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Integrability of det∇u and Evolutionary Wente's Problem Associated to Heat Operator

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Abstract In this note, the authors resolve an evolutionary Wente's problem associated to heat equation, where the special integrability of det∇u for $u \in H^1(\mathbb{R}^2, \mathbb{R}^2)$ is used.

Keywords Jacobian determinant, Heat equation, Wente's problem
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1 Introduction

The classical Wente's problem arises in the study of constant mean curvature immersions (see [9]), for which the scalar version is just the following problem:

$$\begin{cases} -\Delta\psi = \det\nabla v = a_{x_1}b_{x_2} - a_{x_2}b_{x_1}, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $x = (x_1, x_2)$, a, b are functions defined in Ω, a bounded domain in $\mathbb{R}^2$. If $\Omega = \mathbb{R}^2$, we shall replace the boundary condition by the ground state condition

$$\lim_{|x|\to\infty} \psi(x) = 0,$$

where $|x|$ is the Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$. In both cases, when $v = (a, b) \in H^1(\Omega, \mathbb{R}^2)$, it is proved in [10, 3] that ψ, the solution of (1.1), exists and lies in $C(\overline{\Omega})$ and $\nabla\psi$ belongs to $L^2(\Omega, \mathbb{R}^2)$. More precisely, we have

$$\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} \le C(\Omega)\|\nabla a\|_{L^2(\Omega)}\|\nabla b\|_{L^2(\Omega)}. \tag{1.2}$$

Many works have been done to estimate the best constants (see for example [1, 8, 6] and some other generalizations in [2, 4]).

Here, we deal with the following problem: Let $u \in H^1_{\rm loc}(\mathbb{R}^+, H^1(\mathbb{R}^2, \mathbb{R}^2))$ and consider

$$\begin{cases} \partial_t\varphi - \Delta_x\varphi = \det\nabla u(t, x), & \text{in } \mathbb{R}^+ \times \mathbb{R}^2, \\ \lim\limits_{|x|\to\infty} \varphi(t, x) = 0, & \forall\, t > 0, \\ \varphi(0, x) = 0, & \text{on } \mathbb{R}^2. \end{cases} \tag{1.3}$$

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Recall that for an interval I in $\mathbb{R}$ and Banach space E, $H^1(I,E)$ denotes the standard Sobolev space of functions in $L^2(I,E)$ such that the derivative is also in $L^2(I,E)$.

It is not trivial that a solution exists for (1.3), since the second member lies apparently just in $L^1(\mathbb{R}^2)$. But we know that $\det\nabla u$ has a special structure which admits some higher integrability than L^1, and it lies indeed in the Hardy space $\mathcal{H}^1$ (see [5]). Here we will use the special form of Jacobian determinant to show that a unique global solution φ exists. Moreover, $\|\varphi\|_{L^\infty(\mathbb{R}^2)}$ is locally bounded (for the time t) and we can get nearly the best estimate for its L^∞ norm.

First, the linearity of our problem permits us to decompose the solution φ as $\varphi_1+\varphi_2$, where

$$\begin{cases} \partial_t\varphi_2 - \Delta_x\varphi_2 = 0, & \text{in } \mathbb{R}^+\times\mathbb{R}^2,\\ \lim\limits_{|x|\to\infty}\varphi_2(t,x) = 0, & \forall\, t>0,\\ \varphi_2(0,x) = -\varphi_0(x), & \text{on } \mathbb{R}^2, \end{cases} \tag{1.4}$$

$$\begin{cases} \partial_t\varphi_1 - \Delta_x\varphi_1 = \det\nabla u(t,x), & \text{in } \mathbb{R}^+\times\mathbb{R}^2,\\ \lim\limits_{|x|\to\infty}\varphi_1(t,x) = 0, & \forall\, t>0,\\ \varphi_1(0,x) = \varphi_0(x), & \text{on } \mathbb{R}^2, \end{cases} \tag{1.5}$$

where φ_0 is the solution of classical Wente's problem in $\mathbb{R}^2$, associated to $u(0,x)$:

$$\begin{cases} -\Delta\varphi_0 = \det\nabla u(0,x), & \text{in } \mathbb{R}^2,\\ \lim\limits_{|x|\to\infty}\varphi_0(x) = 0. \end{cases} \tag{1.6}$$

It is well-known that $\varphi_2(t,x)$ exists and is given by $\varphi_2 = -E(t,\,\cdot\,)*\varphi_0$, where

$$E(t,x) = \frac{1}{4\pi t}e^{-\frac{|x|^2}{4t}}, \quad t>0,\ x\in\mathbb{R}^2,$$

denotes the fundamental solution of heat operator in $\mathbb{R}^2$, that is, E satisfies $\partial_t E - \Delta_x E = \delta_{(0,0)}$. It is easy to see that $\|\varphi_2\|_\infty \le \|\varphi_0\|_\infty$ and by the ground state condition for φ_0,

$$\lim_{t\to\infty}\|\varphi_2\|_\infty = 0.$$

Thus our study will concentrate on that of φ_1.

Throughout this note, $\|\cdot\|_p$ denotes always the L^p norm over $\mathbb{R}^2$, ∇ and Δ denote always the derivation with respect to the variable x. Our main results state

Theorem 1.1 *Let u be a function in $H^1_{\text{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^2,\mathbb{R}^2))$. Then a unique global solution of* (1.5) *exists and $\varphi_1\in C(\mathbb{R}^+\times\mathbb{R}^2)$. Furthermore,*

$$\sup_{\Sigma(u)\neq 0}\,\sup_{t>0}\frac{\|\varphi_1(t,\,\cdot\,)-\varphi_0\|_\infty}{G_u(t)} = \frac{1}{2\pi} \tag{1.7}$$

with

$$G_u(t) = \int_0^t \|\nabla(\partial_s a)(s,\,\cdot\,)\|_2\|\nabla b(s,\,\cdot\,)\|_2 + \|\nabla a(s,\,\cdot\,)\|_2\|\nabla(\partial_s b)(s,\,\cdot\,)\|_2 ds,$$

where a, b are the two companents of u and $\Sigma(u) = G_u(\infty)$.

Remark 1.1 Consequently, we get $\varphi\in C(\mathbb{R}^+\times\mathbb{R}^2)$.

Theorem 1.2 *The solution φ_1 belongs to $C(\mathbb{R}^+, H^1(\mathbb{R}^2))$ if $u \in H^1_{\mathrm{loc}}(\mathbb{R}^+, H^1(\mathbb{R}^2, \mathbb{R}^2))$. Furthermore, $t \mapsto \varphi_1(t, \cdot)$ is locally Lipschitz with values in $L^2(\mathbb{R}^2)$ and we have the following estimates: for any $t > 0$,*

$$\|\varphi_1(t, \cdot) - \varphi_0\|_2^2 \le \frac{1}{\pi}\int_0^t G_u(s)^2 ds, \tag{1.8}$$

$$\frac{1}{2}\|\nabla\varphi_1(t, \cdot) - \nabla\varphi_0\|_2^2 + \|\partial_t\varphi_1\|^2_{L^2([0,t]\times\mathbb{R}^2)} \le \frac{3G_u(t)^2}{4\pi}. \tag{1.9}$$

2 Proof of Theorem 1.1

If $u \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$, the space of C^∞ functions with compact support, we know that the solution of (1.5) is explicitly given by

$$\varphi_1(t,x) = E(t, \cdot) * \varphi_0(x) + \int_0^t E(s, \cdot) * \det\nabla u(t-s, \cdot)(x)ds. \tag{2.1}$$

We will establish the estimate (1.7) in this case. Then the existence and estimate of φ_1 in general case will come from density argument. First, we consider the value of φ_1 at the point $(t,0)$. We have

$$\varphi_1(t,0) = \int_{\mathbb{R}^2} \frac{e^{-\frac{|y|^2}{4t}}}{4\pi t}\varphi_0(y)dy + \int_0^t\int_{\mathbb{R}^2} \frac{e^{-\frac{|y|^2}{4s}}}{4\pi s}\det\nabla u(t-s,y)dyds = \mathrm{I} + \mathrm{J}. \tag{2.2}$$

Using polar coordinates, we have

$$\begin{aligned}\mathrm{I} &= \int_0^{2\pi}\int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{4\pi t}\varphi_0(r,\theta)rdrd\theta = -\frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \partial_r\big(e^{-\frac{r^2}{4t}}\big)\varphi_0(r,\theta)drd\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty e^{-\frac{r^2}{4t}}\partial_r\varphi_0 drd\theta + \varphi_0(0).\end{aligned}$$

On the other hand, since

$$\det\nabla u = \frac{(a_r b)_\theta - (a_\theta b)_r}{r},$$

we get

$$\begin{aligned}\mathrm{J} &= \int_0^t\int_0^{2\pi}\int_0^{+\infty} \frac{e^{-\frac{r^2}{4s}}}{4\pi s}[(a_r b)_\theta - (a_\theta b)_r](t-s,r,\theta)drd\theta ds \\ &= \int_0^t\int_0^{2\pi}\int_0^\infty \partial_r\Big(\frac{e^{-\frac{r^2}{4s}}}{4\pi s}\Big)(a_\theta b)(t-s,r,\theta)drd\theta ds \\ &= -\int_0^t\int_0^{2\pi}\int_0^\infty \partial_s\Big(\frac{e^{-\frac{r^2}{4s}}}{2\pi r}\Big)(a_\theta b)(t-s,r,\theta)drd\theta ds.\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_1(t,0) - \varphi_0(0) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty e^{-\frac{r^2}{4t}}\partial_r\varphi_0 drd\theta - \int_0^\infty\int_0^{2\pi}\frac{e^{-\frac{r^2}{4t}}}{2\pi r}(a_\theta b)(0,r,\theta)drd\theta \\ - \int_0^t\int_0^{2\pi}\int_0^\infty \frac{e^{-\frac{r^2}{4s}}}{2\pi r}\partial_s(a_\theta b)(t-s,r,\theta)drd\theta ds.\end{aligned} \tag{2.3}$$

Furthermore, $-\Delta\varphi_0 = \det\nabla u(0,x)$ means

$$-\frac{1}{r}\partial_r(r\partial_r\varphi_0) - \frac{1}{r^2}\partial_\theta^2\varphi_0 = \frac{(a_r b)_\theta - (a_\theta b)_r}{r},$$

so

$$-\partial_r[r\partial_r\varphi_0 - a_\theta b] = \partial_\theta\Big(\frac{\partial_\theta\varphi_0}{r} + a_r b\Big).$$

Consequently

$$r\partial_r\varphi_0(r,\theta) - (a_\theta b)(0,r,\theta) = -\int_0^r \partial_\theta\Big[\frac{1}{\sigma}(\partial_\theta\varphi_0)(\sigma,\theta) + (a_r b)(0,\sigma,\theta)\Big]d\sigma.$$

We get finally

$$\begin{aligned}\varphi_1(t,0) - \varphi_0(0) = &\int_0^\infty\int_0^{2\pi} \frac{e^{-\frac{r^2}{4t}}}{2\pi r}[r\partial_r\varphi_0(r,\theta) - a_\theta b(0,r,\theta)]drd\theta\\ &- \int_0^t\int_0^\infty\int_0^{2\pi} \frac{e^{-\frac{r^2}{4s}}}{2\pi r}\partial_s(a_\theta b)(t-s,r,\theta)drd\theta ds\\ = &- \int_0^t\int_0^\infty\int_0^{2\pi} \frac{e^{-\frac{r^2}{4(t-s)}}}{2\pi r}\partial_s(a_\theta b)(s,r,\theta)drd\theta ds. \end{aligned} \tag{2.4}$$

If we denote

$$\overline{b}(s,r) = \frac{1}{2\pi}\int_0^{2\pi} b(s,r,\theta)d\theta,$$

we have

$$\int_0^{2\pi} |b-\overline{b}|^2 d\theta \le \int_0^{2\pi} b_\theta^2 d\theta, \quad \forall\, b \in H^1(0,2\pi).$$

Thus

$$\begin{aligned}|\varphi_1(t,0) - \varphi_0(0)| &\le \frac{1}{2\pi}\int_0^t\int_0^\infty \frac{1}{r}\int_0^{2\pi} |[b-\overline{b}(s,r)]\partial_s a_\theta| + |a_\theta\partial_s[b-\overline{b}(s,r)]|d\theta drds\\ &\le \frac{1}{2\pi}\int_0^t \|\nabla(\partial_s a)(s,\cdot)\|_2\|\nabla b(s,\cdot)\|_2 + \|\nabla a(s,\cdot)\|_2\|\nabla(\partial_s b)(s,\cdot)\|_2 ds\\ &= \frac{G_u(t)}{2\pi}.\end{aligned}$$

The last inequality comes from

$$\begin{aligned}\int_0^t \frac{1}{r}\int_0^{2\pi} |\partial_s a_\theta[b-\overline{b}(s,r)]|drd\theta &\le \int_0^t \frac{1}{r}\|\partial_s a_\theta\|_{L^2(0,2\pi)}\|b-\overline{b}(s,r)\|_{L^2(0,2\pi)}dr\\ &\le \Big[\int_0^t\int_0^{2\pi}\frac{(\partial_s a_\theta)^2}{r}d\theta dr\Big]^{\frac12}\Big[\int_0^t\int_0^{2\pi}\frac{(\partial_\theta b)^2}{r}d\theta dr\Big]^{\frac12}\\ &\le \|\nabla\partial_s a(s,\cdot)\|_2\|\nabla b(s,\cdot)\|_2.\end{aligned}$$

As the equation (1.1) is invariant under translation for the variable x, we obtain the same estimate for all x by considering $\varphi_1(x+\cdot)$, hence

$$|\varphi_1(t,x) - \varphi_0(x)| \le \frac{G_u(t)}{2\pi}, \quad \forall\, t \in \mathbb{R}^+,\ x \in \mathbb{R}^2. \tag{2.5}$$

For the inverse inequality, let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth, decreasing, compactly supported function such that $h(0) = 1$. Take now $u(s, x) = h(s)u_0(x)$ where u_0 will be determined later, and let φ_0 be the solution of classical Wente's problem corresponding to u_0. So the solution of heat equation (1.5) is explicitly given by (2.2). We will look at the value of $\varphi_1(t, 0)$.

If we take $u_0 = (a_0, b_0) = g(r)x$ where g is a smooth radial function with compact support in $\mathbb{R}^2$, then $u_0 \in H^1(\mathbb{R}^2, \mathbb{R}^2)$ and $\det\nabla u(s, x) = h^2(s)\det\nabla u_0(x) = \frac{h^2(s)}{2r}[r^2g^2(r)]'$. According to (2.4), we obtain

$$\begin{aligned}\varphi_1(t, 0) - \varphi_0(0) &= \frac{1}{2\pi}\int_0^\infty\int_0^t\int_0^{2\pi}[h^2(s)]'e^{-\frac{r^2}{4(t-s)}}rg^2(r)\cos^2\theta d\theta ds dr\\ &= \frac{1}{2}\int_0^\infty\int_0^t[h^2(s)]'e^{-\frac{r^2}{4(t-s)}}rg^2(r)ds dr.\end{aligned}$$

Clearly,

$$\lim_{t\to\infty}|\varphi_1(t, 0) - \varphi_0(0)| = -\frac{1}{2}\int_0^\infty[h^2(s)]'ds \times \int_0^\infty rg^2 dr = \frac{1}{2}\int_0^\infty rg^2 dr = \varphi_0(0).$$

Moreover, a direct calculus shows

$$G_u(t) = \int_0^t \|\nabla\partial_s a\|_2\|\nabla b\|_2 + \|\nabla a\|_2\|\nabla\partial_s b\|_2 ds = -2\pi\int_0^t h(s)h'(s)ds \times \int_0^\infty \sigma^3 g'^2 d\sigma.$$

Thus for t large enough, we get

$$G_u(t) = \pi\int_0^\infty \sigma^3 g'^2 d\sigma = \|\nabla a_0\|_2 = \|\nabla b_0\|_2.$$

We find, in this special case,

$$\lim_{t\to\infty}\frac{|\varphi_1(t, 0) - \varphi_0(0)|}{G_u(t)} = \frac{\varphi_0(0)}{\|\nabla a_0\|_2\|\nabla b_0\|_2},$$

where the right-hand side is the expression of L^∞ norm estimate for the stationary situation. Following [1], if we take $g_\varepsilon(r) = r^{\varepsilon-1}e^{-\frac{r}{2}}$ with $\varepsilon > 0$, then the right-hand side tends to $(2\pi)^{-1}$, as ε tends to zero. We conclude then

$$\sup_{\Sigma(u)\neq 0}\sup_{t>0}\frac{\|\varphi_1(t, \cdot) - \varphi_0\|_\infty}{G_u(t)} \geq \sup_{\Sigma(u)\neq 0}\lim_{t\to\infty}\frac{|\varphi_1(t, 0) - \varphi_0(0)|}{G_u(t)} \geq \sup_{u_0 = g(r)x}\frac{\varphi_0(0)}{\|\nabla a_0\|_2\|\nabla b_0\|_2} \geq \frac{1}{2\pi}. \quad (2.6)$$

Combining (2.6) and (2.5), we complete the proof.

3 Proof of Theorem 1.2

We prove first (1.8). Using again the density argument, we can just consider the case where $u \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^2)$. By equations (1.5) and (1.6), if we define $\xi(t, x) = \varphi_1(t, x) - \varphi_0(x)$, we have

$$\partial_t\xi - \Delta\xi = \det\nabla u(t, x) - \det\nabla u(0, x) = \int_0^t \partial_\sigma[\det\nabla u](\sigma, x)d\sigma. \quad (3.1)$$

Multiplying (3.1) by ξ and integrating with respect to t, we have

$$\frac{1}{2}\xi(t, x)^2 - \int_0^t \xi(s, x)\Delta\xi(s, x)ds = \int_0^t \xi(s, x)\int_0^s \partial_\sigma[\det\nabla u](\sigma, x)d\sigma ds. \quad (3.2)$$

Now we integrate (3.2) over $\mathbb{R}^2$. Then

$$\frac{1}{2}\int_{\mathbb{R}^2}\xi(t,x)^2dx+\int_0^t\int_{\mathbb{R}^2}|\nabla\xi(s,x)|^2dxds=\int_{\mathbb{R}^2}\int_0^t\xi(s,x)\int_0^s\partial_\sigma[\det\nabla u](\sigma,x)d\sigma dsdx.$$

Hence, by (2.5)

$$\begin{aligned}\frac{1}{2}\|\xi(t,\,\cdot\,)\|_2^2+\int_0^t\|\nabla\xi(s,\,\cdot\,)\|_2^2ds &\le \int_0^t\int_0^s\|\xi(s,\,\cdot\,)\|_\infty\int_{\mathbb{R}^2}|\partial_\sigma[\det\nabla u](\sigma,x)|dxd\sigma ds\\ &\le \frac{1}{2\pi}\int_0^t\int_0^s G_u(s)\int_{\mathbb{R}^2}|\partial_\sigma[\det\nabla u](\sigma,x)|dsd\sigma dx.\end{aligned} \tag{3.3}$$

Otherwise, we have $\partial_t(\det\nabla u)=\det[\nabla(\partial_t a),\nabla b]+\det[\nabla a,\nabla(\partial_t b)]$, which yields

$$\int_{\mathbb{R}^2}|\partial_t[\det\nabla u](\sigma,x)|dx\le\|\nabla(\partial_t a)(\sigma,\cdot)\|_2\|\nabla b(\sigma,\cdot)\|_2+\|\nabla a(\sigma,\cdot)\|_2\|\nabla(\partial_t b)(\sigma,\cdot)\|_2=G_u'(\sigma). \tag{3.4}$$

Combining (3.3) and (3.4), we have

$$\frac{1}{2}\|\xi(t,\,\cdot\,)\|_2^2+\int_0^t\|\nabla\xi(s,\,\cdot\,)\|_2^2ds\le\frac{1}{2\pi}\int_0^t\int_0^s G_u(s)G_u'(\sigma)d\sigma ds=\frac{1}{2\pi}\int_0^t G_u(s)^2ds,$$

and (1.8) follows. For proving (1.9), multiplying (3.1) by $\partial_t\xi$ and integrating with respect to t and x, we get easily

$$\begin{aligned}&\|\partial_t\xi\|^2_{L^2([0,t]\times\mathbb{R}^2)}+\frac{1}{2}\|\nabla\xi(t,\,\cdot\,)\|_2^2\\ &=-\int_{\mathbb{R}^2}\int_0^t\xi(s,x)\partial_s[\det\nabla u](s,x)dsdx+\int_{\mathbb{R}^2}\xi(t,x)\int_0^t\partial_s[\det\nabla u](s,x)dsdx=\mathrm{I}_1+\mathrm{J}_1.\end{aligned}$$

Using (2.5) and (3.4), we have

$$|\mathrm{I}_1|\le\frac{1}{2\pi}\int_0^t G_u(s)G_u'(s)ds=\frac{G_u(t)^2}{4\pi},\quad |\mathrm{J}_1|\le\frac{G_u(t)}{2\pi}\int_0^t G_u'(s)ds=\frac{G_u(t)^2}{2\pi},$$

which imply immediately (1.9).

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