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On Hilbert Coefficients of Filtrations**

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Abstract Let $\mathcal{F}$ be a Hilbert filtration with respect to a Cohen-Macaulay module M. When $G(\mathcal{F}, M)$ and $F_K(\mathcal{F}, M)$ have almost maximal depths, the Hilbert coefficients $g_i(\mathcal{F}, M)$ is calculated. In the general case, an upper bound for $g_2(\mathcal{F}, M)$ is also given.

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1 Introduction

Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field, M a finitely generated R-module of dimension $d > 0$ and $\mathcal{F} = \{I_n\}_{n\geq 0}$ a Hilbert filtration with respect to M. Let K be an $\mathfrak{m}$-primary ideal of R such that $I_{n+1} \subseteq KI_n$ for all $n \geq 0$. Let $H_K(\mathcal{F}, M, n) = \lambda(M/KI_nM)$ be the Hilbert-Samuel function of $\mathcal{F}$ with respect to M and K, and $P_K(\mathcal{F}, M, n)$ the corresponding polynomial. Then

$$P_K(\mathcal{F}, M, n) = g_0(\mathcal{F}, M)\binom{n+d-1}{d} - g_1(\mathcal{F}, M)\binom{n+d-2}{d-1} + \cdots + (-1)^d g_d(\mathcal{F}, M).$$

In this paper, we are interested in the properties of these Hilbert coefficients $g_i(\mathcal{F}, M)$.

Generalizing Huneke's fundamental lemma in [8], Huckaba [6] gave a d-dimensional extension where an integer $w_n(J, I)$ was introduced. When $(R, \mathfrak{m})$ is Cohen-Macaulay of dimension d and depth $G(I) \geq d-1$, it is proved in [6] (cf. [5]) that

$$e_i(I) = \sum_{n=i-1}^{\infty} \binom{n}{i-1} \lambda(I^{n+1}/JI^n), \quad i = 1, \cdots, d,$$

hold for any $\mathfrak{m}$-primary ideal I and any minimal reduction J of I. Motivated by [6], we define a similar integer $w_n(J, K, \mathcal{F}, M)$ for a filtration. Then, the difference between $P_K(\mathcal{F}, M, n)$ and $H_K(\mathcal{F}, M, n)$ can be presented by $w_n(J, K, \mathcal{F}, M)$. The main result in Section 3 states that if M is Cohen-Macaulay, depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$, then

$$g_i(\mathcal{F}, M) = \sum_{n=i-1}^{\infty} \binom{n}{i-1} \lambda(KI_{n+1}M/JKI_nM) + (-1)^i \lambda(M/KM), \quad i = 1, \cdots, d.$$

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For the second Hilbert coefficient $e_2(I)$ of an $\mathfrak{m}$-primary ideal of a Cohen-Macaulay local ring $(R, \mathfrak{m})$, it is shown in [3] that $e_2(I) \leq \sum_{n=1}^{\infty} n\lambda(I^{n+1}/JI^n)$ where J is a minimal reduction of I. In Section 4, we give a similar upper bound for $g_2(\mathcal{F}, M)$:

$$g_2(\mathcal{F}, M) \leq \sum_{n=1}^{\infty} n\lambda(KI_{n+1}M/JKI_nM) + \lambda(M/KM).$$

2 Preliminaries

Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field. We say that $\mathcal{F} = \{I_n\}_{n\geq 0}$ is a filtration if $I_0 = R \supseteq I_1 \supseteq I_2 \supseteq \cdots$ is a chain of ideals of R such that $I_1 \neq R$ and $I_mI_n \subseteq I_{m+n}$ for all m, n. For any filtration $\mathcal{F} = \{I_n\}_{n\geq 0}$, let

$$R(\mathcal{F}) = \bigoplus_{n\geq 0} I_n \quad \text{and} \quad G(\mathcal{F}) = \bigoplus_{n\geq 0} I_n/I_{n+1}$$

be the Rees ring and associated graded ring of $\mathcal{F}$. $\mathcal{F}$ is said to be a Hilbert filtration if I_1 is $\mathfrak{m}$-primary and $R(\mathcal{F})$ is a finitely generated module over the Rees algebra $R(I_1)$, i.e., $I_1I_n = I_{n+1}$ for large n. Further, let M be a finitely generated R-module. We say that $\mathcal{F}$ is a Hilbert filtration with respect to M if $\lambda(M/I_1M) < \infty$ and $I_1I_nM = I_{n+1}M$ for large n.

Throughout the paper, let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field, M a finitely generated R-module of dimension $d > 0$ and $\mathcal{F}$ a Hilbert filtration with respect to M. Let K be an $\mathfrak{m}$-primary ideal of R such that $I_{n+1} \subseteq KI_n$ for all $n \geq 0$.

Let

$$F_K(\mathcal{F}) = \bigoplus_{n\geq 0} I_n/KI_n$$

be the fiber cone of $\mathcal{F}$ with respect to K. Set

$$G(\mathcal{F}, M) = \bigoplus_{n\geq 0} I_nM/I_{n+1}M \quad \text{and} \quad F_K(\mathcal{F}, M) = \bigoplus_{n\geq 0} I_nM/KI_nM.$$

Then $G(\mathcal{F}, M)$ is a finitely generated $G(\mathcal{F})$-module and $F_K(\mathcal{F}, M)$ is a finitely generated $F_K(\mathcal{F})$-module.

Let $H(\mathcal{F}, M, n) = \lambda(M/I_nM)$ be the Hilbert-Samuel function of $\mathcal{F}$ with respect to M and $P(\mathcal{F}, M, n)$ the corresponding polynomial. We have

$$P(\mathcal{F}, M, n) = e_0(\mathcal{F}, M)\binom{n+d-1}{d} - e_1(\mathcal{F}, M)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}, M).$$

Then

$$g_0(\mathcal{F}, M) = e_0(\mathcal{F}, M).$$

Note that

$$H_K(\mathcal{F}, M, n) = H(\mathcal{F}, M, n) + \lambda(I_nM/KI_nM).$$

As, for large n, $\lambda(I_nM/KI_nM)$ is a polynomial in n of degree $d-1$, we can write, for large n, that

$$\lambda(I_nM/KI_nM) = f_0(\mathcal{F}, M)\binom{n+d-2}{d-1} - f_1(\mathcal{F}, M)\binom{n+d-3}{d-2} + \cdots + (-1)^{d-1} f_{d-1}(\mathcal{F}, M).$$

Then

$$g_i(\mathcal{F},M)=e_i(\mathcal{F},M)-f_{i-1}(\mathcal{F},M),\quad i=1,\cdots,d.$$

An ideal $J\subseteq I_1$ is said to be a reduction of $\mathcal{F}$ with respect to M if there exists an integer $r>0$ such that $JI_nM=I_{n+1}M$ for all $n\geq r$. By the following lemma, minimal reductions exist.

Lemma 2.1 (See [11, Lemma 1]) *There exist $x_1,\cdots,x_d\in I_1$ such that $(x_1,\cdots,x_d)$ is a minimal reduction of $\mathcal{F}$ with respect to M and $e_0(\mathcal{F},M)=e_0((x_1,\cdots,x_d),M)$.*

Therefore, if M is Cohen-Macaulay, then $e_0(\mathcal{F},M)=\lambda(M/(x_1,\cdots,x_d)M)$.

Let $x\in I_1\setminus I_2$ and x^* the initial form of x in $G(\mathcal{F})$. x^* is said to be superficial for $G(\mathcal{F},M)$ if there exists an integer $c>0$ such that $(I_{n+1}M:x)\cap I_cM=I_nM$ for all $n>c$. Similarly, for any $x\in I_1\setminus KI_1$, let x^0 the initial form of x in $F_K(\mathcal{F})$, x^0 is said to be superficial for $F_K(\mathcal{F},M)$ if there exists an integer $c>0$ such that $(KI_{n+1}M:x)\cap I_cM=KI_nM$ for all $n>c$. Superficial sequences are defined inductively.

Suppose that x^0 is superficial for $F_K(\mathcal{F},M)$. Let "$-$" denote modulo (x). Thus,

$$\overline{\mathcal{F}}=\mathcal{F}/(x)=\{I_n+(x)/(x)\}_{n\geq 0},\quad \overline{J}=J/(x),\quad \overline{K}=K/(x),\quad \overline{M}=M/xM.$$

Since

$$H_{\overline{K}}(\overline{\mathcal{F}},\overline{M},n+1)=H_K(\mathcal{F},M,n+1)-H_K(\mathcal{F},M,n)+\lambda((KI_{n+1}M:x)/KI_nM),$$

it follows that

$$g_i(\overline{\mathcal{F}},\overline{M})=g_i(\mathcal{F},M),\quad i=0,1,\cdots,d-1.$$

The following proposition can be shown by similar arguments as in [9] (see [2]), we omit its proof.

Proposition 2.1 *There exist $x_1,\cdots,x_d\in I_1\setminus KI_1$ such that $J=(x_1,\cdots,x_d)$ is a minimal reduction of $\mathcal{F}$ with respect to M, and $x_1^*,\cdots,x_d^*$ $(x_1^0,\cdots,x_d^0)$ is a superficial sequence for $G(\mathcal{F},M)$ $(F_K(\mathcal{F},M))$.*

Furthermore, if depth $G(\mathcal{F},M)\geq k$ and depth $F_K(\mathcal{F},M)\geq k$ for an integer $k>0$, then one may choose the above $x_1,\cdots,x_d$ such that $x_1^*,\cdots,x_k^*$ is a regular $G(\mathcal{F},M)$-sequence and $x_1^0,\cdots,x_k^0$ is a regular $F_K(\mathcal{F},M)$-sequence. In this case, for all $n\geq 0$,

$$(KI_{n+1}M+(x_1,\cdots,x_{i-1})M):x_i=KI_nM+(x_1,\cdots,x_{i-1})M,\quad i=1,2,\cdots,k.$$

3 Formulas for Hilbert Coefficients

In this section, we present some formulas for the Hilbert coefficients $g_i(\mathcal{F},M)$.

If $f:\mathbb{Z}\to\mathbb{Z}$ is a function, let Δ denote the first difference function defined by $\Delta[f(n)]=f(n)-f(n-1)$, and let Δ^i be defined by $\Delta^i[f(n)]=\Delta^{i-1}[\Delta[f(n)]]$. By convention, $\Delta^0[f(n)]=f(n)$.

Definition 3.1 *Let $J=(x_1,\cdots,x_d)$ be a minimal reduction of $\mathcal{F}$ with respect to M where $x_1,\cdots,x_d$ are chosen as in Proposition* 2.1, *$J_i=(x_1,\cdots,x_i)$, $i=0,\cdots,d$ with $J_0=0$. For any* *integer $n \geq 0$, an integer $w_n(J, K, \mathcal{F}, M)$ is defined as follows. If $d = 1$, put $w_n(J, K, \mathcal{F}, M) = 0$. When $d > 1$, set*

$$\begin{aligned} w_n(J,K,\mathcal{F},M) = {} & \Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] - \lambda\Big(\frac{KI_{n+1}M : x_1}{JKI_nM : x_1}\Big) \\ & + \Delta^{d-2}\Big[\lambda\Big(\frac{(KI_{n+1}M + J_1M) : x_2}{KI_nM + J_1M}\Big)\Big] - \lambda\Big(\frac{(KI_{n+1}M + J_1M) : x_2}{(JKI_nM + J_1M) : x_2}\Big) \\ & + \cdots \\ & + \Delta\Big[\lambda\Big(\frac{(KI_{n+1}M + J_{d-2}M) : x_{d-1}}{KI_nM + J_{d-2}M}\Big)\Big] - \lambda\Big(\frac{(KI_{n+1}M + J_{d-2}M) : x_{d-1}}{(JKI_nM + J_{d-2}M) : x_{d-1}}\Big), \end{aligned}$$

where $I_n = R$ for $n < 0$.

Remark 3.1 Let "$-$" denote modulo (x_1). Then, it is easy to see that (see [6]),

$$w_n(J,K,\mathcal{F},M) = w_n(\overline{J},\overline{K},\overline{\mathcal{F}},\overline{M}) + \Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] - \lambda\Big(\frac{KI_{n+1}M : x_1}{JKI_nM : x_1}\Big).$$

Theorem 3.1 *Suppose that M is Cohen-Macaulay. Let J be as in Proposition 2.1. Then for all $n \geq 0$,*

$$\lambda(KI_{n+1}M/JKI_nM) + w_n(J,K,\mathcal{F},M) = \Delta^d[P_K(\mathcal{F},M,n+1) - H_K(\mathcal{F},M,n+1)].$$

Proof Since M is Cohen-Macaulay, we see that $x_1, \cdots, x_d$ is an M-sequence. Then the same argument as in the proof of [6, Theorem 2.4] can be applied. We omit the details.

When $M = R$, the following result is the Lemma 3.1 of [9] which is the fiber cone version of Huneke's fundamental lemma (see [8]).

Corollary 3.1 *Suppose that M is Cohen-Macaulay of dimension 2. Let J be as in Proposition 2.1. Then, for all $n \geq 1$,*

$$\Delta^2[P_K(\mathcal{F},M,n+1) - H_K(\mathcal{F},M,n+1)] = \lambda\Big(\frac{KI_{n+1}M}{JKI_nM}\Big) - \lambda\Big(\frac{KI_nM : J}{KI_{n-1}M}\Big).$$

Proof Let $n \geq 1$. Note that

$$\begin{aligned} w_n(J,K,\mathcal{F},M) &= \Delta\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] - \lambda\Big(\frac{KI_{n+1}M : x_1}{JKI_nM : x_1}\Big) \\ &= \lambda\Big(\frac{JKI_nM : x_1}{KI_nM}\Big) - \lambda\Big(\frac{KI_nM : x_1}{KI_{n-1}M}\Big). \end{aligned}$$

Then, by Theorem 3.1, it is enough to show that

$$\frac{KI_nM : x_1}{KI_nM : J} \cong \frac{JKI_nM : x_1}{KI_nM}.$$

Let f be the homomorphism from $KI_nM : x_1$ to $\frac{JKI_nM:x_1}{KI_nM}$ which is induced by the multiplication by x_2. Then $\mathrm{Ker}(f) = KI_nM : J$, hence, it remains to show that

$$JKI_nM : x_1 = x_2(KI_nM : x_1) + KI_nM.$$

Let $y \in JKI_nM : x_1$. Then $x_1y = x_1y_1 + x_2y_2$ for some $y_1, y_2 \in KI_nM$. As x_1, x_2 is an M-sequence, there is $y' \in M$ such that $y - y_1 = x_2y'$ and $y_2 = x_1y'$. Then $y' \in KI_nM : x_1$ and $y \in x_2(KI_nM : x_1) + KI_nM$. The result follows.

Theorem 3.2 *Assume that M is Cohen-Macaulay, depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Let J be as in Proposition* 2.1*. Then $w_n(J, K, \mathcal{F}, M) = 0$ for $n \geq d-1$, and, for $n \leq d-2$,*

$$w_n(J, K, \mathcal{F}, M) = (-1)^{n+1}\binom{d-1}{n+1}\lambda(M/KM).$$

Proof If $d = 1$, it is trivial. Let $d \geq 2$. By Proposition 2.1,

$$(KI_{n+1}M + J_{i-1}M) : x_i = KI_nM + J_{i-1}M, \quad i = 1, \cdots, d-1, \ n \geq 0.$$

As $KI_nM + J_{i-1}M \subseteq (JKI_nM + J_{i-1}M) : x_i \subseteq (KI_{n+1}M + J_{i-1}M) : x_i$, we have also $(JKI_nM + J_{i-1}M) : x_i = (KI_{n+1}M + J_{i-1}M) : x_i$. It follows that

$$\begin{aligned} w_n(J, K, \mathcal{F}, M) = {} & \Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] + \Delta^{d-2}\Big[\lambda\Big(\frac{(KI_{n+1}M + J_1M) : x_2}{KI_nM + J_1M}\Big)\Big] \\ & + \cdots + \Delta\Big[\lambda\Big(\frac{(KI_{n+1}M + J_{d-2}M) : x_{d-1}}{KI_nM + J_{d-2}M}\Big)\Big]. \end{aligned}$$

Let $f(n) = \lambda\big(\frac{KI_{n+1}M : x_1}{KI_nM}\big)$. Then $f(n) = 0$ for $n \geq 0$ and $f(n) = \lambda(M/KM)$ for all $n < 0$. Thus

$$\begin{aligned} \Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] &= \Delta[\Delta^{d-2}[f(n)]] \\ &= \Delta\Big[\sum_{i=0}^{d-2}(-1)^i\binom{d-2}{i}f(n-i)\Big] \\ &= \sum_{i=0}^{d-2}(-1)^i\binom{d-2}{i}\Delta[f(n-i)]. \end{aligned}$$

It follows that, if $n \geq d-1$ then

$$\Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] = 0,$$

and if $n \leq d-2$ then

$$\Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M : x_1}{KI_nM}\Big)\Big] = (-1)^{n+1}\binom{d-2}{n}\lambda(M/KM).$$

Similarly, we have that, if $n \geq d-2$, then

$$\Delta^{d-2}\Big[\lambda\Big(\frac{(KI_{n+1}M + J_1M) : x_2}{KI_nM + J_1M}\Big)\Big] = 0,$$

and if $n \leq d-3$, then

$$\Delta^{d-1}\Big[\lambda\Big(\frac{(KI_{n+1}M + J_1M) : x_2}{KI_nM + J_1M}\Big)\Big] = (-1)^{n+1}\binom{d-3}{n}\lambda(M/KM),$$

and so on. It turns out that $w_n(J,K,\mathcal{F},M)=0$ for $n\ge d-1$, and, for $n\le d-2$,

$$w_n(J,K,\mathcal{F},M)=(-1)^{n+1}\lambda(M/KM)\sum_{j=n}^{d-2}\binom{j}{n}.$$

But it can be shown by using induction on d that

$$\sum_{j=n}^{d-2}\binom{j}{n}=\binom{d-1}{n+1},$$

the result follows.

We will need the following lemma whose proof is easy (see [6]).

Lemma 3.1 *Let $f:\mathbb{Z}\to\mathbb{Z}$ be a function such that $f(n)=0$ for all $n\gg 0$. Then the following statements hold:*

(1) *For $0<k\le d$ and $0<j\le d$,*

$$\sum_{n=k}^{\infty}\binom{n}{k}\Delta^j[f(n+1)]=-\sum_{n=k-1}^{\infty}(-1)^i\binom{n}{k-1}\Delta^{j-1}[f(n+1)].$$

(2) *If $f(n)=c$ for all $n\le 0$, then*

$$\sum_{n=0}^{\infty}\Delta^j[f(n+1)]=\begin{cases}-c, & j=1,\\ 0, & 1<j\le d.\end{cases}$$

(3) *Suppose that $f(n)$ has the form*

$$f(n)=a_0\binom{n+d-1}{d}+a_1\binom{n+d-2}{d-1}+\cdots+a_d.$$

Then $\Delta^{d-i}[f(0)]=a_i$.

Proposition 3.1

$$\sum_{n=i-1}^{\infty}\binom{n}{i-1}\Delta^d[P_K(\mathcal{F},M,n+1)-H_K(\mathcal{F},M,n+1)]$$
$$=\begin{cases}g_i(\mathcal{F},M), & 1\le i<d,\\ g_i(\mathcal{F},M)+(-1)^{i-1}\lambda(M/KM), & i=d.\end{cases}$$

Proof Let $f(n)=P_K(\mathcal{F},M,n)-H_K(\mathcal{F},M,n)$. Then $f(n)=0$ for all $n\gg 0$. Let i be an integer such that $1\le i\le d$. Then, by Lemma 3.1(1),

$$\begin{aligned}\sum_{n=i-1}^{\infty}\binom{n}{i-1}\Delta^d[f(n+1)]&=(-1)^{i-1}\sum_{n=0}^{\infty}\Delta^{d-i+1}[f(n+1)]\\&=(-1)^{i-1}\Delta^{d-i}\Big[\sum_{n=0}^{\infty}\Delta[f(n+1)]\Big]\\&=(-1)^i\Delta^{d-i}[f(0)]\\&=(-1)^i\Delta^{d-i}[P_K(\mathcal{F},M,0)]-(-1)^i\Delta^{d-i}[H_K(\mathcal{F},M,0)].\end{aligned}$$

But $\Delta^{d-i}[P_K(\mathcal{F},M,0)]=(-1)^i g_i(\mathcal{F},M)$ by Lemma 3.1(3), $\Delta^{d-i}[H_K(\mathcal{F},M,0)]=0$ for $i<d$, and $H_K(\mathcal{F},M,0)=\lambda(M/KM)$. Then the result follows.

Corollary 3.2 *Suppose that M is Cohen-Macaulay. Let J be as in Proposition* 2.1. *Then*

$$g_1(\mathcal{F},M)\le\sum_{n=0}^{\infty}\lambda(KI_{n+1}M/JKI_nM)-\lambda(M/KM).$$

Proof Let $d=1$. Then, by Proposition 3.1,

$$g_1(\mathcal{F},M)=-\lambda(M/KM)+\sum_{n=0}^{\infty}\Delta[P_K(\mathcal{F},M,n+1)-H_K(\mathcal{F},M,n+1)].$$

In virtue of Theorem 3.1, we have

$$\Delta[P_K(\mathcal{F},M,n+1)-H_K(\mathcal{F},M,n+1)]=\lambda(KI_{n+1}M/JKI_nM).$$

Hence, in this case,

$$g_1(\mathcal{F},M)=\sum_{n=0}^{\infty}\lambda(KI_{n+1}M/JKI_nM)-\lambda(M/KM).$$

Now let $d\ge 2$. Then, by Proposition 3.1 and Theorem 3.1,

$$\begin{aligned}g_1(\mathcal{F},M)&=\sum_{n=0}^{\infty}\Delta^d[P_K(\mathcal{F},M,n+1)-H_K(\mathcal{F},M,n+1)]\\&=\sum_{n=0}^{\infty}\lambda(KI_{n+1}M/JKI_nM)+\sum_{n=0}^{\infty}w_n(J,K,\mathcal{F},M).\end{aligned}$$

Thus it is enough to show that

$$\sum_{n=0}^{\infty}w_n(J,K,\mathcal{F},M)\le-\lambda(M/KM).$$

By the definition of $w_n(J,K,\mathcal{F},M)$, it is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty}w_n(J,K,\mathcal{F},M)\le&\sum_{n=0}^{\infty}\Delta^{d-1}\Big[\lambda\Big(\frac{KI_{n+1}M:x_1}{KI_nM}\Big)\Big]\\&+\sum_{n=0}^{\infty}\Delta^{d-2}\Big[\lambda\Big(\frac{(KI_{n+1}M+J_1M):x_2}{KI_nM+J_1M}\Big)\Big]\\&+\cdots\\&+\sum_{n=0}^{\infty}\Delta\Big[\lambda\Big(\frac{(KI_{n+1}M+J_{d-2}M):x_{d-1}}{KI_nM+J_{d-2}M}\Big)\Big].\end{aligned}$$

According to Lemma 3.1(2), we have

$$\sum_{n=0}^{\infty}\Delta^{d-j}\Big[\lambda\Big(\frac{(KI_{n+1}M+J_{j-1}M):x_j}{KI_nM+J_{i-1}M}\Big)\Big]=0$$

for $j \leq d-2$, and

$$\sum_{n=0}^{\infty} \Delta[\lambda(\frac{(KI_{n+1}M + J_{d-2}M) : x_{d-1}}{KI_nM + J_{d-2}M})] = -\lambda(M/KM).$$

The result follows.

Now, we can prove the main result of this section.

Theorem 3.3 *Assume that M is Cohen-Macaulay, depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Let J be as in Proposition 2.1. Then*

$$g_i(\mathcal{F}, M) = \sum_{n=i-1}^{\infty} \binom{n}{i-1} \lambda(KI_{n+1}M/JKI_nM) + (-1)^i \lambda(M/KM), \quad 1 \leq i \leq d.$$

Proof Firstly, notice that

$$\sum_{n=i-1}^{\infty} \binom{n}{i-1} \lambda(KI_{n+1}M/JKI_nM) + \sum_{n=i-1}^{\infty} \binom{n}{i-1} w_n(J, K, \mathcal{F}, M)$$
$$= \begin{cases} g_i(\mathcal{F}, M), & 1 \leq i < d, \\ g_i(\mathcal{F}, M) + (-1)^i \lambda(M/KM), & i = d \end{cases}$$

by Theorem 3.1 and Proposition 3.1. Then it is enough to show that

$$\sum_{n=i-1}^{\infty} \binom{n}{i-1} w_n(J, K, \mathcal{F}, M) = (-1)^{i-1} \lambda(M/KM)$$

for $i < d$, and

$$\sum_{n=d-1}^{\infty} \binom{n}{d-1} w_n(J, K, \mathcal{F}, M) = 0.$$

From Theorem 3.2, we have that $w_n(J, K, \mathcal{F}, M) = 0$ for $n \geq d-1$, and $w_n(J, K, \mathcal{F}, M) = (-1)^{n+1} \binom{d-1}{n+1} \lambda(M/KM)$ for $n \leq d-2$. It follows that

$$\sum_{n=d-1}^{\infty} \binom{n}{d-1} w_n(J, K, \mathcal{F}, M) = 0,$$

and for $i < d$,

$$\sum_{n=i-1}^{\infty} \binom{n}{i-1} w_n(J, K, \mathcal{F}, M) = \sum_{n=i-1}^{d-2} \binom{n}{i-1} w_n(J, K, \mathcal{F}, M)$$
$$= \lambda(M/KM) \sum_{n=i-1}^{d-2} (-1)^{n+1} \binom{n}{i-1} \binom{d-1}{n+1}.$$

But it can be shown by using induction on d that

$$\sum_{n=i-1}^{d-2} (-1)^{n+1} \binom{n}{i-1} \binom{d-1}{n+1} = (-1)^i.$$

The theorem follows.

From Theorem 3.3, we have the following

Corollary 3.3 *Assume that M is Cohen-Macaulay, depth $G(\mathcal{F}, M) \geq d-1$ and depth $F_K(\mathcal{F}, M) \geq d-1$. Then the following two statements are true:*

(1) $g_i(\mathcal{F}, M) \geq (-1)^i \lambda(M/KM)$, $i = 1, \cdots, d$;

(2) *If* $g_i(\mathcal{F}, M) = (-1)^i \lambda(M/KM)$ *for some* $i \geq 1$, *then* $g_j(\mathcal{F}, M) = (-1)^j \lambda(M/KM)$ *for* $j = i, \cdots, d$.

4 The Second Hilbert Coefficient

In this section, we will give an upper bound for $g_2(\mathcal{F}, M)$. As is done in [3], we need to reduce the dimension of the module.

Let us generalize the definition of $g_i(\mathcal{F}, M)$ to the case $i > d$ by using the same arguments as in [5].

Write

$$P_K(\mathcal{F}, M, n) = g'_0(\mathcal{F}, M)\binom{n+d}{d} - g'_1(\mathcal{F}, M)\binom{n+d-1}{d-1} + \cdots + (-1)^d g'_d(\mathcal{F}, M).$$

Then

$$g'_0(\mathcal{F}, M) = g_0(\mathcal{F}, M) \quad \text{and} \quad g'_i(\mathcal{F}, M) = g_i(\mathcal{F}, M) + g_{i-1}(\mathcal{F}, M), \quad i = 1, \cdots, d.$$

Let $P_{\mathcal{F}}(M, z) = \sum\limits_{n=0}^{\infty} H_K(\mathcal{F}, M, n) z^n$ be the Hilbert series of $\mathcal{F}$ with respect to M and K. Then there exists a unique polynomial $f_{\mathcal{F}}(M, z) \in \mathbb{Z}[z]$ such that

$$P_{\mathcal{F}}(M, z) = \frac{f_{\mathcal{F}}(M, z)}{(1-z)^{d+1}}.$$

It turns out that

$$g'_i(\mathcal{F}, M) = \frac{\Delta^i[f_{\mathcal{F}}(M, 1)]}{i!}, \quad i = 0, 1, \cdots, d.$$

For any $i > d$, set

$$g'_i(\mathcal{F}, M) = \frac{\Delta^i[f_{\mathcal{F}}(M, 1)]}{i!}.$$

Let

$$f_{\mathcal{F}}(M, z) = \sum_{n=0}^{s} a_n z^n.$$

Then

$$g'_i(\mathcal{F}, M) = \sum_{n=i}^{s} \binom{n}{i} a_n, \quad i = 0, 1, \cdots$$

(see [5]).

An argument similar to that of Proposition 1.5 of [5] can be used to get the following

Proposition 4.1 *Let J be as in Proposition 2.1 with x_1 being regular. Let "$\overline{\ \ }$" denote modulo (x_1). Then*

$$g'_d(\mathcal{F}, M) = g'_d(\overline{\mathcal{F}}, \overline{M}) - \sum_{n=0}^{\infty}(-1)^d \lambda((KI_{n+1}M : x_1)/KI_nM).$$

When M has dimension one, we can calculate $g'_2(\mathcal{F}, M)$.

Lemma 4.1 *Suppose that M is Cohen-Macaulay of dimension one. Then*

$$g'_2(\mathcal{F}, M) = \sum_{n=0}^{\infty}(n+1)\lambda(KI_{n+1}M/JKI_nM).$$

Proof From

$$\sum_{n=0}^{\infty}\lambda(M/KI_nM)z^n = \frac{f_{\mathcal{F}}(M,z)}{(1-z)^2} = \frac{\sum\limits_{n=0}^{s} a_n z^n}{(1-z)^2},$$

we get that

$$\begin{aligned} a_0 &= \lambda(M/KM), \\ a_1 &= \lambda(M/KI_1M) - 2\lambda(M/KM), \\ a_n &= \lambda(M/KI_nM) - 2\lambda(M/KI_{n-1}M) + \lambda(M/KI_{n-2}M), \quad n = 2, \cdots, s, \end{aligned}$$

and for $n \geq s+1$,

$$\lambda(M/KI_nM) - 2\lambda(M/KI_{n-1}M) + \lambda(M/KI_{n-2}M) = 0.$$

Let $n \geq 2$. Note that

$$\lambda(M/KI_nM) - 2\lambda(M/KI_{n-1}M) + \lambda(M/KI_{n-2}M) = \lambda\Big(\frac{KI_{n-1}M}{KI_nM}\Big) - \lambda\Big(\frac{KI_{n-2}M}{KI_{n-1}M}\Big).$$

Thus

$$a_n = \lambda(KI_{n-1}M/KI_nM) - \lambda(KI_{n-2}M/KI_{n-1}M), \quad n = 2, \cdots, s,$$

and for $n \geq s+1$,

$$\lambda(KI_{n-1}M/KI_nM) = \lambda(KI_{n-2}M/KI_{n-1}M).$$

Set

$$\rho_n = \lambda(KI_{n+1}M/JKI_nM) = \lambda(KI_{n+1}M/x_1KI_nM).$$

Then

$$\begin{aligned} &\lambda(KI_nM/KI_{n+1}M) \\ &= \lambda(M/x_1M) + \lambda(x_1M/x_1KI_nM) - \lambda(KI_{n+1}M/x_1KI_nM) - \lambda(M/KI_nM) \\ &= \lambda(M/x_1M) - \lambda(KI_{n+1}M/x_1KI_nM) \\ &= g'_0(\mathcal{F}, M) - \rho_n. \end{aligned}$$

It follows that

$$a_n = \rho_{n-2} - \rho_{n-1}, \quad n = 2, \cdots, s \quad \text{and} \quad \rho_{n-2} = \rho_{n-1}, \quad n > s.$$

Hence

$$g_0'(\mathcal{F}, M) = \sum_{n=0}^{s} a_n = a_0 + a_1 + \sum_{n=2}^{s} (\rho_{n-2} - \rho_{n-1}) = a_0 + a_1 + \rho_0 - \rho_{s-1}.$$

But

$$\begin{aligned} a_0 + a_1 + \rho_0 &= \lambda(M/x_1KM) - \lambda(M/KM) \\ &= \lambda(KM/x_1KM) \\ &= \lambda(M/KM) - \lambda(x_1M/x_1KM) + \lambda(KM/x_1KM) \\ &= \lambda(M/KM) + \lambda(KM/x_1M) \\ &= \lambda(M/x_1M) \\ &= g_0'(\mathcal{F}, M), \end{aligned}$$

so $\rho_{s-1} = 0$. Hence $\rho_n = 0$ for all $n \geq s-1$. Thus

$$\begin{aligned} g_2'(\mathcal{F}, M) &= \sum_{n=2}^{s} \binom{n}{2} a_n = \sum_{n=2}^{s} \binom{n}{2} (\rho_{n-2} - \rho_{n-1}) \\ &= \rho_0 + \sum_{n=1}^{s-2} \left(\binom{n+2}{2} - \binom{n+1}{2} \right) \rho_n \\ &= \sum_{n=0}^{s-2} (n+1)\rho_n = \sum_{n=0}^{\infty} (n+1)\rho_n \\ &= \sum_{n=1}^{\infty} (n+1)\lambda(KI_{n+1}M/JKI_nM). \end{aligned}$$

Now we show the main theorem of this section.

Theorem 4.1 *Suppose that $d \geq 2$ and M is Cohen-Macaulay. Let J be as in Proposition 2.1. Then*

$$g_2(\mathcal{F}, M) \leq \sum_{n=1}^{\infty} n\lambda(KI_{n+1}M/JKI_nM) + \lambda(M/KM).$$

Proof Let "$-$" denote modulo $(x_1, \cdots, x_{d-2})$ and "$\widetilde{\ }$" denote modulo $(x_1, \cdots, x_{d-1})$. Then $g_2(\mathcal{F}, M) = g_2(\overline{\mathcal{F}}, \overline{M})$, $g_1(\overline{\mathcal{F}}, \overline{M}) = g_1(\widetilde{\mathcal{F}}, \widetilde{M})$, and by Proposition 4.1, $g_2'(\overline{\mathcal{F}}, \overline{M}) \leq g_2'(\widetilde{\mathcal{F}}, \widetilde{M})$. In virtue of Theorem 3.3 and Lemma 4.1, we have

$$\begin{aligned} g_1(\widetilde{\mathcal{F}}, \widetilde{M}) &= \sum_{n=0}^{\infty} \lambda(\widetilde{K}\widetilde{I}_{n+1}\widetilde{M}/\widetilde{J}\widetilde{K}\widetilde{I}_n\widetilde{M}) - \lambda(\widetilde{M}/\widetilde{K}\widetilde{M}) \\ &= \sum_{n=0}^{\infty} \lambda(\widetilde{K}\widetilde{I}_{n+1}\widetilde{M}/\widetilde{J}\widetilde{K}\widetilde{I}_n\widetilde{M}) - \lambda(M/KM), \\ g_2'(\widetilde{\mathcal{F}}, \widetilde{M}) &= \sum_{n=0}^{\infty} (n+1)\lambda(\widetilde{K}\widetilde{I}_{n+1}\widetilde{M}/\widetilde{J}\widetilde{K}\widetilde{I}_n\widetilde{M}). \end{aligned}$$

Therefore,

$$\begin{aligned} g_2(\mathcal{F}, M) &= g_2'(\overline{\mathcal{F}}, \overline{M}) - g_1(\overline{\mathcal{F}}, \overline{M}) \\ &\leq g_2'(\widetilde{\mathcal{F}}, \widetilde{M}) - g_1(\widetilde{\mathcal{F}}, \widetilde{M}) \\ &= \sum_{n=1}^{\infty} n\lambda(\widetilde{K}\widetilde{I}_{n+1}\widetilde{M}/\widetilde{J}\widetilde{K}\widetilde{I}_n\widetilde{M}) + \lambda(M/KM) \\ &\leq \sum_{n=0}^{\infty} (n+1)\lambda(KI_{n+1}M/JKI_nM) + \lambda(M/KM). \end{aligned}$$

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