Chin. Ann. Math.
28B(4), 2007, 463–474
DOI: 10.1007/s11401-005-0234-5

Chinese Annals of Mathematics, Series B

Lefschetz Formulae for *p*-Adic Groups

Anton DEITMAR*

Abstract In this paper, Lefschetz formulae for torus actions on p-adic groups are proven. These are similar to comparable formulae for real Lie groups. Applications lie in the realm of dynamical zeta functions.

Keywords Arithmetic groups, Trace formula, Lefschetz formula
2000 MR Subject Classification 11S40, 11F72, 11F75, 22E40

0 Introduction

In paper [5], the author has extended the theory of Selberg-type zeta functions to higher rank p-adic groups. This extension remained incomplete insofar as within a higher rank group only elements of splitrank one were considered.

In the analogous setting of real Lie groups, it emerged in recent years that the role of the Selberg zeta function in higher rank spaces is played by certain Lefschetz formulae attached to torus actions (see [6]).

We explain this in more detail. Recall for rank one groups the relation between the Selberg zeta function and the trace formula. It is clear that the analytical properties of the zeta function are derived by means of the trace formula. Less well known is the fact that one can deduce the trace formula from the location of poles and zeros of the zeta function by evaluating a contour integral. Likewise, in the case of higher rank groups and splitrank one elements, the Selberg zeta function corresponds to a Lefschetz formula attached to the action of a minimal split torus. There is no proper analogue of the zeta function for higher rank elements, but there is a Lefschetz formula for that case, too. A special version of the Lefschetz formula in the real setting was shown in [6] and a general version in [7]. In the present paper, we give the general Lefschetz formula in the p-adic setting.

1 The Trace Formula

Let F be a nonarchimedean local field with valuation ring $\mathcal{O}$ and uniformizer ϖ. Denote by G a semisimple linear algebraic group over F. Let $K \subset G$ be a good maximal compact subgroup. Choose a parabolic subgroup $P = LN$ of G with Levi component L. Let A denote the largest split torus in the center of L. Then A is called the split component of P. Let $\Phi = \Phi(G, A)$ be the root system of the pair (G, A), i.e., Φ consists of all homomorphisms

Manuscript received June 9, 2005. Published online July 9, 2007.
*Mathematisches Institut, Auf der Morgenstelle 10, 72076 Tübingen, Germany.
E-mail: deitmar@uni-tuebingen.de

$\alpha : A \to \mathrm{GL}_1$ such that there is X in the Lie algebra of G with $\mathrm{Ad}(a)X = a^\alpha X$ for every $a \in A$. Given α, let $\mathfrak{n}_\alpha$ be the Lie algebra generated by all such X and let N_α be the closed subgroup of N corresponding to $\mathfrak{n}_\alpha$. Let $\Phi^+ = \Phi(P, A)$ be the subset of Φ consisting of all positive roots with respect to P. Let $\Delta \subset \Phi^+$ be the subset of simple roots. Let $A^- \subset A$ be the set of all $a \in A$ such that $|a^\alpha| < 1$ for any $\alpha \in \Delta$.

There is a reductive subgroup M of L with compact center such that MA has finite index in L. We can choose M such that $K_M = M \cap K$ is a good maximal subgroup of M. An element g of G is called elliptic, if it is contained in a compact torus. Let M_{ell} denote the set of elliptic elements in M.

Let $X^*(A) = \mathrm{Hom}(A, \mathrm{GL}_1)$ be the group of all homomorphisms as algebraic groups from A to GL_1. This group is isomorphic to $\mathbb{Z}^r$ with $r = \dim A$. Likewise let $X_*(A) = \mathrm{Hom}(\mathrm{GL}_1, A)$. There is a natural $\mathbb{Z}$-valued pairing

$$\begin{aligned} X^*(A) \times X_*(A) &\to \mathrm{Hom}(\mathrm{GL}_1, \mathrm{GL}_1) \cong \mathbb{Z}, \\ (\alpha, \eta) &\mapsto \alpha \circ \eta. \end{aligned}$$

For every root $\alpha \in \Phi(A, G) \subset X^*(A)$, let $\breve{\alpha} \in X_*(A)$ be its coroot. Then $(\alpha, \breve{\alpha}) = 2$. The valuation v of F gives a group homomorphism $\mathrm{GL}_1(F) \to \mathbb{Z}$. Let A_c be the unique maximal compact subgroup of A. Let $\Sigma = A/A_c$. Then Σ is a $\mathbb{Z}$-lattice of rank $r = \dim A$. By composing with the valuation v the group $X^*(A)$ can be identified with

$$\Sigma^* = \mathrm{Hom}(\Sigma, \mathbb{Z}).$$

Let

$$\mathfrak{a}_0^* = \mathrm{Hom}(\Sigma, \mathbb{R}) \;\cong\; X^*(A) \otimes \mathbb{R}$$

be the real vector space of all group homomorphisms from Σ to $\mathbb{R}$ and let $\mathfrak{a}^* = \mathfrak{a}_0^* \otimes \mathbb{C} = \mathrm{Hom}(\Sigma, \mathbb{C}) \cong X^*(A) \otimes \mathbb{C}$. For $a \in A$ and $\lambda \in \mathfrak{a}^*$, let

$$a^\lambda = q^{-\lambda(a)},$$

where q is the number of elements in the residue class field of F. In this way we get an identification

$$\mathfrak{a}^* \Big/ \frac{2\pi i}{\log q} \Sigma^* \;\cong\; \mathrm{Hom}(\Sigma, \mathbb{C}^\times).$$

A quasicharacter $\nu : A \to \mathbb{C}^\times$ is called unramified if ν is trivial on A_c. The set $\mathrm{Hom}(\Sigma, \mathbb{C}^\times)$ can be identified with the set of unramified quasicharacters on A. Any unramified quasicharacter ν can thus be given a unique real part $\mathrm{Re}(\nu) \in \mathfrak{a}_0^*$. This definition extends to not necessarily unramified quasicharacters $\chi : A \to \mathbb{C}^\times$ as follows. Choose a splitting $s : \Sigma \to A$ of the exact sequence

$$1 \to A_c \to A \to \Sigma \to 1.$$

Then $\nu = \chi \circ s$ is an unramified character of A. Set

$$\mathrm{Re}(\chi) = \mathrm{Re}(\nu).$$

This definition does not depend on the choice of the splitting s. For quasicharacters χ, χ' and $a \in A$, we will frequently write a^χ instead of $\chi(a)$ and $a^{\chi+\chi'}$ instead of $\chi(a)\chi'(a)$. Note that the absolute value satisfies $|a^\chi| = a^{\mathrm{Re}(\chi)}$ and that a quasicharacter χ actually is a character if and only if $\mathrm{Re}(\chi) = 0$.

Let $\Delta_P : P \to \mathbb{R}_+$ be the modular function of the group P. Then there is $\rho \in \mathfrak{a}_0^*$ such that $\Delta_P(a) = |a^{2\rho}|$. For $\nu \in \mathfrak{a}^*$ and a root α let

$$\nu_\alpha = (\nu, \check{\alpha}) \in X^*(\mathrm{GL}_1) \otimes \mathbb{C} \cong \mathbb{C}.$$

Note that $\nu \in \mathfrak{a}_0^*$ implies $\nu_\alpha \in \mathbb{R}$ for every α. For $\nu \in \mathfrak{a}_0^*$, we say that ν is positive, $\nu > 0$, if $\nu_\alpha > 0$ for every positive root α.

Example 1.1 Let $G = \mathrm{GL}_n(F)$ and $\varpi_j \in G$ be the diagonal matrix $\varpi_j = \mathrm{diag}(1, \cdots, 1, \varpi, 1, \cdots, 1)$ with the ϖ on the j-th position. Let $\nu \in \mathfrak{a}^*$ and

$$\nu_j = \nu(\varpi_j A_c) \in \mathbb{C}.$$

Let α be a root, say $\alpha(\mathrm{diag}(a_1, \cdots, a_n)) = \frac{a_i}{a_j}$. Then

$$\nu_\alpha = \nu_i - \nu_j.$$

Hence $\nu \in \mathfrak{a}_0^*$ is positive if and only if $\nu_1 > \nu_2 > \cdots > \nu_n$.

We will fix Haar-measures of G and its reductive subgroups as follows. For $H \subset G$ being a torus, there is a unique maximal compact subgroup U_H which is open. Then we fix a Haar measure on H such that $\mathrm{vol}(U_H) = 1$. If H is connected reductive with compact center, then we choose the unique positive Haar-measure which up to sign coincides with the Euler-Poincaré measure (see [8]). So in the latter case, our measure is determined by the following property: For any discrete torsionfree cocompact subgroup $\Gamma_H \subset H$, we have

$$\mathrm{vol}(\Gamma_H \backslash H) = (-1)^{q(H)} \chi(\Gamma_H, \mathbb{Q}),$$

where $q(H)$ is the k-rank of the derived group H_{der} and $\chi(\Gamma_H, \mathbb{Q})$ the Euler-Poincaré characteristic of $H^\bullet(\Gamma_H, \mathbb{Q})$. For the applications, recall that centralizers of tori in connected groups are connected (see [1]).

Assume that we are given a discrete subgroup Γ of G such that the quotient space $\Gamma\backslash G$ is compact. Let (ω, V_ω) be a finite dimensional unitary representation of Γ and $L^2(\Gamma\backslash G, \omega)$ be the Hilbert space consisting of all measurable functions $f : G \to V_\omega$ such that $f(\gamma x) = \omega(\gamma) f(x)$ and $|f|$ is square integrable over $\Gamma\backslash G$ (modulo null functions). Let R denote the unitary representation of G on $L^2(\Gamma\backslash G, \omega)$ defined by right shifts, i.e., $R(g)\varphi(x) = \varphi(xg)$ for $\varphi \in L^2(\Gamma\backslash G, \omega)$. It is known that as a G-representation this space splits as a topological direct sum:

$$L^2(\Gamma\backslash G, \omega) = \bigoplus_{\pi \in \widehat{G}} N_{\Gamma,\omega}(\pi)\pi$$

with finite multiplicities $N_{\Gamma,\omega}(\pi) < \infty$.

Let f be integrable over G, so f is in $L^1(G)$. The integral

$$R(f) := \int_G f(x)R(x)dx$$

defines an operator on the Hilbert space $L^2(\Gamma\backslash G, \omega)$.

For $g \in G$ and any function f on G, we define the orbital integral

$$\mathcal{O}_g(f) := \int_{G_g\backslash G} f(x^{-1}gx)dx,$$

whenever the integral exists. Here G_g is the centralizer of g in G. It is known that the group G_g is unimodular, so we have an invariant measure on $G_g\backslash G$.

A function f on G or any of its closed subgroups is called smooth if it is locally constant. It is called uniformly smooth if there is an open subgroup U of G such that f factors over $U\backslash G/U$. This is in particular the case if f is smooth and compactly supported.

Proposition 1.1 (Trace Formula) *Let f be integrable and uniformly smooth. Then we have*

$$\sum_{\pi\in\widehat{G}} N_{\Gamma,\omega}(\pi)\ \operatorname{tr}\pi(f) = \sum_{[\gamma]} \operatorname{tr}\omega(\gamma)\ \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\ \mathcal{O}_\gamma(f),$$

where the sum on the right hand side runs over the set of Γ-conjugacy classes $[\gamma]$ in Γ and Γ_γ denotes the centralizer of γ in Γ. Both sides converge absolutely and the left hand side actually is a finite sum.

Proof At first fix a fundamental domain $\mathcal{F}$ for $\Gamma\backslash G$ and let $\varphi \in L^2(\Gamma\backslash G, \omega)$. Then

$$\begin{aligned} R(f) &= \int_G f(y)\varphi(xy)dy = \int_G f(x^{-1}y)\varphi(y)dy = \sum_{\gamma\in\Gamma}\int_{\mathcal{F}} f(x^{-1}\gamma y)\varphi(\gamma y)dy \\ &= \int_{\Gamma\backslash G}\Big(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\omega(\gamma)\Big)\varphi(y)dy. \end{aligned}$$

We want to show that the sum $\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\omega(\gamma)$ converges in $\operatorname{End}(V_\omega)$ absolutely and uniformly in x and y. Since y can be replaced by γy, $\gamma \in \Gamma$ and since ω is unitary, we only have to show the convergence of $\sum_{\gamma\in\Gamma} |f(x^{-1}\gamma y)|$ locally uniformly in y. Let γ and τ be in Γ and assume that $x^{-1}\gamma y$ and $x^{-1}\tau y$ lie in the same class in G/U. Then it follows that $\tau yU \cap \gamma yU \neq \emptyset$, so with $V = yUy^{-1}$ we have $\gamma^{-1}\tau V \cap V \neq \emptyset$. It is clear that V depends on y only up to U, so to show locally uniform convergence in y it suffices to fix V. Since V is compact also $V^2 = \{vv' \mid v, v' \in V\}$ is compact and so $\Gamma \cap V^2$ is finite. This implies that there are only finitely many $\gamma \in \Gamma$ with $\gamma V \cap V \neq \emptyset$. Hence the map $\Gamma \to G/U$, $\gamma \mapsto x^{-1}\gamma yU$ is finite to one with fibers having $\leq n$ elements for some natural number n. For y fixed modulo U, we get

$$\sum_{\gamma\in\Gamma} |f(x^{-1}\gamma y)| \leq n\int_{G/U} |f(x)|\ dx = \frac{n}{\operatorname{vol}(U)}\ \| f \|^1 .$$

We have shown the uniform convergence of the sum

$$k_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\omega(\gamma).$$

Observe that $R(f)$ factors over $L^2(\Gamma\backslash G,\omega)^U = L^2(\Gamma\backslash G/U,\omega)$, which is finite dimensional since $\Gamma\backslash G/U$ is a finite set. So $R(f)$ acts on a finite dimensional space and $k_f(x,y)$ is the matrix of this operator. We infer that $R(f)$ is of trace class, its trace equals

$$\sum_{\pi\in\widehat{G}} N_{\Gamma,\omega}(\pi)\ \operatorname{tr}\pi(f),$$

and the sum is finite. Further, since $k_f(x,y)$ is the matrix of $R(f)$, this trace also equals

$$\begin{aligned}\int_{\Gamma\backslash G} \operatorname{tr} k_f(x,x)\ dx &= \sum_{\gamma\in\Gamma}\int_{\mathcal{F}} f(x^{-1}\gamma x)dx\ \operatorname{tr}\omega(\gamma)\\ &= \sum_{[\gamma]}\sum_{\sigma\in\Gamma_\gamma\backslash\Gamma}\int_{\mathcal{F}} f((\sigma x)^{-1}\gamma(\sigma x))dx\ \operatorname{tr}\omega(\gamma)\\ &= \sum_{[\gamma]}\int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx\ \operatorname{tr}\omega(\gamma)\\ &= \sum_{[\gamma]}\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\ \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx\ \operatorname{tr}\omega(\gamma).\end{aligned}$$

2 The Covolume of a Centralizer

Suppose that $\gamma\in\Gamma$ is G-conjugate to some $a_\gamma m_\gamma\in A^-M_{\text{ell}}$. We want to compute the covolume

$$\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma).$$

An element x of G is called neat if for every representation $\rho: G\to \mathrm{GL}_n(F)$ of G the matrix $\rho(x)$ has no eigenvalue which is a root of unity different from 1. A subset A of G is called neat if each element of it is neat. Every arithmetic Γ has a finite index subgroup which is neat (see [2]).

Lemma 2.1 *Let $x\in G$ be neat and semisimple. Let G_x denote its centralizer in G. Then for every $k\in\mathbb{N}$, we have $G_x = G_{x^k}$.*

Proof Since G is linear algebraic, it is a subgroup of some $H = \mathrm{GL}_n(\overline{F})$, where $\overline{F}$ is an algebraic closure of F. If we can show the claim for H then it follows for G as well since $G_x = H_x\cap G$. In H we can assume x to be a diagonal matrix. Since x is neat, this implies the claim.

We suppose that Γ is neat. This implies that for any $\gamma\in\Gamma$, the Zariski closure of the group generated by γ is a torus. It then follows that G_γ is a connected reductive group (see [1]).

An element $\gamma\in\Gamma$ is called primitive if $\gamma=\sigma^n$ with $\sigma\in\Gamma$ and $n\in\mathbb{N}$ implies $n=1$. It is a property of discrete cocompact torsion free subgroups Γ of G that every $\gamma\in\Gamma$, $\gamma\neq 1$, is a positive power of a unique primitive element. In other words, given a nontrivial $\gamma\in\Gamma$, there exists a unique primitive γ_0 and a unique $\mu(\gamma)\in\mathbb{N}$, such that

$$\gamma = \gamma_0^{\mu(\gamma)}.$$

Let Σ be a group of finite cohomological dimension $\mathrm{cd}(\Sigma)$ over $\mathbb{Q}$. We write

$$\chi(\Sigma)=\chi(\Sigma,\mathbb{Q}):=\sum_{p=0}^{\mathrm{cd}(\Sigma)}(-1)^p\dim H^p(\Sigma,\mathbb{Q})$$

for the Euler-Poincaré characteristic. We also define the higher Euler characteristic as

$$\chi_r(\Sigma)=\chi_r(\Sigma,\mathbb{Q}):=\sum_{p=0}^{\mathrm{cd}(\Sigma)}(-1)^{p+r}\binom{p}{r}\dim H^p(\Sigma,\mathbb{Q})\quad\text{for } r=1,2,3,\cdots.$$

It is known that Γ has finite cohomological dimension over $\mathbb{Q}$.

We denote by $\mathcal{E}_P(\Gamma)$ the set of all conjugacy classes $[\gamma]$ in γ such that γ is in G conjugate to an element $a_\gamma m_\gamma\in AM$, where m_γ is elliptic and $a_\gamma\in A^-$.

Let $\gamma\in\mathcal{E}_P(\Gamma)$. To simplify the notation we assume that $\gamma=a_\gamma m_\gamma\in A^-M_{\mathrm{ell}}$. Let C_γ be the connected component of the center of G_γ. Then $C_\gamma=AB_\gamma$, where B_γ is the connected center of M_{m_γ}. The latter group will also be written as M_γ. Let M_γ^{der} be the derived group of M_γ. Then $M_\gamma=M_\gamma^{\mathrm{der}}B_\gamma$.

Lemma 2.2 *B_γ is compact.*

Proof Since m_γ is elliptic, there is a compact Cartan subgroup T of M containing m_γ. Since M modulo its center is a connected semisimple linear algebraic group, it follows that T is a torus and therefore abelian. Therefore $T\subset M_{m_\gamma}$. Let $b\in B_\gamma$. Then b commutes with every $t\in T$. Therefore b lies in the centralizer of T in M which equals T. So we have shown $B_\gamma\subset T$.

Let $\Gamma_{\gamma,A}=A\cap\Gamma_\gamma B_\gamma$ and $\Gamma_{\gamma,M}=M_\gamma^{\mathrm{der}}\cap\Gamma_\gamma AB_\gamma$. Similarly to the proof of Lemma 3.3 of [9], one shows that $\Gamma_{\gamma,A}$ and $\Gamma_{\gamma,M}$ are discrete cocompact subgroups of A and M_γ^{der} respectively. Let

$$\lambda_\gamma\stackrel{\mathrm{def}}{=}\mathrm{vol}(\Gamma_{\gamma,A}\backslash A).$$

Proposition 2.1 *Assume that Γ is neat and let $\gamma\in\Gamma$ be G-conjugate to an element of A^+M_{ell}. Then we get*

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)=\lambda_\gamma\ (-1)^{q(G)+r}\ \chi_r(\Gamma_\gamma),$$

where $r=\dim A$.

Proof We normalize the volume of B_γ to be 1. Then

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)=\mathrm{vol}(\Gamma_\gamma\backslash AM_\gamma)=\mathrm{vol}(\Gamma_\gamma B_\gamma\backslash AM_\gamma).$$

The space $\Gamma_\gamma B_\gamma\backslash AM_\gamma$ is the total space of a fibration with fibre $\Gamma_{\gamma,A}\backslash A$ and base space $\Gamma_\gamma AB_\gamma\backslash M_\gamma A\cong\Gamma_{\gamma,M}\backslash M_\gamma^{\mathrm{der}}$. Hence

$$\mathrm{vol}(\Gamma_\gamma B_\gamma\backslash AM_\gamma)=\mathrm{vol}(\Gamma_{\gamma,A}\backslash A)\,\mathrm{vol}(\Gamma_{\gamma,M}\backslash M_\gamma^{\mathrm{der}}).$$

Since $\lambda_\gamma=\mathrm{vol}(\Gamma_{\gamma,A}\backslash A)$, it remains to show

$$\mathrm{vol}(\Gamma_{\gamma,M}\backslash M_\gamma^{\mathrm{der}})=(-1)^r\chi_r(\Gamma_\gamma).$$

We know that

$$\operatorname{vol}(\Gamma_{\gamma,M}\backslash M_\gamma^{\mathrm{der}}) = (-1)^{q(M_\gamma)}\chi(\Gamma_{\gamma,M}) = (-1)^{q(G)+r}\chi(\Gamma_{\gamma,M}).$$

So it remains to show that $\chi(\Gamma_{\gamma,M}) = \chi_r(\Gamma_\gamma)$. The group $\Gamma_{\gamma,M}$ is isomorphic to Γ_γ/Σ, where $\Sigma = \Gamma \cap AB_\gamma$ is isomorphic to $\mathbb{Z}^r$. So the proposition follows from the next lemma.

Lemma 2.3 *Let Γ, Λ be of finite cohomological dimension over $\mathbb{Q}$. Let C_r be a group isomorphic to $\mathbb{Z}^r$ and assume that there is an exact sequence*

$$1 \to C_r \to \Gamma \to \Lambda \to 1.$$

Assume that C_r is central in Γ. Then

$$\chi(\Lambda, \mathbb{Q}) = \chi_r(\Gamma, \mathbb{Q}).$$

Proof We first consider the case $r = 1$. In this case, we want to prove for every r,

$$\chi_{r-1}(\Lambda, \mathbb{Q}) = \chi_r(\Gamma, \mathbb{Q}).$$

For this consider the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(\Lambda, H^q(C_1, \mathbb{Q})),$$

which abuts to

$$H^{p+q}(\Gamma, \mathbb{Q}).$$

Since $C_1 \cong \mathbb{Z}$, it follows that

$$H^q(C_1, \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } q = 0, 1, \\ 0, & \text{else.} \end{cases}$$

Since C_1 is infinite cyclic and central, it is an exercise to see that the spectral sequence degenerates at E_2. Therefore,

$$\begin{aligned} \chi_r(\Gamma) &= \sum_{j \geq 0} (-1)^{j+r} \binom{j}{r} \dim H^j(\Gamma) \\ &= \sum_{j \geq r} (-1)^{j+r} \binom{j}{r} (\dim H^j(\Lambda) + \dim H^{j-1}(\Lambda)) \\ &= \sum_{j \geq r} (-1)^{j+r} \binom{j}{r} \dim H^j(\Lambda) - \sum_{j \geq r-1} (-1)^{j+r} \binom{j+1}{r} \dim H^j(\Lambda). \end{aligned}$$

Now replace $\binom{j+1}{r}$ by $\binom{j}{r} + \binom{j}{r-1}$ to get the claim. For the general case, write $C_r = C_1 \oplus C^1$, where C_1 is cyclic and $C^1 \cong \mathbb{Z}^{r-1}$. Apply the above to C_1 and iterate this to get the lemma and hence the proposition.

3 The Lefschetz Formula

For a representation π of G, let π^∞ denote the subrepresentation of smooth vectors, i.e., π^∞ is the representation on the space $\bigcup_{H\subset G} \pi^H$, where H ranges over the set of all open subgroups of G. Further, let π_N denote the Jacquet module of π. By definition, π_N is the largest quotient MAN-module of π^∞ on which N acts trivially. One can achieve this by factoring out the vector subspace consisting of all vectors of the form $v - \pi(n)v$ for $v \in \pi^\infty$, $n \in N$. It is known that if π is an irreducible admissible representation, then π_N is an admissible MA-module of finite length. For a smooth M-module V let $H_c^\bullet(M,V)$ denote the continuous cohomology with coefficients in V as in [3].

Theorem 3.1 (Lefschetz Formula) *Let Γ be a neat discrete cocompact subgroup of G. Let φ be a uniformly smooth function on A with support in A^-. Suppose that the function $a \mapsto \varphi(a)|a^{-2\rho}|$ is integrable on A. Let σ be a finite dimensional representation of M. Let q be the F-splitrank of G and $r = \dim A$. Then*

$$\sum_{\pi\in\widehat{G}} N_{\Gamma,\omega}(\pi) \sum_{q=0}^{\dim M} (-1)^a \int_{A^-} \varphi(a)\, \mathrm{tr}\,(a|H_c^q(M,\pi_N\otimes\sigma))\, da$$

equals

$$(-1)^{q+r} \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \lambda_\gamma\, \chi_r(\Gamma_\gamma)\, \mathrm{tr}\,\omega(\gamma)\, \mathrm{tr}\,\sigma(m_\gamma)\, \varphi(a_\gamma)\, |a_\gamma^{2\rho}|.$$

Both outer sums converge absolutely and the sum over $\pi \in \widehat{G}$ actually is a finite sum, i.e., the summand is zero for all but finitely many π. For a given compact open subgroup U of A, both sides represent a continuous linear functional on the space of all functions φ as above which factor over A/U, where this space is equipped with the norm $\| \varphi \| = \int_A |\varphi(a)||a^{-2\rho}|\, da$.

Let A^* denote the set of all continuous group homomorphisms $\lambda\colon A \to \mathbb{C}^\times$. For $\lambda \in A^*$ and an A-module V, let V_λ denote the generalized λ-eigenspace, i.e.,

$$V_\lambda \stackrel{\text{def}}{=} \bigcup_{k=1}^{\infty} \{v \in V \mid (a-\lambda(a))^k v = 0,\ \forall\, a \in A\}.$$

Then

$$\int_{A^-} \varphi(a)\, \mathrm{tr}\,(a \mid H_c^q(M,\pi_N\otimes\sigma))\, da = \sum_{\lambda\in A^*} \dim H_c^q(M,\pi_N\otimes\sigma)_\lambda \int_{A^-} \varphi(a)\, \lambda(a)\, da.$$

For $\lambda \in A^*$, define

$$m_\lambda^{\sigma,\omega} \stackrel{\text{def}}{=} \sum_{\pi\in\widehat{G}} N_{\Gamma,\omega}(\pi) \sum_{q=0}^{\dim M} (-1)^q \dim H_c^q(M,\pi_N\otimes\sigma)_\lambda.$$

The sum is always finite.

On the other hand, for $[\gamma] \in \mathcal{E}_P(\Gamma)$, let

$$c_\gamma \stackrel{\text{def}}{=} \lambda_\gamma\, \chi_r(\Gamma_\gamma)\, |a_\gamma^{2\rho}|.$$

Then the theorem is equivalent to the following Corollary.

Corollary 3.1 (Lefschetz Formula) *As an identity of distributions on A^-, we have*

$$\sum_{\lambda\in A^*} m_\lambda^{\sigma,\omega}\,\lambda = \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} c_\gamma \,\mathrm{tr}\,\omega(\gamma)\,\mathrm{tr}\,\sigma(m_\gamma)\,\delta_{a_\gamma}.$$

Proof of Theorem 3.1 Let f_{EP} be an Euler-Poincaré function on M which is K_M-central (see [8]). For regular $m \in M$, we have

$$\mathcal{O}_m^M(f_{EP}) = \begin{cases} 1, & \text{if } m \text{ is elliptic,} \\ 0, & \text{otherwise.} \end{cases}$$

For $g \in G$ and a finite dimensional F vector space V on which g acts linearly, let $E(g\,|\,V)$ be the set of all absolute values $|\mu|$, where μ ranges over the eigenvalues of g in the algebraic closure $\overline{F}$ of F. Let $\lambda_{\min}(g\,|\,V)$ denote the minimum and $\lambda_{\max}(v\,|\,V)$ the maximum of $E(g\,|\,V)$. For $am \in AM$, define

$$\lambda(am) \overset{\mathrm{def}}{=} \frac{\lambda_{\min}(a\,|\,\overline{\mathfrak{n}})}{\lambda_{\max}(m\,|\,\mathfrak{g})^2}.$$

Note that $\lambda_{\max}(m\,|\,\mathfrak{g})$ is always ≥ 1 and that $\lambda_{\max}(m\,|\,\mathfrak{g})\lambda_{\min}(m\,|\,\mathfrak{g}) = 1$. We will consider the set

$$(AM)^\sim := \{am \in AM \mid \lambda(am) > 1\}.$$

Let M_{ell} denote the set of elliptic elements in M.

Lemma 3.1 *The set $(AM)^\sim$ has the following properties:*

(1) $A^- M_{\mathrm{ell}} \subset (AM)^\sim$,

(2) $am \in (AM)^\sim \Rightarrow a \in A^-$,

(3) $am,\ a'm' \in (AM)^\sim,\ g \in G$ *with* $a'm' = gamg^{-1} \Rightarrow a = a',\ g \in AM$.

Proof The first two are immediate. For the third, let $am, a'm' \in (AM)^\sim$ and $g \in G$ with $a'm' = gamg^{-1}$. Observe that by the definition of $(AM)^\sim$, we have

$$\begin{aligned} \lambda_{\min}(am\,|\,\overline{\mathfrak{n}}) &\geq \lambda_{\min}(a\,|\,\overline{\mathfrak{n}})\lambda_{\min}(m\,|\,\mathfrak{g}) \\ &> \lambda_{\max}(m\,|\,\mathfrak{g})^2\lambda_{\min}(m\,|\,\mathfrak{g}) \\ &= \lambda_{\max}(m\,|\,\mathfrak{g}) \\ &\geq \lambda_{\max}(m\,|\,\mathfrak{a}+\mathfrak{m}+\mathfrak{n}) \\ &\geq \lambda_{\max}(am\,|\,\mathfrak{a}+\mathfrak{m}+\mathfrak{n}), \end{aligned}$$

that is, any eigenvalue of am on $\overline{\mathfrak{n}}$ is strictly bigger than any eigenvalue on $\mathfrak{a}+\mathfrak{m}+\mathfrak{n}$. Since $\mathfrak{g} = \mathfrak{a}+\mathfrak{m}+\mathfrak{n}+\overline{\mathfrak{n}}$ and the same holds for $a'm'$, which has the same eigenvalues as am, we infer that $\mathrm{Ad}(g)\overline{\mathfrak{n}} = \overline{\mathfrak{n}}$. So g lies in the normalizer of $\overline{\mathfrak{n}}$, which is $\overline{P} = MA\overline{N} = \overline{N}AM$. Now suppose $g = nm_1a_1$ and $\widehat{m} = m_1mm_1^{-1}$. Then

$$gamg^{-1} = na\widehat{m}n^{-1} = a\widehat{m}\ (a\widehat{m})^{-1}n(a\widehat{m})\ n^{-1}.$$

Since this lies in AM, we have $(a\widehat{m})^{-1}n(a\widehat{m}) = n$ which, since $am \in (AM)^\sim$, implies $n = 1$. The lemma is proven.

Let G act on itself by conjugation. Write $g.x = gxg^{-1}$, and write $G.x$ for the orbit, so $G.x = \{gxg^{-1} \mid g \in G\}$ as well as $G.S = \{gsg^{-1} \mid s \in S, g \in G\}$ for any subset S of G.

Fix a smooth function η on N which has compact support, is positive, invariant under $K \cap MAN$ and satisfies $\int_N \eta(n)dn = 1$. Extend the function φ from A^- to a conjugation invariant smooth function $\widetilde{\varphi}$ on AM such that $\widetilde{\varphi}(am) = \varphi(a)$ whenever m is elliptic and such that there is a compact subset $C \subset A^-$ such that $\widetilde{\varphi}$ is supported in $CM \cap (AM)^\sim$. It follows that the function

$$am \;\mapsto\; f_{EP}(m)\,\mathrm{tr}\,\sigma(m)\,\widetilde{\varphi}(am)\,|a^{2\rho}|$$

is smooth and integrable on AM. Given these data let $f = f_{\eta,\tau,\varphi} : H \to \mathbb{C}$ be defined by

$$f(knma(kn)^{-1}) := \eta(n) f_{EP}(m)\,\mathrm{tr}\,\sigma(m)\,\widetilde{\varphi}(am)\,|a^{2\rho}|$$

for $k \in K, n \in N, m \in M, a \in \overline{A^-}$. Further $f(x) = 0$ if x is not in $G.(AM)^\sim$.

Lemma 3.2 *The function f is well defined.*

Proof By the decomposition $G = KP = KNMA$ every element $x \in G.(AM)^\sim$ can be written in the form $knma(kn)^{-1}$. Now suppose that two such representations coincide, that is,

$$knma(kn)^{-1} = k'n'm'a'(k'n')^{-1}$$

or $gmag^{-1} = m'a'$, where $g = (n')^{-1}(k')^{-1}kn$. Lemma 3.1 implies that $g \in MA$ and $a = a'$, so that $gmg^{-1} = m'$. Write $k_1 = (k')^{-1}k$. Then $k_1 \in K \cap n'MAn^{-1}$, and hence $k_1 \in K \cap MAN$. So $k_1 = m_1a_1n_1$ with $a_1, m_1, n_1 \in K$. Now write $g = (n')^{-1}m_1a_1n_1n = m_1a_1(n')^{-a_1m_1}n_1n$. Since $g \in MA$, it follows that $g = m_1a_1 \in K \cap MA$, and the well-definedness of f follows.

We will plug f into the trace formula. For the geometric side, let $\gamma \in \Gamma$. We have to calculate the orbital integral:

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)dx.$$

By the definition of f it follows that $\mathcal{O}_\gamma(f) = 0$ if $\gamma \notin G.(AM)^\sim$. It remains to compute $\mathcal{O}_{am}(f)$ for $am \in (AM)^\sim$. Again by the definition of f, it follows that

$$\begin{aligned}\mathcal{O}_{am}(f) &= \mathcal{O}_m^M(f_{EP})\,\mathrm{tr}\,\sigma(m)\,\widetilde{\varphi}(am)\,|a^{2\rho}| \\ &= \begin{cases} \mathrm{tr}\,\sigma(m)\,\varphi(a)\,|a^{2\rho}|, & \text{if } m \text{ is elliptic,} \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Here $\mathcal{O}_m^M$ denotes the orbital integral in the group M. Recall that Proposition 2.1 says

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) = (-1)^{q(G)+r}\,\lambda_\gamma\,\chi_r(\Gamma_\gamma),$$

so that for $\gamma \in \mathcal{E}_P(\Gamma)$,

$$\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\,\mathcal{O}_\gamma(f) = (-1)^{q(G)+r}\,\lambda_\gamma\,\chi_r(\Gamma_\gamma)\,\mathrm{tr}\,\sigma(m_\gamma)\,\varphi(a_\gamma)\,|a_\gamma^{2\rho}|.$$

To compute the spectral side let $\pi \in \widehat{G}$. We want to compute $\operatorname{tr}\pi(f)$. Let Θ_π^G be the locally integrable conjugation invariant function on G such that

$$\operatorname{tr}\pi(f) = \int_G f(x)\, \Theta_\pi^G(x) dx.$$

This function Θ_π is called the character of π. It is known that the Jacquet module π_N is a finitely generated admissible module for the group MA and therefore it has a character $\Theta_{\pi_N}^{MA}$. In [4], it is shown that

$$\Theta_\pi(am) = \Theta_{\pi_N}^{MA}(ma) \quad \text{for } ma \in A^- M_{\text{ell}}.$$

Let h be a function in $L^1(G)$ which is supported in the set $G.MA$. Comparing invariant differential forms as in the proof of the Weyl integration formula one gets that the integral $\int_G h(x)\, dx$ equals

$$\frac{1}{|W(G,A)|} \int_A \int_M \int_{G/AM} h(yamy^{-1})\, |\det(1 - am|\mathfrak{n} + \overline{\mathfrak{n}})| dy\, da\, dm,$$

where $W(G,A)$ is the Weyl group of A in G.

For $a \in A^-$ and $m \in M_{\text{ell}}$ every eigenvalue of am on $\mathfrak{n}$ is of absolute value < 1 and > 1 on $\overline{\mathfrak{n}}$. By the ultrametric property this implies

$$|\det(1 - am \,|\, \mathfrak{n} + \overline{\mathfrak{n}})| = |\det(1 - am \,|\, \overline{\mathfrak{n}})| = |\det(am \,|\, \overline{\mathfrak{n}})| = |\det(a \,|\, \overline{\mathfrak{n}})| = |a^{-2\rho}|.$$

We apply this to $h(x) = f(x)\Theta_\pi^G(x)$ and use conjugation invariance of Θ_π^G to get that $\operatorname{tr}\pi(f)$ equals

$$\frac{1}{|W(G,A)|} \int_{AM} f_{EP}(m)\, \operatorname{tr}\sigma(m)\, \widetilde{\varphi}(am)\, \Theta_{\pi_N}^{M}(am)\, da\, dm,$$

which is the same as

$$\int_{A^- M} f_{EP}(m)\, \operatorname{tr}\sigma(m)\, \widetilde{\varphi}(am)\, \Theta_{\pi_N}^{M}(am)\, da\, dm.$$

We recall the Weyl integration formula for M. Let $(H_j)_j$ be a maximal family of pairwise non-conjugate Cartan subgroups of M. Let W_j be the Weyl group of H_j in M. For $h \in H_j$, let $D_j(h) = \det(1 - h|\mathfrak{m}/\mathfrak{h}_j)$, where $\mathfrak{m}$ and $\mathfrak{h}_j$ are the Lie algebras of M and H_j respectively. Then, for every $h \in L^1(M)$,

$$\begin{aligned}\int_M h(m)\, dm &= \sum_j \frac{1}{|W_j|} \int_{H_j^{\text{reg}}} \int_{M/H_j} h(mxm^{-1})\, D_j(x)\, dm\, dx\\ &= \sum_j \frac{1}{|W_j|} \int_{H_j} \mathcal{O}_x^M(h)\, dx,\end{aligned}$$

where H_j^{reg} is the set of $x \in H_j$ which is regular in M. We fix $a \in A^-$ and apply this to $h(m) = f_{EP}(m)\widetilde{\varphi}(am)\operatorname{tr}\sigma(m)\Theta_{\pi_N}^M(am)$. Since $\widetilde{\varphi}$ is conjugation invariant we get for $x \in H_j^{\text{reg}}$,

$$\mathcal{O}_x^M(h) = \mathcal{O}_x^M(f_{EP})\, \widetilde{\varphi}(am)\, \operatorname{tr}\sigma(m)\, \Theta_{\pi_N}^{AM}(ax).$$

This is non-zero only if x is elliptic. If x is elliptic, then $\widetilde{\varphi}(ax)$ equals $\varphi(a)$. So we can replace $\widetilde{\varphi}(ax)$ by $\varphi(a)$ throughout. Thus $\operatorname{tr}\pi(f)$ equals

$$\int_{A^- M} f_{EP}(m)\,\varphi(a)\,\operatorname{tr}\sigma(m)\,\Theta_{\pi_N}^M(am)\,da\,dm.$$

The trace $\operatorname{tr}\pi(f)$ therefore equals

$$\int_{A^- M} f_{EP}(m)\,\varphi(a)\,\Theta_{\pi_N\otimes\sigma}(am)\,da\,dm.$$

We write $H_c^\bullet(M,V)$ for the continuous cohomology of M with coefficients in the M-module V. By [8, Theorem 2],

$$\operatorname{tr}(\pi_N\otimes\sigma)(f_{EP}) = \sum_{q=0}^{\dim M} (-1)^q\,\dim H_c^q(M,\pi_N\otimes\sigma).$$

The cohomology groups $H_c^q(M,\pi_N\otimes\sigma)$ are finite dimensional A-modules and

$$\operatorname{tr}\pi(f) = \sum_{q=0}^{\dim M} (-1)^q \int_{A^-} \operatorname{tr}(a|H_c^q(M,\pi_N\otimes\sigma))\,\varphi(a)\,da.$$

The Lefschetz Theorem follows.

References

[1] Borel, A., Linear Algebraic Groups, W. A. Benjamin Inc., New York, 1969.

[2] Borel, A., Introduction Aux Groupes Arithmétiques, Hermann, Paris, 1969.

[3] Borel, A. and Wallach, N., Continuous Cohomology, Discrete Groups, and Representations of Reductive Groups, Ann. Math. Stud., **94**, Princeton, 1980.

[4] Casselman, W., Characters and jacquet modules, *Math. Ann.*, **230**, 1977, 101–105.

[5] Deitmar, A., Geometric zeta-functions on p-adic groups, *Math. Japon.*, **47**(1), 1998, 1–17.

[6] Deitmar, A., A prime geodesic theorem for higher rank spaces, *Geometric and Functional Analysis*, **14**, 2004, 1238–1266.

[7] Deitmar, A., A Lefschetz formula for higher rank, preprint. http://arxiv.org/abs/math/0505403

[8] Kottwitz, R., Tamagawa numbers, *Ann. Math.*, **127**, 1988, 629–646.

[9] Wolf, J., Discrete groups, symmetric spaces and global holonomy, *Amer. J. Math.*, **84**, 1962, 527–542.