}\,[\frac{g_{20}}{i\omega_0}e^{-i\omega_0\tau_0}-\frac{\overline{g}_{20}}{3i\omega_0}e^{i\omega_0\tau_0}]-2g_{20}+\frac{2}{3}\overline{g}_{20}+H_{20}}{\tau_0\sqrt{1+4\mu\xi^2}e^{-2i\omega_0\tau_0}+2i\omega_0\tau_0}.$$

So, $w_{20}(0) = \frac{g_{20}}{i\omega_0\tau_0}-\frac{\overline{g}_{20}}{3i\omega_0\tau_0}+E_1$.

Then, by (3.6) and (3.7), we have

$$w_{20}(-1) = \frac{2g_{20} - \frac{2}{3}\overline{g}_{20} + 2i\omega_0\tau_0 E_1 - H_{20}(0)}{-\tau_0\sqrt{1+4\mu\xi^2}}.$$

So,

$$g_{21} = -2\mu\xi\overline{B}i\Big[\frac{-2H_{11}(0)}{\sqrt{1+4\mu\xi^2}} + \frac{g_{20}}{i\omega_0} - \frac{\overline{g}_{20}}{3i\omega_0} + E_1\tau_0\Big].$$

Thus, we can compute the following quantities

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0\tau_0}\Big[g_{11}g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\Big] + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}C_1(0)}{\alpha'(\tau_0)}, \\ \beta_2 &= 2\mathrm{Re}C_1(0), \\ \tau_2 &= -\frac{\mathrm{Im}C_1(0) + \mu_2 w'(\tau_0)}{\omega_0\tau_0}. \end{aligned} \tag{3.9}$$

Then, we give the main result in this section.

Theorem 3.1 *In formulas* (3.9)*, the direction of Hopf bifurcation is determined by* μ_2*; the stability of bifurcating periodic solutions is determined by* β_2*; the period of the bifurcating periodic solutions is determined by* T_2*.*

4 Numerical Simulation

In this section, we consider an example of system (2.1) with $\xi = 1,\ \mu = 1$. By (2.4), we find that $\tau_0 = 0.7025$.

From (3.9), it follows that $\mu_2 = 0.2847,\ T_2 = 0.5926,\ \beta_2 = -0.8212$.

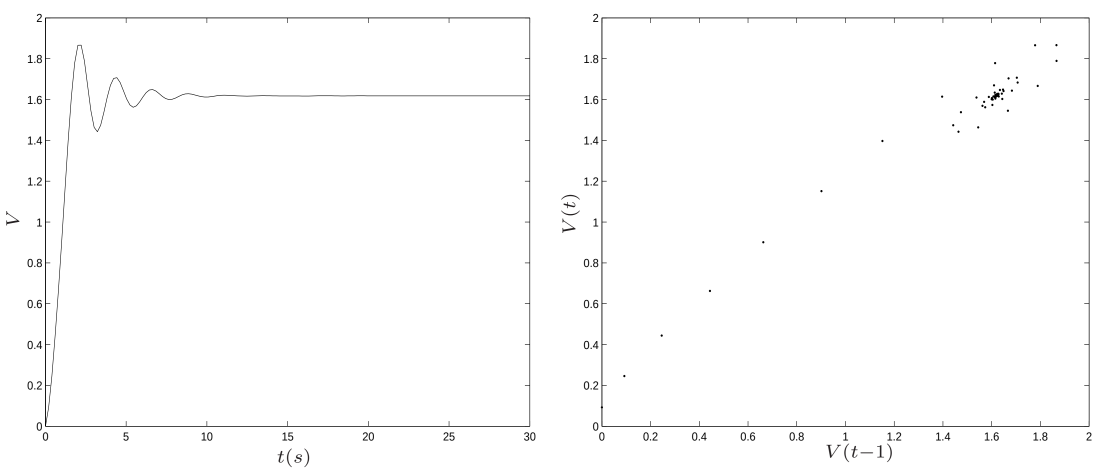

Figure 2 Waveform plot and phase plot for system (2.1) with $\tau = 0.5$.

These calculations show that the equilibrium is stable when $\tau < \tau_0$ as is illustrated by computer simulations (Figure 2: $\tau = 0.5$, Figure 3: $\tau = 0.7$). When τ passes through the critical value $\tau = 0.7025$, the equilibrium loses its stability and a Hopf bifurcation occurs, i.e.,

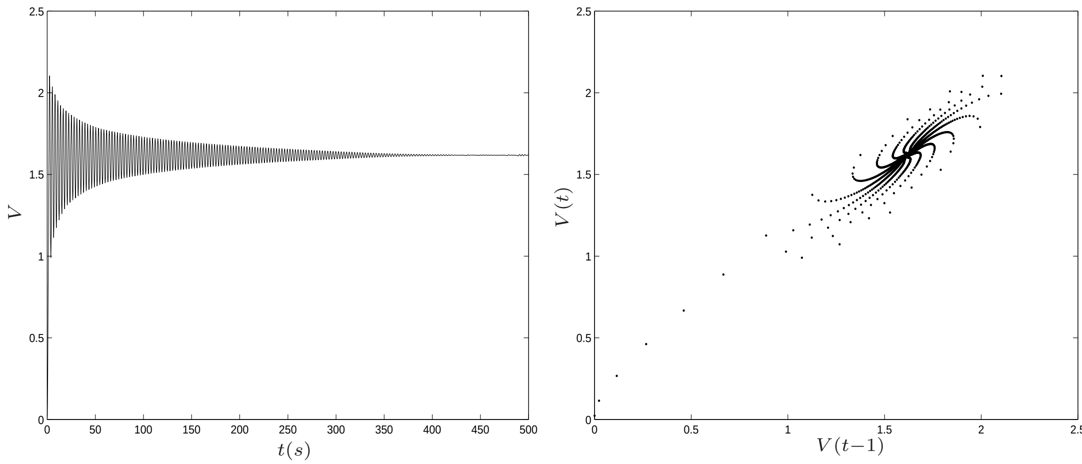

Figure 3 Waveform plot and phase plot for system (2.1) with $\tau = 0.7$.

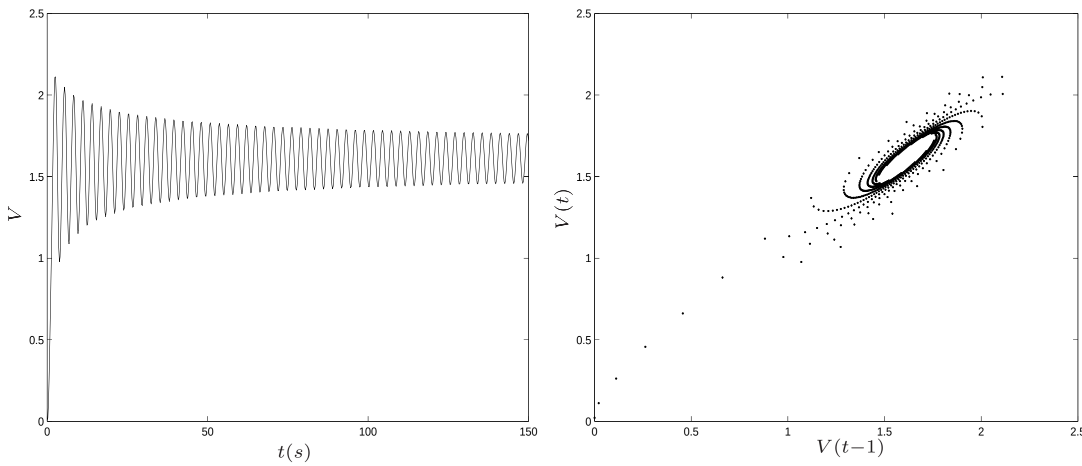

Figure 4 Waveform plot and phase plot for system (2.1) with $\tau = 0.705$.

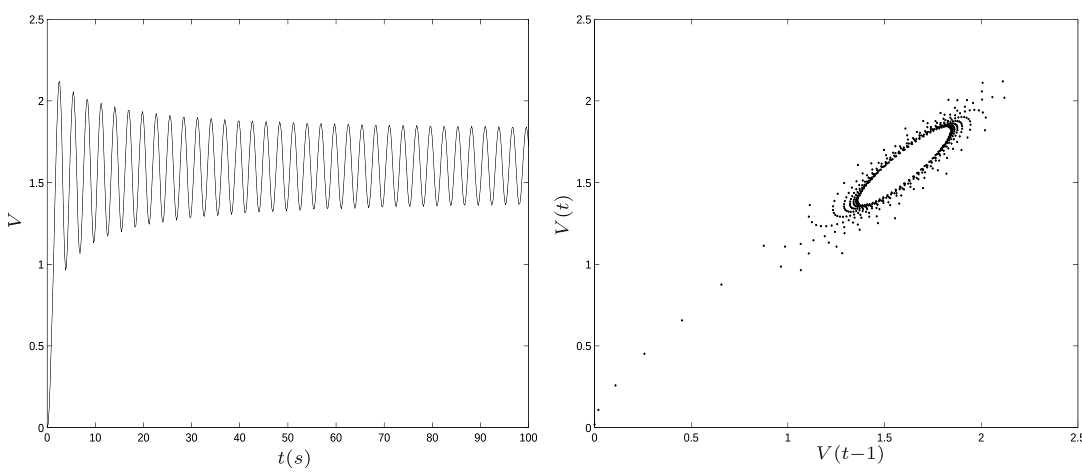

Figure 5 Waveform plot and phase plot for system (2.1) with $\tau = 0.71$.

a periodic solution bifurcates from the equilibrium. The periodic orbit is stable since $\beta_2 < 0$. Since $\mu_2 > 0$, the bifurcating periodic solutions exist at least for the values of τ slightly larger than the critical value $\tau_0 = 0.7025$. For $\tau = 0.705$, as predicted by the theory, Figure 4 shows that there is an orbitally stable limit cycle. Since $T_2 > 0$, the period of the periodic solutions increases as τ increases. For $\tau = 0.71$, the phase plot and the waveform plot are shown in Figure 5. Comparing Figure 4 with Figure 5, we can see that the period of the orbit with $\tau = 0.71$ is larger than that with $\tau = 0.705$.

5 Conclusions

In this paper, a small-world network with nonlinear interaction and time delay in the one-dimensional case and higher dimensional case have been studied. By using the delay time as the bifurcation parameter, we have shown that a Hopf bifurcation occurs when this parameter passes through a critical value. We have also determined the stability and the direction of the bifurcating periodic orbits by applying the normal form theory and the center manifold reduction. Our simulation results have verified and demonstrated the correctness of the theoretical results. We will further investigate the complex dynamics behavior of higher dimensional small-world networks and its control.

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