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Volume of Domains in Symmetric Spaces***

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Abstract The authors derive a formula for the volume of a compact domain in a symmetric space from normal sections through a special submanifold in the symmetric space. This formula generalizes the volume of classical domains as tubes or domains given as motions along the submanifold. Finally, some stereological considerations regarding this formula are provided.

Keywords Curvature-adapted submanifold, Lie triple systematic normal bundle, Root decomposable normal bundle, Symmetric space, Volume
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1 Introduction

It is usual to find in the literature expressions for the volume of a domain, from normal sections of the domain through to a given submanifold. For instance, when the domain is a tube around a submanifold M in a Riemannian manifold N, the precise expression of its volume is well-known when N is a rank 1 symmetric space and M is a compatible (curvature-adapted) submanifold (see [5]). The volume of the tube is also known for generic rank k symmetric spaces, when M is a special totally geodesic submanifold of the symmetric space (see [12]). Additionally, an older formula for the volume of some domains in $\mathbb{R}^3$ was given by Pappus and is commonly known as Guldin's theorem. This theorem gives the volume of a solid generated by a motion of a plane set along the circle described by its center of mass. The formula has been generalized for domains obtained as motions along a curve in some rank 1 symmetric spaces (see [6, 4]), and domains obtained as motions along a submanifold in a space form (see [3]).

Applied stereology, however, uses the estimation of volume from a systematic set of planes (the estimation of volume from Cavalieri's principle) and the estimation of volume using lines through a fixed point (see [1]).

In this work, we give an expression for the volume of domains in a symmetric space from normal sections through special submanifolds (see Theorem 4.1). This formula generalizes the known expressions for the volume of tubes and domains given as motions in some symmetric spaces. Moreover, in the last section we particularize the formula for surfaces in $\mathbb{R}^3$ and we obtain a result (see Corollary 5.1) from which we deduce some basic formulae of known estimators in applied stereology.

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2 General Results about Symmetric Spaces

Let $N = G/K$ be an n-dimensional symmetric space, (g, τ) be the orthogonal symmetric Lie algebra associated with N and

$$p = \{X \in g \mid \tau(X) = -X\}.$$

Let Δ be the root system with respect to a maximal abelian subspace h in p. The subspace

$$p_\alpha = \{X \in p \,|\, (\mathrm{ad}Y)^2 X = -\alpha^2(Y)X, \ \forall Y \in h\}, \quad \alpha \in \Delta \tag{2.1}$$

is called the root space for α. Here we note that, in case where N is non-compact, these roots imply $\sqrt{-1}$ multiplies of the usual roots.

Elements of p_α are called root vectors for α and h is interpreted as the root space for 0.

Let Δ_+ be the set of all positive roots with respect to some lexicographic ordering of h. Then we have

$$p = h \oplus \sum_{\alpha \in \Delta^+} p_\alpha, \tag{2.2}$$

which is called the root space decomposition with respect to h. Without loss of generality, given $X \in h$, we assume that

$$0 = \alpha_1(X) = \cdots = \alpha_q(X) < \alpha_{q+1}(X) \leq \cdots \leq \alpha_n(X), \tag{2.3}$$

where q is the rank of the symmetric space N. From now on, p will be identified with the tangent space T_pN for each $p \in N$.

3 A Special Class of Submanifolds in Symmetric Spaces

Let M be an s-dimensional submanifold in the symmetric space N. If, for each $x \in M$, $T_x^\perp M$ is a Lie triple system in $T_xN \equiv p$, then we shall say that M has a Lie triple systematic normal bundle. In this case, there exists a totally geodesic submanifold M'_x of N such that $T_xM'_x = T_x^\perp M$ (see [10, p. 237], [9, p. 224]). Note that every totally geodesic submanifold can be translated by an element of G so as to contain 0.

Moreover, if, for each $\xi (\neq 0) \in T_x^\perp M$, there exists a maximal abelian subspace h in p containing ξ such that

$$T_x^\perp M = (h \cap T_x^\perp M) \oplus \sum_{\alpha \in \Delta_+} (p_\alpha \cap T_x^\perp M), \tag{3.1}$$

then we shall say that M has a root decomposable normal bundle.

Remark 3.1 M has a root decomposable normal bundle if and only if the operator $R(\,\cdot\,, v)v$ leaves T_xM invariant (x is the base point of v) (see [11]). Therefore, if M is a curvature-adapted submanifold of N, it has a root decomposable normal bundle.

Suppose that M has a Lie triple systematic normal bundle. Then, since for each $x \in M$, M'_x is totally geodesic in N, it will be a Riemannian symmetric space. We will therefore have the following root space decomposition

$$T_xM'_x = T_x^\perp M = h' \oplus \sum_{\alpha' \in \Delta'_+} p'_{\alpha'} \tag{3.2}$$

with respect to a maximal Abelian subspace h' of $T_xM'_x$.

Remark 3.2 If M has a root decomposable and a Lie triple systematic normal bundle, the decompositions given in (3.1) and (3.2) are equivalent in the sense that $h \cap T_x^\perp M$ is a maximal Abelian subspace of $T_x^\perp M$ and $p_\alpha \cap T_x^\perp M$ is a root space of M'_x with respect to $h \cap T_x^\perp M$; that is, a root $\alpha \in \Delta_+$ restricted to $T_x^\perp M$ corresponds to a root $\alpha' \in \Delta'_+$.

From now on, M will denote a submanifold of N which has a root decomposable and a Lie triple systematic normal bundle, such that $\forall x \in M$, M'_x is a rank 1 symmetric space; that is, $\dim(h \cap T_x^\perp M) = 1$.

Remark 3.3 All submanifolds in a real space form satisfy the conditions imposed above (see [11]).

Notation Given a vector $X \in h \cap T_x^\perp M$, the roots of M'_x are supposed to be ordered as

$$0 = \alpha'_1(X) < \alpha'_2(X) \le \cdots \le \alpha'_{n-s}(X), \tag{3.3}$$

where each α'_i corresponds to a root α_j restricted to $T_x^\perp M$.

The remaining positive roots α_j of N, which are not included in (3.3), will be denoted as

$$\beta_1(X), \cdots, \beta_s(X) \quad \text{with} \quad \beta_1(X) = \cdots = \beta_{q-1}(X) = 0. \tag{3.4}$$

4 Volume of Domains in the Symmetric Space *N*

Since, for each $x \in M$ there exists a totally geodesic submanifold M'_x of N such that $T_xM \oplus T_xM'_x = T_xN$, we will consider compact domains D in N such that, for each $x \in M$, there exists a diffeomorphism

$$\phi : \bigcup_{x \in M} D_x \to D, \tag{4.1}$$

where $D_x = D \cap M'_x$ and, for each $(x, z) \in M \times D_x$, $\phi(x, z) = \exp_x(tN_z)$, where N_z is the unitary vector at $x \in M$, tangent to the minimal geodesic of M'_x from x to z, and t is the distance from x to z (note that the distance from x to z in N coincides with this distance in M'_x).

Remark 4.1 Particular examples of domains that are expressed as (4.1) are tubes around the submanifold M (see [12]) and domains given as motions along M (see [3]).

Our aim is to obtain a new expression for the volume of D from the diffeomorphism in (4.1). Let ω denote the volume element of D (that is, the volume element of the symmetric space N), dx that of M and dz that of D_x (that is, dz is the volume element of the totally geodesic submanifold M'_x).

Furthermore, since we have supposed that M'_x is a rank 1 symmetric space, there will exist only one unitary vector in $h \cap T_x^\perp M$. Therefore the roots in (3.3) and (3.4) will be constants α'_i and β_j, respectively, with respect to this unitary vector. Hence, for every $\beta \in \mathbb{R}$, $s_\beta : \mathbb{R} \to \mathbb{R}$ will denote the solution to the equation $s'' + \beta s = 0$ with the initial conditions $s(0) = 0$ and $s'(0) = 1$; and $c_\beta = s'_\beta$; i.e.,

$$s_\beta(s) = \begin{cases} \dfrac{\sin(s\sqrt{\beta})}{\sqrt{\beta}}, & \beta > 0, \\ s, & \beta = 0, \\ \dfrac{\sinh(s\sqrt{\beta})}{\sqrt{\beta}}, & \beta < 0. \end{cases} \tag{4.2}$$

Theorem 4.1 *Let M be a submanifold of N as before and let $k_1, \cdots, k_s$ denote the eigenvalues of the Weingarten map of M with respect to the unitary vector in $h \cap T_x^{\perp} M$. Then,*

$$\mathrm{Vol}(D) = \int_D \omega = \int_M \int_{D_x} \prod_{i=1}^{s} (c_{\beta_i}(t) - s_{\beta_i}(t) k_i)\, dz dx. \tag{4.3}$$

Proof Let $\{e_1, \cdots, e_s\}$ and $\{e'_1, \ldots, e'_{n-s}\}$ be the orthonormal basis of $T_x M$ and $T_z D_x = \tau_t(T_x^{\perp} M)$, respectively, which are root vectors associated to the roots in (3.3) and (3.4), respectively. (τ_t denotes the parallel transport along the minimal geodesic γ_z from x to z.)

$$\mathrm{Vol}(D) = \int_D \omega = \int_M \int_{D_x} \phi^* \omega, \tag{4.4}$$

$$\phi^* \omega = |\phi_* e_1 \wedge \cdots \wedge \phi_* e_s \wedge \phi_* e'_1 \wedge \cdots \wedge \phi_* e'_{n-s}| dz dx. \tag{4.5}$$

For $1 \leq a \leq s$, let c_a be a curve in M with $c'_a(0) = e_a$. Then $\phi_{*(x,z)}(e_a) = Y_a(t)$, where Y_a is the Jacobi field along $\gamma_z(t) = \exp_x(tN_z)$ such that $Y_a(0) = e_a$ and $Y'_a(0) = \nabla_{e_a} N_z$. Since $N_z \in T_x^{\perp} M = T_x M'_x$ and M'_x is a rank 1 symmetric space we may suppose that $N_z = e'_1$, that is, the unique unitary vector in the maximal Abelian subspace $h \cap T_x^{\perp} M$. Then, the Jacobi fields Y_a are given by

$$Y_a(t) = \tau_t \Big(c_{\beta_a}(t) e_a + \sum_{i=1}^{s} s_{\beta_i}(t) (\nabla_{e_a} N_z)_i^T + \sum_{j=2}^{n-s} s_{\alpha'_j}(t) (\nabla_{e_a} N_z)_j^{\perp} \Big), \tag{4.6}$$

where $(\nabla_{e_a} N_z)_i^T$ denotes the component of $\nabla_{e_a} N_z$ tangent to M at x in the direction of e_i and $(\nabla_{e_a} N_z)_j^{\perp}$ denotes the component of $\nabla_{e_a} N_z$ orthogonal to M at x in the direction of e'_j.

On the other hand, from (4.1), $\{\phi_* e'_1, \cdots, \phi_* e'_{n-s}\}$ will be an orthonormal basis of $T_z D_x$; so

$$\phi^* \omega = |Y_1(t) \wedge \cdots \wedge Y_s(t) \wedge \phi_* e'_1 \wedge \cdots \wedge \phi_* e'_{n-s}| dz dx. \tag{4.7}$$

Now, bearing in mind that $\{\phi_* e'_1, \cdots, \phi_* e'_{n-s}\}$ are normal to M, the exterior multiplication of $\phi_* e'_1 \wedge \cdots \wedge \phi_* e'_{n-s}$ by $(\nabla_{e_a} N_z)^{\perp}$ will be zero. Moreover, $(\nabla_{e_a} N_z)^T = -L_{N_z} e_a$, where L_{N_z} denotes the Weingarten map of M at x in the direction of N_z. Then, from (4.5) and (4.6) we have

$$\begin{aligned} \phi^* \omega = \Big| & \Big(c_{\beta_1}(t) \tau_t e_1 - \sum_{i=1}^{s} s_{\beta_i}(t) \tau_t (L_{N_z} e_1)_i \Big) \wedge \cdots \wedge \Big(c_{\beta_1}(t) \tau_t e_s - \sum_{i=1}^{s} s_{\beta_i}(t) \tau_t (L_{N_z} e_s)_i \Big) \\ & \wedge \phi_* e'_1 \wedge \cdots \wedge \phi_* e'_{n-s} \Big| dz dx. \end{aligned} \tag{4.8}$$

Now, if we consider an orthonormal basis $\{\overline{e}_1, \cdots, \overline{e}_s\}$ given by eigenvectors of the Weingarten map; i.e., $L_{N_z} \overline{e}_i = k_i \overline{e}_i$ $(i = 1, \cdots, s)$, using the properties of the wedge product and the fact that the transformation matrix A from $\{e_1, \cdots, e_s\}$ to $\{\overline{e}_1, \cdots, \overline{e}_s\}$ is orthogonal ($\det A = 1$) we obtain, from (4.8) and (4.4), the result.

Remark 4.2 Equation (4.3) generalizes the volumes of tubes around totally geodesic submanifolds M obtained in [12] and the volumes of domains in space forms given as motions along M (see [6, 3]).

5 Volume of Domains in $\mathbb{R}^3$: Stereological Implications

Let S be a regular orientable surface in $\mathbb{R}^3$ and D a domain in $\mathbb{R}^3$ which satisfies (4.1). In this case D_x are domains in $\mathbb{R}$ and, from Theorem 4.1, we obtain

Corollary 5.1 *The volume of the domain D is given by*

$$\mathrm{Vol}\,(D) = \int_S \mathrm{Length}(D_x)\sigma - 2\int_S H\Big(\int_{D_x} tdt\Big)\sigma + \int_S K\Big(\int_{D_x} t^2 dt\Big)\sigma, \tag{5.1}$$

where σ is the area element of S, dt is the line element of $\mathbb{R}$ and H and K denote the mean and Gauss curvatures of S, respectively. (*Only the sign in the second term* $(-2\int_S H(\int_{D_x} tdt)\sigma)$ *depends on the choice of the normal vector to S.*)

Properties 5.1 (1) *Suppose that S is flat. Then $H = 0$ and $K = 0$, and*

$$\mathrm{Vol}\,(D) = \int_S \mathrm{Length}(D_x)\sigma, \tag{5.2}$$

which is the basic formula for the 'fakir' estimator of the volume (*see* [2]).

(2) *Suppose that $S = S^2(r)$ is a sphere of radius r in $\mathbb{R}^3$. Then $H = \frac{1}{r}$ and $K = \frac{1}{r^2}$. Now, we consider the local parametrization of S given by the spherical coordinates and we obtain*

$$\begin{aligned}\mathrm{Vol}\,(D) = r^2 \int_0^{2\pi}\int_0^{\pi} (\mathrm{Length}(D_x))\sin(\phi)d\phi d\theta - 2r\int_0^{2\pi}\int_0^{\pi}\Big(\int_{D_x} tdt\Big)\sin(\phi)d\phi d\theta \\ + \int_0^{2\pi}\int_0^{\pi}\Big(\int_{D_x} t^2dt\Big)\sin(\phi)d\phi d\theta.\end{aligned} \tag{5.3}$$

Now, supposing that the diffeomorphism defined in (4.1) *exists for each sphere $S^2(r)$ when r tends to zero, we obtain, by taking limits in the above formula, that*

$$\mathrm{Vol}(D) = \int_0^{2\pi}\int_0^{\pi}\Big(\int_{D_x} t^2dt\Big)\sin(\phi)d\phi d\theta, \tag{5.4}$$

which is the basic formula for the 'nucleator' estimator of volume (*see* [8]).

(3) *Now, we consider that $S = P(r)$ is a tubular surface of radius r around a unit speed curve $\beta : [a, b] \longrightarrow \mathbb{R}^3$. We consider the parametrization of S given by*

$$X(u, v) = \beta(u) + r\cos(v)\mathbf{n}(u) + r\sin(v)\mathbf{b}(u), \quad u \in [a, b],\ v \in [0, 2\pi), \tag{5.5}$$

where $\mathbf{n}(u)$ and $\mathbf{b}(u)$ are the normal and binormal vectors to $\beta(u)$. Let $\kappa(u)$ denote the curvature of $\beta(u)$. Then, using the Frenet formulas, we have that the principal curvatures of S are

$$k_1 = \frac{-1}{r} \quad and \quad k_2 = \frac{\kappa(u)\cos(v)}{1 - r\kappa(u)\cos(v)}, \tag{5.6}$$

$$\|X_u \times X_v\| = r(1 - r\kappa(u)\cos(v)). \tag{5.7}$$

Then

$$\begin{aligned}\mathrm{Vol}(D) = r\int_a^b\int_0^{2\pi}(1 - r\kappa(u)\cos(v))(\mathrm{Length}(D_x))dvdu \\ + \int_a^b\int_0^{2\pi}\Big(\int_{D_x} tdt\Big)dvdu - 2r\int_a^b\int_0^{2\pi}\kappa(u)\cos(v)\Big(\int_{D_x} tdt\Big)dvdu \\ - \int_a^b\int_0^{2\pi}\kappa(u)\cos(v)\Big(\int_{D_x} t^2dt\Big)dvdu.\end{aligned} \tag{5.8}$$

Now, supposing that the diffeomorphism defined in (4.1) *exists for each tubular surface* $P(r)$ *when* r *tends to zero, we obtain, by taking limits in the above formula, that*

$$\mathrm{Vol}\,(D) = \int_a^b \Big(\int_0^{2\pi}\int_{D_x} t dt dv\Big) du - \int_a^b \Big(\int_0^{2\pi}\int_{D_x} \cos(v) t^2 dt dv\Big)\kappa(u) du. \tag{5.9}$$

If P_u *is the plane orthogonal to* $\beta(u)$ *and* D_u *denotes the intersection* $P_u \cap D$*, using polar coordinates in* P_u *centered at* $\beta(u)$*, we have that*

$$\mathrm{Area}(D_u) = \int_0^{2\pi}\int_{D_x} t dt dv \tag{5.10}$$

and, on the other hand,

$$M_{(\mathbf{n}(u))^\perp}(D_u) = -\int_0^{2\pi}\int_{D_x} \cos(v) t^2 dt dv \tag{5.11}$$

is the moment of D_u *with respect to the line in* P_u *orthogonal to* $\mathbf{n}(u)$*; so formula* (5.3) *coincides with formula* (5.1) *of* [7].

(4) *When* $D_x = [0, r]$ *for all* $p \in S$*, we have*

$$\mathrm{Vol}\,(D) = r\mathrm{Area}(S) - r^2 \int_S H\sigma + \frac{r^3}{3}\int_S K\sigma, \tag{5.12}$$

which is a well-known formula for the volume of half-tubes related to a classical formula by Steiner that was proved in 1840 (*see* [5]).

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