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Elements of Small Orders in K_2F II

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Abstract In "Elements of small orders in $K_2(F)$" (Algebraic K-Theory, Lecture Notes in Math., **966**, 1982, 1–6.), the author investigates elements of the form $\{a,\ \Phi_n(a)\}$ in the Milnor group K_2F of a field F, where $\Phi_n(x)$ is the n-th cyclotomic polynomial. In this paper, these elements are generalized. Applying the explicit formulas of Rosset and Tate for the transfer homomorphism for K_2, the author proves some new results on elements of small orders in K_2F.

Keywords Cyclotomic elements in K_2F, Transfer in K-theory, Milnor group
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1 Introduction

In an earlier paper with the same title (see [1]), we investigated elements of the form $c_n(a) := \{a,\ \Phi_n(a)\}_F \in K_2F$, where F is a field of characteristic prime to n,

$$\Phi_n(x) = \prod_{\substack{k=1 \\ (k,n)=1}}^{n} (1 - \zeta_n^k x)$$

is the n-th cyclotomic polynomial, and $a,\ \Phi_n(a) \in F^*$. We call such $c_n(a)$ cyclotomic elements.

In [1], it is proved that for $n = 1, 2, 3, 4$ and 6, the set

$$G_n(F) := \{c_n(a) : a, \Phi_n(a) \in F^*\}$$

is a subgroup of K_2F. Moreover, for $F = \mathbb{Q}$, every element of order $n|12$ has the form

$$\prod_{d|n} c_d(a_d) \quad \text{for some } a_d \in F^*.$$

The proofs in [1] are quite elementary, and the results are obtained by manipulation on symbols only. We did not use the transfer homomorphism for K_2.

J. Urbanowicz [2] generalized some of our results to arbitrary fields applying the transfer and his transfer symbols. He proved also that for arbitrary field F of characteristic $\neq 3$ every element of order 3 in K_2F equals $c_3(a)$ for some $a \in F^*$ (see [2, Corollary 3.4]).

In the present, paper we extend some results in [1] and [2] applying the effective formulas for the transfer given by S. Rosset and J. Tate [3].

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2 The Transfer for K_2

We state basic properties of the transfer homomorphism (see [3–5]).

Let E be an extension of degree d of a field F. Then there are homomorphisms

$$j_{F/E}: \ K_2F \longrightarrow K_2E, \quad j_{F/E}\{a,\ b\}_F = \{a,\ b\}_E \quad \text{for } a,b \in F^*,$$
$$\mathrm{Tr}_{E/F}: \ K_2E \longrightarrow K_2F, \quad \text{for the definition (see[4, §14])}.$$

Then

$$(\mathrm{Tr}_{E/F} \circ j_{F/E})(\alpha) = \alpha^d \quad \text{for } \alpha \in K_2F,$$

and if E is a Galois extension of F with the Galois group G, then

$$(j_{F/E} \circ \mathrm{Tr}_{E/F})(\beta) = \prod_{\sigma \in G} \sigma(\beta) \quad \text{for } \beta \in K_2E.$$

The transfer is transitive: If $F \subset E \subset L$, then

$$\mathrm{Tr}_{L/F} = \mathrm{Tr}_{E/F} \circ \mathrm{Tr}_{L/E}.$$

If $b \in E^*$, but $a \in F^*$, then the transfer of $\{a,\ b\}_E$ is easy to describe

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \{a,\ N_{E/F}\, b\}_F, \tag{2.1}$$

where $N_{E/F}: E^* \to F^*$ is the norm.

If both $a, b \in E^* \setminus F$, then the reciprocity law of Rosset and Tate [3] makes it possible to express $\mathrm{Tr}_{E/F}\{a,\ b\}_E$ as the product of symbols in K_2F.

We give below the details. First, we make some reduction steps. Taking in (2.1) $F = F(a)$ we get

$$\mathrm{Tr}_{E/F(a)}\{a,\ b\}_E = \{a,\ N_{E/F(a)}\, b\}_{F(a)}.$$

Thus, by the transitivity of transfer, to get $\mathrm{Tr}_{E/F}\{a, b\}_E$ it is sufficient to compute

$$\mathrm{Tr}_{F(a)/F}\{a,\ b'\}_{F(a)}, \quad \text{where } b' = N_{E/F(a)}\, b.$$

In other words, we can assume that $E = F(a)$, and we have to write the element $\mathrm{Tr}_{F(a)/F}\{a, b\}_{F(a)}$ for $b \in F(a)$ as the product of symbols in K_2F.

Since $b \in F[a]$, we have $b = f(a)$ for some $f \in F[x]$ with $0 \le \deg f < (F(a) : F)$. Let $f(x) = cf_1(x) \cdots f_r(x)$, where $c \in F^*$ and $f_j(x) \in F[x]$ for $1 \le j \le r$ are monic and irreducible, $r \ge 0$.

Then

$$\mathrm{Tr}_{F(a)/F}\{a,\ b\}_{F(a)} = \mathrm{Tr}_{F(a)/F}\{a,\ c\}_{F(a)} \cdot \prod_{j=1}^{r} \mathrm{Tr}_{F(a)/F}\{a,\ f_j(a)\}_{F(a)}.$$

The first factor on the right equals $\{N_{F(a)/F}\, a,\ c\}_F$, by (2.1). Thus it remains to express $\mathrm{Tr}_{F(a)/F}\{a,\ f_j(a)\}_{F(a)}$ as the product of symbols in K_2F. The reciprocity law solves this problem.

We state the reciprocity law of Rosset and Tate in the following form (see [3]).

Reciprocity Law *Let $f, g \in F[x]$ be monic, irreducible and relatively prime polynomials, let $g(a) = f(b) = 0$ and assume that $ab \neq 0$. Then*

$$\mathrm{Tr}_{F(a)/F}\{a,\ f(a)\}_{F(a)} = \{g(0),\ f(0)\}_F \cdot \{(-1)^{\deg g},\ (-1)^{\deg f}\}_F \cdot \mathrm{Tr}_{F(b)/F}\{b,\ g(b)\}_{F(b)}. \quad (2.2)$$

We leave it as an easy exercise for the reader to deduce this form of the reciprocity law from its statement in [3].

Since

$$\deg f_j \leq \deg f < \deg g, \quad \text{where } g \in F[x],\ g(a) = 0,$$

by applying to $\mathrm{Tr}_{F(a)/F}\{a,\ f_j(a)\}_{F(a)}$ the reciprocity law several times we get eventually the product of symbols in K_2F.

The reciprocity law simplifies if one of the polynomials f, g is of degree 1.

Proposition 2.1 *If a polynomial $g \in F[x]$ is monic and irreducible with a root $a \neq 0$, and $b \in F^*$, $g(b) \neq 0$, then*

$$\mathrm{Tr}_{F(a)/F}\{a,\ b - a\}_{F(a)} = \{b,\ g(b)/g(0)\}_F. \quad (2.3)$$

Proof We have

$$\{a,\ b - a\}_{F(a)} = \{a,\ -1\}_{F(a)} \cdot \{a,\ a - b\}_{F(a)}. \quad (2.4)$$

Since $g(0) = (-1)^{\deg g} N_{F(a)/F}\, a$, by (2.1) we have

$$\mathrm{Tr}_{F(a)/F}\{a,\ -1\}_{F(a)} = \{N_{F(a)/F}\, a,\ -1\}_F = \{(-1)^{\deg g} g(0),\ -1\}_F. \quad (2.5)$$

Let $f(x) = x - b$. Then

$$\{g(0), f(0)\}_F = \{g(0),\ -b\}_F = \{g(0),\ -1\}_F \cdot \{b,\ g(0)\}_F^{-1},$$
$$\{(-1)^{\deg g},\ (-1)^{\deg f}\}_F = \{(-1)^{\deg g},\ -1\}_F.$$

Therefore, in view of $F(b) = F$ and (2.5), from (2.2) we get

$$\begin{aligned}\mathrm{Tr}_{F(a)/F}\{a,\ a - b\}_{F(a)} &= \mathrm{Tr}_{F(a)/F}\{a,\ f(a)\}_{F(a)} \\ &= \{b,\ g(b)\}_F \cdot \{(-1)^{\deg g} g(0),\ -1\}_F \cdot \{b,\ g(0)\}_F^{-1} \\ &= \{b,\ g(b)/g(0)\}_F \cdot \mathrm{Tr}_{F(a)/F}\{a,\ -1\}_{F(a)}.\end{aligned}$$

Then (2.4) implies the result.

Proposition 2.2 *Let $n \geq 2$ and assume that the cyclotomic polynomial $\Phi_n(x)$ is irreducible over F, where F is a field of characteristic prime to n. Then for $b \in F^*$ satisfying $\Phi_n(b) \neq 0$ we have*

$$\mathrm{Tr}_{F(\zeta_n)/F}\{\zeta_n,\ b - \zeta_n\}_{F(\zeta_n)} = \{b,\ \Phi_n(b)\}_F = c_n(b).$$

Proof In Proposition 2.1, put $g(x) = \Phi_n(x)$ and $a = \zeta_n$. Since $\Phi_n(x)$ is monic for $n \geq 2$ and $\Phi_n(0) = 1$, (2.3) gives the result.

From Proposition 2.2, it follows that $c_n(b)^n = 1$, which was proved directly in [1] without the assumption that $\Phi_n(x) \in F[x]$ is irreducible.

3 Generalized Cyclotomic Elements

We give another example of elements of order n in K_2F.

We assume, as always, that n is prime to the characteristic of the field F. Let $\zeta_n^{(1)} = \zeta_n, \zeta_n^{(2)}, \dots, \zeta_n^{(r_n)}$ be the maximal set of primitive n-th roots of unity pairwise nonconjugate over F. Then $\zeta_n^{(j)} = \zeta_n^{k_j}$ for some k_j, $(k_j, n) = 1$. Let $\Phi_n^{(j)}(x)$ be the minimal monic polynomial for $\zeta_n^{(j)}$ over F.

Obviously, $F(\zeta_n^{k_j}) = F(\zeta_n)$. Hence

$$\deg \Phi_n^{(j)} = (F(\zeta_n^{k_j}) : F) = (F(\zeta_n) : F) =: d_n$$

does not depend on j.

We have

$$\Phi_n(x) = \prod_{j=1}^{r_n} \Phi_n^{(j)}(x). \tag{3.1}$$

Hence $\varphi(n) = \deg \Phi_n = d_n \cdot r_n$.

We define generalized cyclotomic elements in K_2F as follows:

$$c_n^{(j)}(a) := \{a,\ \Phi_n^{(j)}(a)\}_F, \quad \text{where } 1 \le j \le r_n, \quad a,\ \Phi_n^{(j)}(a) \in F^*.$$

The generalized cyclotomic elements have similar properties as the ordinary ones. If $\Phi_n \in F[x]$ is irreducible, then there are no generalized cyclotomic elements. It is the case for $n \le 2$. Therefore in what follows we assume that $n \ge 3$.

By (3.1),

$$c_n(a) = \{a,\ \Phi_n(a)\}_F = \prod_{j=1}^{r_n} \{a,\ \Phi_n^{(j)}(a)\}_F = \prod_{j=1}^{r_n} c_n^{(j)}(a),$$

so every cyclotomic element is the product of generalized ones.

Proposition 3.1 *If $n \ge 3$ is prime to the characteristic of the field F, then for $a \in F^*$ satisfying $\Phi_n^{(j)}(a) \ne 0$ and $E = F(\zeta_n)$ we have*

$$\mathrm{Tr}_{E/F}\{\zeta_n^{k_j},\ a - \zeta_n^{k_j}\}_E = \{a,\ \Phi_n^{(j)}(a)/\Phi_n^{(j)}(0)\}_F = \{\Phi_n^{(j)}(0),\ a\}_F \cdot c_n^{(j)}(a). \tag{3.2}$$

In particular, if ζ_n and ζ_n^{-1} are conjugate over F, then

$$\mathrm{Tr}_{E/F}\{\zeta_n^{k_j},\ a - \zeta_n^{k_j}\}_E = c_n^{(j)}(a).$$

Proof The formula (3.2) follows immediately from Proposition 2.1. If $\sigma(\zeta_n) = \zeta_n^{-1}$ for some $\sigma \in \mathrm{Gal}(E/F)$, then $\sigma(\zeta_n^{k_j}) = \zeta_n^{-k_j}$, and it follows that the polynomial $\Phi_n^{(j)}(x)$ is symmetric. Hence $\Phi_n^{(j)}(0) = 1$, since $\Phi_n^{(j)}(x)$ is monic. Then the second part of the proposition follows from (3.2).

Corollary 3.1 *We have $c_n^{(j)}(a)^{2n} = 1$ for $a \in E^*$, $1 \le j \le r_n$, and if ζ_n and ζ_n^{-1} are conjugate over F, then $c_n^{(j)}(a)^n = 1$.*

Proof Since $\Phi_n^{(j)}(0) = \pm\zeta_n^r$ for some r, and $\Phi_n^{(j)}(0) = 1$, if ζ_n and ζ_n^{-1} are conjugate over F, then the corollary follows from Proposition 3.1.

4 Elements in K_2F of Order n, Where $n|12$

We give a direct proof of Corollary 3.4 in [2] based on the similar ideas as the original one.

Theorem 4.1 *Let F be a field with* $\operatorname{char} F \neq 3$. *Then every element of order* 3 *in K_2F has the form $c_3(b)$ for some $b \in F^*$.*

Proof Let $\alpha \in K_2F$ be an element of order 3. Then α^{-1} also has order 3. Put $E = F(\zeta_3)$. By a theorem of A. Suslin [6, Theorem 1.8] we have $j_{F/E}(\alpha^{-1}) = \{\zeta_3,\ a\}_E$ for some $a \in E^*$.

If $\zeta_3 \in F$, then $E = F$. Hence $\alpha = j_{F/E}(\alpha) = \{\zeta_3,\ a^{-1}\}_F$, and the result follows from [1, Theorem 1].

Assume that $\zeta_3 \notin F$. Then $(E : F) = 2$. Hence

$$(\mathrm{Tr}_{E/F} \circ j_{F/E})\,(\alpha^{-1}) = (\alpha^{-1})^2 = \alpha.$$

On the other hand

$$(\mathrm{Tr}_{E/F} \circ j_{F/E})\,(\alpha^{-1}) = \mathrm{Tr}_{E/F}\{\zeta_3,\ a\}_E, \quad \text{where } a \in F(\zeta_n)^* = E^*.$$

If $a \in F$, then

$$\mathrm{Tr}_{E/F}\{\zeta_3,\ a\}_E = \{N_{E/F}\,\zeta_3,\ a\}_F = \{1,\ a\}_F = 1,$$

i.e., $\alpha^{-1} = 1$. It is a contradiction, since α has order 3.

Hence $a \notin F,\ a = u\zeta_3 + v,\ u, v \in F,\ u \neq 0$. Then $a = -u(b - \zeta_3)$, where $b = -\frac{v}{u} \in F$. Similarly as above we get $\mathrm{Tr}_{E/F}\{\zeta_3,\ -u\}_E = 1$. If $b = 0$, then $\alpha = \mathrm{Tr}_{E/F}\{\zeta_3,\ -\zeta_3\}_E = 1$, a contradiction.

Then $b \neq 0$ and

$$\alpha = \mathrm{Tr}_{E/F}\{\zeta_3,\ a\}_E = \mathrm{Tr}_{E/F}\{\zeta_3,\ b - \zeta_3\}_E = c_3(b),$$

by Proposition 2.2 with $n = 3$.

Now we prove an analogous theorem for elements of order 4.

Theorem 4.2 *Let F be a field with* $\operatorname{char} F \neq 2$. *Then every element of order* 4 *in K_2F is of the form $c_4(a')\,c_2(a'')$ for some $a', a'' \in F^*$.*

Proof If $\zeta_4 = i \in F$, then the result follows from [6, Theorem 1.8] and [1, Theorem 1].

Assume that $i \notin F$ and denote $E = F(i)$. Let $\alpha \in K_2F$ be an element of order 4. Then α^2 has order 2. Hence $\alpha^2 = \{-1,\ a\}_F$ for some $a \in F^*$.

We have

$$j_{F/E}(\alpha)^2 = j_{F/E}(\alpha^2) = j_{F/E}\{-1,\ a\}_F = \{-1,\ a\}_E = \{i,\ a\}_E^2.$$

It follows that the element $j_{F/E}(\alpha) \cdot \{i,\ a\}_E^{-1}$ has order ≤ 2, so it equals $\{-1,\ b\}_E$ for some $b \in E^*$. Then

$$j_{F/E}(\alpha) = \{i,\ a\}_E \cdot \{-1,\ b\}_E = \{i,\ a\}_E \cdot \{i,\ b\}_E^2.$$

Since $(E : F) = 2$, we get $\mathrm{Tr}_{E/F}(j_{F/E}(\alpha)) = \alpha^2$.

From $a \in F^*$, by (2.1), it follows that

$$\mathrm{Tr}_{E/F}\{i,\ a\}_E = \{N_{E/F}\,i,\ a\}_F = \{1,\ a\}_F = 1.$$

If $b \in F^*$, we get

$$\alpha^2 = \mathrm{Tr}_{E/F}(j_{F/E}(\alpha)) = \mathrm{Tr}_{E/F}\{i,\ b\}_E^2 = \{N_{E/F}\, i,\ b\}_F^2 = 1.$$

Hence $b \notin F,\ b = ui + v,\ u, v \in F,\ u \neq 0$. Then $b = -u(a' - i)$, where $a' = -\frac{v}{u} \in F$. Similarly as above we get $\mathrm{Tr}_{E/F}\{i,\ -u\} = 1$.

If $a' = 0$, then $\alpha^2 = \mathrm{Tr}_{E/F}\{i,\ -i\}_E = 1$, a contradiction. Then $a' \neq 0$ and from Proposition 2.2 it follows that $\mathrm{Tr}_{E/F}\{i,\ a' - i\} = c_4(a')$.

Thus we have proved that

$$\alpha^2 = \mathrm{Tr}_{E/F}(j_{F/E}(\alpha)) = \mathrm{Tr}_{E/F}\{i,\ b\}_E^2 = c_4(a')^2.$$

Consequently the element $\alpha\, c_4(a')^{-1}$ has order ≤ 2, so it equals $c_2(a'')$ for some $a'' \in F^*$. Hence $\alpha = c_4(a')\, c_2(a'')$.

Theorem 4.3 *If F is a field of characteristic $\neq 2, 3$ and $n|12$, then every element of K_2F of order n has the form*

$$\prod_{\substack{d|n,\\ d\leq 4}} c_d(a_d) \quad \textit{for some } a_d \in F^*.$$

Proof For $n \leq 4$ the theorem follows from [1, Theorem 1] and Theorems 4.1 and 4.2 above.

For $n = 6$ it is sufficient to observe that every element a of order 6 can be written as the product $a^3 \cdot a^4$ of elements of orders 2 and 3, respectively. Similarly, every element a of order 12 can be written as the product $a^4 \cdot a^9$ of elements of orders 3 and 4.

It is not true in general that every element of order 4 in K_2F has the form $c_4(a)$ for some $a \in F^*$, as the following example shows.

Example 4.1 For $F = \mathbb{Q}$ and $a \in \mathbb{Q}^*$ we have $\Phi_4(a) = a^2 + 1 > 0$. Consequently the real Hilbert symbol $\eta\ :\ K_2\mathbb{Q} \to \mu(\mathbb{R})$ satisfies $\eta(c_4(a)) = \eta\{a,\ a^2 + 1\}_{\mathbb{Q}} = 1$. On the other hand, if $c_4(a)$ has order 4, then $\{-1,\ -1\}_{\mathbb{Q}} \cdot c_4(a)$ has also order 4, but $\eta(\{-1,\ -1\}_{\mathbb{Q}} \cdot c_4(a)) = -1$.

We can take e.g. $a = 2$. Then $c_4(2) = \{2,\ 5\}_{\mathbb{Q}}$ has order 4, and the element $\{-1,\ -1\}_{\mathbb{Q}} \cdot \{2,\ 5\}_{\mathbb{Q}}$ cannot be expressed as $c_4(a)$ for some $a \in \mathbb{Q}^*$, as the real Hilbert symbol shows.

5 Transfer from Quadratic and Cubic Extensions

First we prove a formula for $\mathrm{Tr}_{F(a)/F}\{a,\ b\}_{F(a)}$, where $b \in F(a)^*$ has a simple form $b = ua + v,\ u, v \in F$. Next we apply it to fields $F(a)$ quadratic or cubic over F.

Proposition 5.1 *If $E = F(a)$ is an extension of F of degree d prime to the characteristic of F, and $g \in F[x]$ is the monic and irreducible polynomial with a root a, then for $b \in E^*$ of the form $b = ua + v$, where $u, v \in F$, we have*

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \begin{cases} \{(-1)^d g(0),\ v\}_F, & \textit{if } u = 0, \\ \{(-1)^d g(0),\ -u\}_F, & \textit{if } v = 0, \\ \{(-1)^d,\ -u\}_F \cdot \{g(0),\ v\}_F \cdot \{w,\ g(w)\}_F, & \textit{if } uv \neq 0, \end{cases}$$

where $w = -\frac{v}{u}$.

Proof We have $N_{F(a)/F}\, a = (-1)^d g(0)$.

If $u = 0$, then $b = v \in F^*$ and, by (2.1), we get

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \{N_{E/F}\,a,\ v\}_F = \{(-1)^d g(0),\ v\}_F.$$

If $v = 0$, then $b = ua$. Hence $\{a,\ b\}_E = \{a,\ ua\}_E = \{a,\ -u\}_E$, and we get similarly

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \mathrm{Tr}_{E/F}\{a,\ -u\}_E = \{(-1)^d g(0),\ -u\}_F.$$

Finally, let $uv \neq 0$. Then $b = -u(w - a)$, where $w = -\frac{v}{u} \in F^*$. Hence, by Proposition 2.1

$$\begin{aligned}
\mathrm{Tr}_{E/F}\{a,\ b\}_E &= \mathrm{Tr}_{E/F}\{a,\ -u\}_E \cdot \mathrm{Tr}_{E/F}\{a,\ w - a\}_E \\
&= \{(-1)^d g(0),\ -u\}_F \cdot \left\{w,\ \frac{g(w)}{g(0)}\right\}_F \\
&= \{(-1)^d,\ -u\}_F \cdot \{g(0),\ -uw\}_F \cdot \{w,\ g(w)\}_F \\
&= \{(-1)^d,\ -u\}_F \cdot \{g(0),\ v\}_F \cdot \{w,\ g(w)\}_F,
\end{aligned}$$

since $-uw = v$.

Corollary 5.1 *If $E = F(a)$ is a quadratic extension of F, $\mathrm{char}\,F \neq 2$, and $g \in F[x]$ is the monic irreducible polynomial with a root a, then for $b = ua + v \in E^*$, where $u, v \in F$, we have*

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \begin{cases} \{g(0),\ v\}_F, & \text{if } u = 0, \\ \{g(0),\ -u\}_F, & \text{if } v = 0, \\ \{g(0),\ v\}_F \cdot \{w,\ g(w)\}_F, & \text{if } uv \neq 0, \end{cases}$$

where $w = -\frac{v}{u} \in F^*$.

Proof Since $d = (E : F) = 2$, the corollary follows from Proposition 5.1.

To prove an analogous result for cubic extensions we need the following well-known lemma (see [7, p. 93]).

Lemma 5.1 *If $E = F(a)$ is a cubic extension of the field F, then every element $b \in E$ can be written in the form*

$$b = \frac{ua + v}{ta + z}, \quad \text{where } u, v, t, z \in F. \tag{5.1}$$

We can assume that $t = 0$ or -1, and $z = 1$ if $t = 0$. Then the representation (5.1) is unique.

Proof Every $b \in E$ can be written in the standard form $b = b_0 + b_1 a + b_2 a^2$, where $b_0, b_1, b_2 \in F$. Let $g(x) = g_0 + g_1 x + g_2 x^2 + x^3 \in F[x]$ be the minimal polynomial for a.

Then we have to find $u, v, t, z \in F$ satisfying $(ta + z)(b_0 + b_1 a + b_2 a^2) = ua + v$. Equivalently

$$zb_0 - b_2 t g_0 = v, \quad zb_1 + tb_0 - b_2 t g_1 = u, \quad zb_2 + b_1 t - b_2 g_2 t = 0.$$

To satisfy the last equation we take $t = 0,\ z = 1$, if $b_2 = 0$, and $t = -1,\ z = \frac{b_1 - b_2 g_2}{b_2}$, if $b_2 \neq 0$. Then u and v are determined by the first two equations.

The uniqueness is obvious.

Corollary 5.2 *If $E = F(a)$ is a cubic extension of a field F, $\mathrm{char}\,F \neq 3$, and $g \in F[x]$ is the monic irreducible polynomial with a root a, then for $b \in E^*$ we have*

(i) *If* $b = ua + v$, *or* $b = \frac{ua+v}{-a}$, *where* $u, v \in F$ (*i.e.*, $t = 0$, $z = 1$ *or* $t = -1$, $z = 0$ *in* (5.1)), *then*

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \begin{cases} \{-g(0),\ v\}_F, & \text{if } u = 0, \\ \{-g(0),\ -u\}_F, & \text{if } v = 0, \\ \{-1,\ -u\}_F \cdot \{g(0),\ v\}_F \cdot \{w,\ g(w)\}_F, & \text{if } uv \neq 0, \end{cases}$$

where $w = -\frac{v}{u}$.

(ii) *If* $b = \frac{ua+v}{z-a}$, $u, v, z \in F$, $z \neq 0$, (*i.e.*, $t = -1$, $z \neq 0$ *in* (5.1)), *then*

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \mathrm{Tr}_{E/F}\{a,\ ua + v\}_E \cdot \{z,\ g(z)/g(0)\}_F^{-1},$$

where the first factor is given in (i).

Proof (i) Since $d = (E : F) = 3$ is odd, we have $(-1)^d = -1$ and the corollary in the case $b = ua + v$ follows from Proposition 3.1.

If $z = 0$, then $\{a,\ z - a\}_E = \{a,\ -a\}_E = 1$. Hence for $b = \frac{ua+v}{z-a}$ the element $\mathrm{Tr}_{E/F}\{a,\ b\}_E = \mathrm{Tr}_{E/F}\{a,\ ua + v\}_E$ is given in (i).

(ii) If $z \neq 0$, then by the formula for b, we have

$$\mathrm{Tr}_{E/F}\{a,\ b\}_E = \mathrm{Tr}_{E/F}\{a,\ ua + v\}_E \cdot \mathrm{Tr}_{E/F}\{a,\ z - a\}_E^{-1}.$$

The first factor is given in (i); to the second one we apply Proposition 2.1. We get $\mathrm{Tr}_{E/F}\{a,\ z - a\}_E = \left\{z,\ \frac{g(z)}{g(0)}\right\}_F$. Hence the result follows.

6 Elements of Order 5 in K_2F

For the convenience, we include the following well-known result.

Lemma 6.1 *Let* n *be prime to the characteristic of the field* F. *Let* $E = F(\zeta_n)$ *and* $d = (E : F)$.

If $(n, d) = 1$, *then every element of order* n *in* K_2F *has the form*

$$\mathrm{Tr}_{E/F}\{\zeta_n,\ a\}_E \quad \text{for some } a \in E^*.$$

Proof Let $\alpha \in K_2F$ be an element of order n. Then

$$(\mathrm{Tr}_{E/F} \circ j_{F/E})(\alpha) = \alpha^d.$$

Since $(n, d) = 1$, we have $dd' \equiv 1 \pmod n$ for some d'. From $\alpha^n = 1$ it follows that $j_{F/E}(\alpha)^n = 1$. Hence by the Suslin theorem (see [6, Theorem 1.8]) we get

$$j_{F/E}(\alpha) = \{\zeta_n, b\}_E \quad \text{for some } b \in E^*.$$

Then $a := b^{d'}$ satisfies

$$\mathrm{Tr}_{E/F}\{\zeta_n, a\}_E = (\mathrm{Tr}_{E/F}\{\zeta_n, b\}_E)^{d'} = ((\mathrm{Tr}_{E/F} \circ j_{F/E})(\alpha))^{d'} = \alpha^{dd'} = \alpha.$$

Let F be a field of characteristic $\neq 5$. We shall determine all elements of order 5 in K_2F.

By Suslin's theorem, it is sufficient to consider the case $\zeta_5 \notin F$. Let $E = F(\zeta_5)$ and denote by σ the automorphism of E satisfying $\sigma(\zeta_5) = \zeta_5^{-1}$ and $\sigma|F = \mathrm{id}$. Then the subfield of E fixed by σ is $F(t)$, where $t = \zeta_5 + \zeta_5^{-1}$.

From $\sigma^2 = \mathrm{id}$ it follows that E is a quadratic extension of $F(t)$ generated by ζ_5. The minimal polynomial for ζ_5 over $F(t)$ is

$$\Phi_5^{(1)}(x) := (x - \zeta_5)(x - \zeta_5^{-1}) = x^2 - tx + 1.$$

Lemma 6.2 (See [2, Corollary 3.3]) *Every element $\alpha \in K_2F$ of order 5 has the form*

$$\alpha = \mathrm{Tr}_{F(t)/F}\{w, \Phi_5^{(1)}(w)\}_{F(t)} \quad \textit{for some } w \in F(t)^*.$$

In particular, if $t \in F$, then $\alpha = \{w, \Phi_5^{(1)}(w)\}_F = c_5^{(1)}(w)$ for some $w \in F^*$, and if $t \notin F$ and $w \in F^*$, then $\alpha = c_5(w)$.

Proof By Lemma 6.1, every element $\alpha \in K_2F$ of order 5 has the form

$$\alpha = \mathrm{Tr}_{E/F}\{\zeta_5,\ a\}_E \quad \text{for some } a \in E^*.$$

From the transitivity of transfer we get

$$\alpha = \mathrm{Tr}_{F(t)/F}(\mathrm{Tr}_{E/F(t)}\{\zeta_5,\ a\}_E).$$

Thus it is sufficient to find $\mathrm{Tr}_{E/F(t)}\{\zeta_5,\ a\}_E$.

Every $a \in E$ can be written in the form $a = u\zeta_5 + v$ for some $u, v \in F(t)$. We apply Corollary 3.1 to the quadratic extension E of $F(t)$ with $a, b, F, g(x)$ replaced by $\zeta_5, a, F(t), \Phi_5^{(1)}(x)$ respectively. Since $\Phi_5^{(1)}(0) = 1$, the corollary gives

$$\mathrm{Tr}_{E/F(t)}\{\zeta_5, a\}_E = \begin{cases} 1, & \text{if } uv = 0, \\ \{w, \Phi_5^{(1)}(w)\}_{F(t)}, & \text{if } uv \neq 0, \text{ where } w = -\dfrac{v}{u} \in F(t)^*. \end{cases}$$

Hence the first part of the lemma follows.

If $t \in F$, then the claim is obvious.

Assume that $t \notin F$. Then the automorphism $\tau \in \mathrm{Gal}(F(\zeta_5)/F)$ satisfying $\tau(\zeta_5) = \zeta_5^2$ has order 4, and hence it induces the nontrivial automorphism of $F(t)$ over F.

We have $\Phi_5(x) = \Phi_5^{(1)}(x) \cdot \tau(\Phi_5^{(1)}(x))$. Consequently $N_{F(t)/F}\ \Phi_5^{(1)}(w) = \Phi_5(w)$. If $w \in F^*$, then by the first part of the lemma we get

$$\alpha = \mathrm{Tr}_{F(t)/F}\{w,\ \Phi_5^{(1)}(w)\}_{F(t)} = \{w,\ N_{F(t)/F}\,\Phi_5^{(1)}(w)\}_F = \{w,\ \Phi_5(w)\}_F = c_5(w),$$

which proves the last part of the lemma.

In what follows we can assume that $t \notin F$ and $w \in F(t)^* \setminus F$. Hence $F(t) = F(w)$, and $\Phi_5^{(1)}(w)$ is an F-linear combination of 1 and w.

More precisely, we have

Lemma 6.3 *Let $t \notin F$, $w = rt + s$, where $r, s \in F$, $r \neq 0$. Then*

(i) $\Phi_5^{(1)}(w) = Aw + B$, *where*

$$A = 1 + 2s - r - \frac{s}{r}, \quad B = 1 - \left(1 - \frac{1}{r}\right)C, \quad C = -r^2 - rs + s^2.$$

(ii) *The minimal polynomial for w over F is*

$$g_w(x) = x^2 + (r-2s)x + C, \quad \text{hence } N_{F(t)/F}\, w = C.$$

Equivalently, $g_w(x) = r^2 \cdot \Psi_5\left(\frac{x-s}{r}\right)$, *where* $\Psi_5(x) = x^2 + x - 1 \in F[x]$ *is the minimal polynomial for t over F.*

Proof (i) Substituting $w = rt + s$ into $\Phi_5^{(1)}(w) = w^2 - tw + 1$ and applying $t^2 + t - 1 = 0$, we get $\Phi_5^{(1)}(w) = \varphi(r,s)\,t + \psi(r,s)$ with $\varphi, \psi \in F[x,y]$. Then put $t = \frac{w-s}{r}$; this gives the explicit formulas for A and B.

(ii) Since $t + \tau(t) = t \cdot \tau(t) = -1$, we get $w + \tau(w) = -r + 2s$ and $w \cdot \tau(w) = (rt+s)(r\tau(t) + s) = -r^2 - rs + s^2 = C$. Hence the minimal polynomial for w over F is

$$g_w(x) = (x-w)(x-\tau(w)) = x^2 + (r-2s)x + C.$$

Evidently both polynomials $g_w(x)$ and $r^2 \cdot \Psi_5\left(\frac{x-s}{r}\right)$ are quadratic, monic, irreducible and have a common root $x = w = rt + s$. Hence they are equal.

From the above lemmas we get immediately

Theorem 6.1 *Let F be a field of characteristic $\neq 5$, and assume that $\zeta_5 \notin F$. Then every element α of order 5 in K_2F can be written in the form*

$$\alpha = \alpha(w) = \mathrm{Tr}_{F(t)/F}\{w,\ \Phi_5^{(1)}(w)\}_{F(t)} \quad \text{for some } w \in F(t)^*,$$

where $t := \zeta_5 + \zeta_5^{-1}$ *and* $\Phi_5^{(1)}(x) = x^2 - tx + 1$.

More precisely,

(1) *If* $t \in F$, *then* $\alpha(w) = \{w,\ \Phi_5^{(1)}(w)\}_F = c_5^{(1)}(w)$ *for some* $w \in F^*$.

(2) *If* $t \notin F$ *and* $w \in F^*$, *then* $\alpha(w) = \{w, \Phi_5(w)\}_F = c_5(w)$.

(3) *If* $t, w \notin F$, *then, in the notation of Lemma 6.3,*

$$\alpha(w) = \begin{cases} \{C,\ B\}_F, & \text{if } A = 0, \\ \{C,\ -A\}_F, & \text{if } B = 0, \\ \{C,\ B\}_F \cdot \left\{ -\dfrac{B}{A},\ g_w\left(-\dfrac{B}{A}\right)\right\}_F, & \text{if } AB \neq 0. \end{cases}$$

Here $w = rt + s$ *with* $r, s \in F$, $r \neq 0$, $g_w(x) = x^2 + (r-2s)x + C \in F[x]$ *is the minimal polynomial for w over F, and A, B, C are defined in Lemma 6.3.*

Conversely, every such $\alpha(w)$ *satisfies* $\alpha(w)^5 = 1$.

Proof The first part of the theorem follows from Lemma 6.2. The case (1) is then obvious and the case (2) follows from the last part of Lemma 6.2.

To prove the case (3) we observe that, by Lemma 6.3,

$$\alpha(w) = \mathrm{Tr}_{F(t)/F}\{w,\ \Phi_5^{(1)}(w)\}_{F(t)} = \mathrm{Tr}_{F(t)/F}\{w,\ Aw + B\}_{F(t)}.$$

Now we can apply Corollary 5.1 with a, u, v, E replaced by $w, A, B, F(t)$, respectively, and we get the result.

Thus Theorem 6.1 gives a parametric description of all elements α of order 5 in K_2F with two parameters r, s. Namely $\alpha = \alpha(w) = \alpha(rt + s)$ is the product of at most two symbols in

K_2F with entries being rational functions of r and s. The formulas obtained are complicated, but in some particular cases, as shown in the corollaries below, they can be simplified.

Corollary 6.1 *In the notation of Theorem* 6.1 (*with s replaced by $-s$ for convenience*), *we have: If $w = t - s$ for some $s \in F^*$, then $\alpha(w) = c_5(s)^{-1}$.*

Proof We have $r = 1,\ A = -s,\ B = 1,\ C = s^2 + s - 1$, $g_w(x) = \Psi_5(x + s)$. Hence

$$g_w\Big(-\frac{B}{A}\Big) = \Psi_5(s + s^{-1}) = s^{-2}\Phi_5(s).$$

Then, by the case (3) of Theorem 6.1, we get

$$\alpha(w) = \Big\{-\frac{B}{A},\ g_w\Big(-\frac{B}{A}\Big)\Big\}_F = \{s^{-1},\ s^{-2}\,\Phi_5(s)\}_F = \{s^{-1},\ \Phi_5(s)\}_F = c_5(s)^{-1}.$$

Corollary 6.2 *In the notation of Theorem* 6.1 (*with r replaced by $-r$ for convenience*), *we have: If $w = -rt$ for some $r \in F^*$, then*

$$\alpha(w) = \{-1,\ \Phi_3(r)\}_F \cdot \Big\{-\frac{\Phi_3(r)}{r+1},\ \Psi_5(\Phi_3(r))\Big\}_F.$$

Proof We have $s = 0$, so $A = 1 + r,\ B = r^2 + r + 1,\ C = -r^2$. We claim that $AB \neq 0$.

In fact, if $A = 0$, i.e., $r = -1$, then $B = 1,\ C = -1$ and, by the case (3) of Theorem 6.1, we get $\alpha = \{C,\ B\}_F = 1$, a contradiction.

If $B = 0$, i.e., $r^2 + r + 1 = 0$, then $A = 1 + r = -r^2 = C$ and, by the case (3) of Theorem 6.1, $\alpha = \{C, -A\}_F = \{A, -A\}_F = 1$, a contradiction.

Thus we have $AB \neq 0$, and we can apply the case (3) of Theorem 6.1 once more.

$$\{C,\ B\}_F = \{-r^2,\ r^2 + r + 1\}_F = \{-1,\ \Phi_3(r)\}_F \cdot \Big\{\Phi_3(r),\ \frac{1}{r^2}\Big\}_F. \tag{6.1}$$

From

$$\frac{B}{A} = \frac{r^2 + r + 1}{r+1} = \frac{\Phi_3(r)}{r+1}, \quad g_c(x) = r^2\,\Psi_5\Big(-\frac{x}{r}\Big)$$

and the identity

$$\Psi_5(x+1) = x^2\,\Psi_5(1 + x^{-1}),$$

it follows that

$$g_w\Big(-\frac{B}{A}\Big) = r^2\,\Psi_5\Big(\frac{B}{Ar}\Big) = r^2\,\Psi_5\Big(1 + \frac{1}{r(r+1)}\Big) = \frac{1}{(r+1)^2}\,\Psi_5(\Phi_3(r)). \tag{6.2}$$

In view of $\{r+1,\ (r+1)^2\}_F = 1$, $\{r+1,\ r^2\}_F = 1$ and $\{\Phi_3(r),\ r^2(r+1)^2\}_F = \{\Phi_3(r),\ (1 - \Phi_3(r))^2\}_F = 1$ from (6.1) and (6.2) we get, by the case (3) of Theorem 6.1,

$$\begin{aligned}\alpha &= \{C,\ B\}_F \cdot \Big\{-\frac{B}{A},\ g_w\Big(-\frac{B}{A}\Big)\Big\}_F \\ &= \{-1,\ \Phi_3(r)\}_F \cdot \Big\{\Phi_3(r),\ \frac{1}{r^2}\Big\}_F \cdot \Big\{-\frac{\Phi_3(r)}{r+1},\ \frac{1}{(r+1)^2}\,\Psi_5(\Phi_3(r))\Big\}_F\end{aligned}$$

$$
\begin{aligned}
&= \{-1,\ \Phi_3(r)\}_F \cdot \Big\{\frac{\Phi_3(r)}{r+1},\ \frac{1}{r^2}\Big\}_F \\
&\quad \cdot \{-1,\ \Psi_5(\Phi_3(r))\}_F \cdot \Big\{\ \frac{\Phi_3(r)}{r+1},\ \frac{1}{(r+1)^2}\,\Psi_5(\Phi_3(r))\ \Big\}_F \\
&= \{-1,\ \Phi_3(r)\}_F \cdot \Big\{\frac{\Phi_3(r)}{r+1},\ \Psi_5(\Phi_3(r))\Big\}_F \cdot \{-1,\ \Psi_5(\Phi_3(r))\}_F \\
&= \{-1,\ \Phi_3(r)\}_F \cdot \Big\{-\frac{\Phi_3(r)}{r+1},\ \Psi_5(\Phi_3(r))\Big\}_F,
\end{aligned}
$$

and the corollary follows.

Remark 6.1 The last corollary implies that

$$\Big\{\frac{\Phi_3(r)}{r+1},\ \Psi_5(\Phi_3(r))\ \Big\}^{10} = 1$$

for every $r \in F^*$ such that $r \neq -1,\ \Phi_3(r) \neq 0,\ \Psi_5(\Phi_3(r)) \neq 0$.

Corollary 6.3 *In the notation of Theorem* 6.1*, we have: If* $A = 0$*, i.e.,* $s = \frac{r(r-1)}{2r-1}$*, and* $w = rt + s = r\big(t + \frac{r-1}{2r-1}\big)$*, then*

$$\alpha = \Big\{-\frac{r^2}{(2r-1)^2}(5r^2 - 5r + 1),\ \frac{1}{(2r-1)^2}(5(r^2 - r)^2 + 5(r^2 - r) + 1)\Big\}_F.$$

Proof It is sufficient to substitute $s = \frac{r(r-1)}{2r-1}$ in the formulas for B and C given in Lemma 6.3, and to apply case (1) of Theorem 6.1.

Remark 6.2 Let us note that $\frac{1}{5}\Psi_5(5(x+2)) = 5x^2 + 5x + 1$, so the formula for α in the last corollary can be rewritten in the form

$$\alpha = \Big\{-\frac{r^2}{(2r-1)^2} \cdot \frac{1}{5}\,\Psi_5(5(-r+2)),\ \frac{1}{(2r-1)^2} \cdot \frac{1}{5}\,\Psi_5(5(r^2 - r + 2))\Big\}_F.$$

Example 6.1 Put $F = \mathbb{Q},\ r = 2$ in Corollary 6.3. Then $s = \frac{2}{3}$ and $w = 2t + \frac{2}{3}$. Consequently the corollary gives

$$\alpha = \alpha(w) = \Big\{-\frac{4}{9} \cdot 11,\ \frac{1}{9} \cdot 31\Big\}_F.$$

Now the corresponding tame symbols satisfy

$$
\begin{aligned}
\tau_\infty(\alpha) &= 1, \\
\tau_3(\alpha) &= \Big(-\frac{4}{9} \cdot 11\Big)^2 \cdot \Big(\frac{1}{9} \cdot 31\Big)^{-2} \pmod 3 = \Big(\frac{4 \cdot 11}{31}\Big)^2 \pmod 3 = 1 \pmod 3, \\
\tau_{11}(\alpha) &= \Big(\frac{31}{9}\Big)^{-1} \pmod{11} = 1 \pmod{11}, \\
\tau_{31}(\alpha) &= -\frac{4}{9} \cdot 11 \pmod{31} = \frac{27}{9} \cdot 11 \pmod{31} = 2 \pmod{31}.
\end{aligned}
$$

Hence $\tau_{31}(\alpha^5) = 2^5 \pmod{31} = 1 \pmod{31}$. Therefore, in fact, α is an element of order 5 in $K_2\mathbb{Q}$.

7 Elements of Order 7 in K_2F

We give a short description of an argument leading to explicit formulas for all elements of order 7 in K_2F, where $\operatorname{char} F \neq 7$. We assume that $\Phi_7(x) \in F[x]$ is irreducible. In the other case we have $(F(\zeta_7) : F) \leq 3$, and the argument is much easier.

There are two subfields of $E = F(\zeta_7)$ containing F, namely $F_1 = F(\sqrt{-7}\,)$ quadratic over F, and $F_2 = F(t)$, where $t = \zeta_7 + \zeta_7^{-1}$, cubic over F.

From Lemma 6.1, it follows that every element $\alpha \in K_2F$ of order 7 has the form $\alpha = \operatorname{Tr}_{E/F}\{\zeta_7,\ a\}_E$ for some $a \in E^*$. We assume that $E = F(a)$, and the other cases are easier.

By the transitivity of transfer, we can write the element α in two ways

$$\alpha = \operatorname{Tr}_{F_j/F}(\operatorname{Tr}_{E/F_j}\{\zeta_7,\ a\}_E), \quad \text{where } j = 1 \text{ or } 2. \tag{7.1}$$

We give more details in both cases.

Case 1 The minimal polynomial for ζ_7 over F_1 is

$$\Phi_7^{(1)}(x) = (x - \zeta_7)(x - \zeta_7^2)(x - \zeta_7^4) = x^3 - \xi x^2 - (1 + \xi)x - 1,$$

where $\xi = \zeta_7 + \zeta_7^2 + \zeta_7^4 = \frac{-1+\sqrt{-7}}{2}$ is a root of $f(x) = x^2 + x + 2$.

We can apply Corollary 5.2 with $a, F, b, g(x)$ replaced by $\zeta_7, F_1, a, \Phi_7^{(1)}(x)$, respectively, and we get a formula for $\operatorname{Tr}_{E/F_1}\{\zeta_7,\ a\}_E$.

For example, if $a = u\frac{w'-\zeta_7}{w''-\zeta_7}$ with $u, w', w'' \in F_1^*$, then

$$\begin{aligned}\operatorname{Tr}_{E/F_1}\{\zeta_7,\ a\}_E &= \{-1,\ uw'w''\}_{F_1} \cdot \{w',\ \Phi_7^{(1)}(w')\}_{F_1} \cdot \{w'',\ \Phi_7^{(1)}(w'')\}_{F_1}^{-1} \\ &= \{-1,\ uw'w''\}_{F_1} \cdot c_7^{(1)}(w') \cdot c_7^{(1)}(w'').\end{aligned} \tag{7.2}$$

Now, in view of (7.1), we have to apply $\operatorname{Tr}_{F_1/F}$ to all factors on the r.h.s. of (7.2).

The case of the first factor is easy:

$$\operatorname{Tr}_{F_1/F}\{-1,\ uw'w''\}_{F_1} = \{-1,\ N_{F_1/F}(uw'w'')\}_F.$$

To investigate the next factors we need a lemma similar to Lemma 6.3.

Lemma 7.1 *If $w = u\xi + v$, where $u, v \in F$, $u \neq 0$ and ξ is a root of $f(x) = x^2 + x + 2$, then $\Phi_7^{(1)}(w) = Aw + B$, where*

$$\begin{aligned}A &= -u^2 + (-3v + 1)u + (3v^2 + 2v) - \frac{v^2 + v}{u}, \\ B &= 2u^3 - (5v + 2)u^2 + (3v^2 + 3v + 2)u - (2v^3 + 2v^2 + v + 1) + \frac{v^3 + v^2}{u}.\end{aligned}$$

Proof We have

$$\Phi_7^{(1)}(w) = \Phi_7^{(1)}(u\xi + v) = \varphi(u, v) + \xi\psi(u, v),$$

for some polynomials $\varphi,\ \psi \in F[x, y]$, which can be written explicitly. Then substituting $\xi = \frac{w-v}{u}$ we get the result.

Now we can apply Lemma 7.1 and Corollary 5.1 to the last two factors of (7.2) to get effective but complicated formulas for α.

Case 2 is similar. The minimal polynomial for ζ_7 over F_2 is $g(x) = x^2 - tx + 1$, where $t = \zeta_7 + \zeta_7^{-1}$, and the minimal polynomial for t over F is $\Psi_7(x) = x^3 + x^2 - 2x - 1$.

If $a = u\zeta_7 + v$, where $u, v \in F(t)^*$, then $a = -u(b - \zeta_7)$ with $b = -\frac{v}{u} \in F(t)^*$. We omit the easier case where $uv = 0$.

Then

$$\mathrm{Tr}_{E/F_2}\{\zeta_7,\ a\}_E = \mathrm{Tr}_{E/F_2}\{\zeta_7,\ -u\}_E \cdot \mathrm{Tr}_{E/F_2}\{\zeta_7,\ b - \zeta_7\}_E = \{b,\ g(b)\}_{F_2}$$

by Corollary 5.1, since $N_{E/F_2}\,\zeta_7 = 1$.

It remains to determine $\mathrm{Tr}_{F_2/F}\{b,\ g(b)\}_{F_2}$. Since $F_2 = F(t) = F(b)$ is a cubic extension of F, by Lemma 5.1, $g(b)$ can be written in the form

$$g(b) = \frac{pb + q}{rb + s}, \quad \text{where } p, q, r, s \in F,$$

and we can apply Corollary 5.2. We leave the details to the reader.

There is the following simple particular case similar to that given in Corollary 6.1.

Corollary 7.1 *In the above notation, if $a = ut + v$, where $u, v \in F(t)^*$ and $b = -\frac{v}{u}$ has the simple form $b = t + r$, for some $r \in F$, then*

$$\alpha = \mathrm{Tr}_{E/F}\{\zeta_7,\ a\}_E = c_7\Big(-\frac{1}{r}\Big).$$

Proof By the above, we have $\alpha = \mathrm{Tr}_{F_2/F}\{b,\ g(b)\}_{F_2}$. From $b = t + r$ and $g(x) = x^2 - tx + 1$ we get $g(b) = br + 1$.

The minimal polynomial for b over F is $\Psi_7(x - r) =: g_b(x)$. By Corollary 5.2, with $a, b, u, v, E, g(x)$ replaced by $b, br + 1, v, 1, F_2, g_b(x)$, respectively, we get

$$\alpha = \mathrm{Tr}_{F_2/F}\{b,\ br + 1\}_{F_2} = \{-1,\ -r\}_F \cdot \Big\{-\frac{1}{r},\ g_b\Big(-\frac{1}{r}\Big)\Big\}_F.$$

Now,

$$g_b\Big(-\frac{1}{r}\Big) = \Psi_7\Big(-r - \frac{1}{r}\Big) = (-r)^3\Phi_7\Big(-\frac{1}{r}\Big)$$

in view of the identity

$$\Psi_7(x + x^{-1}) = x^{-3}\Phi_7(x).$$

Hence

$$\Big\{-\frac{1}{r},\ g_b\Big(-\frac{1}{r}\Big)\Big\}_F = \Big\{-\frac{1}{r},\ -\frac{1}{r^3}\Big\}_F \cdot \Big\{-\frac{1}{r},\ \Phi_7\Big(-\frac{1}{r}\Big)\Big\}_F = \{-1,\ -r\}_F \cdot c_7\Big(-\frac{1}{r}\Big).$$

Consequently $\alpha = c_7\big(-\frac{1}{r}\big)$.

References

[1] Browkin, J., Elements of small orders in $K_2(F)$, Algebraic K-Theory, Lecture Notes in Math., **966**, 1982, 1–6.

[2] Urbanowicz, J., On elements of given order in K_2F, *J. Pure Appl. Algebra*, **50**, 1988, 295–307.

[3] Rosset, S. and Tate, J., A reciprocity law for K_2-traces, *Comment. Math. Helvetici*, **58**, 1983, 38–47.

[4] Milnor, J., Introduction to Algebraic K-Theory, Ann. of Math. Studies, **72**, Princeton Univ. Press, Princeton, 1971.

[5] Bass, H., K_2 des corps globaux, Semin. Bourbaki 1970/71, Exp. 382–399; Lecture Notes in Math, **244**, 1971, 233–255.

[6] Suslin, A. A., Torsion in K_2 of fields, *K-Theory*, **1**, 1987, 5–29.

[7] Delone, B. N. and Faddeev, D. K., The theory of irrationalities of the third degree (in Russian), *Trudy Mat. Inst. Steklov*, **11**, 1940, 1–340; English translation, Providence, 1964.