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Codimension Two PL Embeddings of Spheres with Nonstandard Regular Neighborhoods***

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Abstract For a given polyhedron $K \subset M$, the notation $R_M(K)$ denotes a regular neighborhood of K in M. The authors study the following problem: find all pairs (m, k) such that if K is a compact k-polyhedron and M a PL m-manifold, then $R_M(f(K)) \cong R_M(g(K))$ for each two homotopic PL embeddings $f, g : K \to M$. It is proved that $R_{S^{k+2}}(S^k) \not\cong S^k \times D^2$ for each $k \geq 2$ and some PL sphere $S^k \subset S^{k+2}$ (even for any PL sphere $S^k \subset S^{k+2}$ having an isolated non-locally flat point with the singularity $S^{k-1} \subset S^{k+1}$ such that $\pi_1(S^{k+1} - S^{k-1}) \not\cong \mathbb{Z}$).

Keywords Embedding, Regular neighborhood, Dehn surgery, Fundamental group
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1 Introduction

This paper is devoted to the uniqueness of regular neighborhoods problem for distinct embeddings of one manifold into the other. Definitions of regular neighborhood and other notions of PL topology can be found in [28]. For any subpolyhedron $X \subset M$ of a PL manifold M, we denote the regular neighborhood of X in M by $R_M(X)$. The following is the main result of the present paper.

Theorem 1.1 *$R_{S^{k+2}}(S^k)$ is not homeomorphic to $S^k \times D^2$ for any $k \geq 2$ and*

(a) *any PL sphere $S^k \subset S^{k+2}$ which is the suspension over a locally flat PL knot $S^{k-1} \subset S^{k+1}$ such that $G = \pi_1(S^{k+1} - S^{k-1}) \not\cong \mathbb{Z}$; or*

(b) *any PL sphere $S^k \subset S^{k+2}$ having an isolated non-locally flat point with the singularity $S^{k-1} \subset S^{k+1}$ such that $G = \pi_1(S^{k+1} - S^{k-1}) \not\cong \mathbb{Z}$.*

For $k = 2$ in Theorem 1.1(b) one can take any non-locally flat PL sphere $S^2 \subset S^4$. Recall that for $x \in S^k \subset S^{k+2}$ the singularity at point x is the isotopy class of the submanifold $\partial D^{k+2} \cap S^k$ of ∂D^{k+2}. Here $(D^{k+2}, D^{k+2} \cap S^k)$ is any PL $(k+2, k)$ ball pair which is a regular

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neighborhood of x in the pair (S^{k+2}, S^k). Of course, Theorem 1.1(a) follows from Theorem 1.1(b), but the proof of the former is simpler (in particular for $k = 2$, because our argument uses the fundamental group whereas the proof of Theorem 1.1(b) in this case uses [7, 8]).

We now give some motivation for Theorem 1.1. An important question in the topology of manifolds (see [17]) asks for which cases the normal vector bundle of a smooth k-manifold K is independent of the choice of the smooth embedding $K \to S^m$. This is the case when

(a) $m = k + 1$; or

(b) $m = k + 2$ and K is orientable (see [15, 26]); or

(c) K is a homotopy k-sphere and either $m = k + 3$ or $m \geq \frac{3k}{2} + 1$ (see [9, 17, 14]); or

(d) $K = \mathbb{R}P^2$ and $m = 4$ (see [18]).

On the other hand, there exist smooth embeddings $K \subset S^m$ with distinct normal bundles (see [12, Haefliger's example in §2], [18, 20, 25]). For example, K can be the Klein bottle and $f_0, f_4 : K \to S^4$ can be any smooth embeddings with the normal Euler classes equal to 0 and 4, respectively. Then $\pi_1(R_{S^4}(f_0(K))) \cong \mathbb{Z}^3$ while $\pi_1(R_{S^4}(f_4(K)))$ is not virtually abelian, i.e. it does not contain any abelian subgroups of finite index. This argument, due to Hillman, shows that for such embeddings even the spaces of normal bundles are nonhomeomorphic.

Any locally flat PL embedding has a normal block bundle (see [27]). Thus the Massey question makes sense in the locally flat PL category. In this category the normal bundle does not depend on the embedding when $K = S^k$ (by the Zeeman Unknotting Balls Theorem) in the cases (a), (b), (d) above (see [27, Corollary 6.8], [29, Theorem 2]). In particular, $R_{S^4}(S^2) \cong S^2 \times D^2$ for any locally flat PL embedding $S^2 \subset S^4$ (see Theorem 1.1). Note that a PL sphere $S^n \subset \mathbb{R}^m$ has a PL standard regular neighborhood if either $m - n \neq 2$ or the embedding $S^n \subset \mathbb{R}^m$ is locally flat (see [29, 30]).

From now on we shall work in the PL (not locally flat PL) category, unless otherwise specified. As another polyhedral version of the Massey problem we can consider the question: For embeddings $f, g : K \to S^m$ can the homeomorphism $g \circ f^{-1} : f(K) \to g(K)$ be extended to a homeomorphism $R_M(f(K)) \cong R_M(g(K))$ (see [16])? The analogues of Theorem 1.1 and conjectures (e), (f), (g) below for this problem are trivial. Theorem 1.1 was motivated by the following related problem, raised independently by Chernavski and Shtanko (private communication) and also in [10, 11, 23]:

Find all pairs (m, k) such that if K is a compact k-polyhedron and M a PL m-manifold, then

$(*)$ $R_M(f(K)) \cong R_M(g(K))$ for each two homotopic PL embeddings $f, g : K \to M$.

A special case of this problem for $M = S^m$ (then f and g are always homotopic) attracted special attention. The property $(*)$ holds for $m \geq 2k + 2$ by general position. In general, many invariants of $R_M(f(K))$ and $R_M(g(K))$ coincide: the homotopy types, the intersection rings, the higher Massey products, the (classifying maps of) tangent bundles (and hence also all invariants deduced from the tangent bundle, e.g. characteristic classes and numbers). This implies that $(*)$ also holds for $m = 2k + 1$ (because in this case an m-thickening is completely determined by its tangent bundle (see [16])) and for $m = 2$ (because a 2-manifold N with boundary is completely determined by $H_1(N, \mathbb{Z}_2)$ and its intersection form). Also,

$$R_M(f(K)) \times I^l \cong R_{M\times I^l}(f(K)) \cong R_{M\times I^l}(g(K)) \cong R_M(g(K)) \times I^l \quad \text{for } l \geq 2k+1-m.$$

The boundary $\partial R_M(f(K))$ is l-connected for $m \geq k+l+2$ and l-connected K. All this suggests that distinguishing $R_M(f(K))$ and $R_M(g(K))$ is a nontrivial problem for $m \geq k+3$.

The property $(*)$ holds for $m = k+1 \geq 3$ and a fake surface K (see [1, 2, 19, 23, 24]). The property $(*)$ fails for:

(a) $M = S^m$, $m = k+1 \geq 3$ and $K = S^k \vee S^1 \vee S^1$ (see [23]); or

(b) $M = S^3$, K a common spine of the granny knot and the square knot (see [3, 4, 21, 25]); or

(c) $M = S^4$ and the Dunce Hat K (see [10, 11, 31]); or

(d) $M = S^{k+2}$, $k \geq 2$ and $K = S^k$ by Theorem 1.1.

We conjecture that the property $(*)$ also fails for:

(e) $M = S^{2k}$ and $K = \Sigma(S^{k-1} \sqcup S^{k-1})$ (one can take f and g to be the suspensions over the trivial link and the Hopf link, respectively); or

(f) $M = S^4$ and $K = \mathbb{R}P^2$ (f and g should be non-locally flat); or

(g) each $k+3 \leq m \leq 2k$ and some polyhedron or even manifold K.

The case (a) above was proved by using the number of connected components of $\partial R_M(f(K))$; the cases (c) and (d) were and will be proved by using $\pi_1(\partial R_M(f(K)))$.

2 Proof of Theorem 1.1

If a PL embedding $S^2 \subset S^4$ is locally flat everywhere except at n points with singularities $L_1, \cdots, L_n$, then $\partial R_{S^4}(S^2)$ is obtained from ∂D^4 by the Dehn surgery over the knot $L_1\#\cdots\#L_n$ with certain framing (the details are analogous to the proof of Theorem 2.1(a) below). Hence the case $k=2$ of Theorem 1.1(b) follows by a deep result of Gabai [7], [8, Remark after Corollary 5], to the effect that the Dehn surgery along any non-trivial knot $S^1 \subset S^3$ can never yield $S^1 \times S^2$. (The argument of [7, 8] uses foliation theory and cannot be generalized to obtain Theorem 1.1(b) for $k > 2$.)

Theorem 1.1(b) for $k > 2$ and Theorem 1.1(a) are implied by the following result.

Theorem 2.1 *For $k \geq 2$ and given embedding $S^k \subset S^{k+2}$ set $N = R_{S^{k+2}}(S^k)$. Then $G := \pi_1(S^{k+1} - S^{k-1})$ maps into $\pi_1(\partial N)$ in each of the following two cases:*

(a) *the embedding $S^k \subset S^{k+2}$ is the suspension over a locally flat PL knot $S^{k-1} \subset S^{k+1}$.*

(b) *$k \geq 3$ and the embedding $S^k \subset S^{k+2}$ has an isolated non-locally flat point with the singularity $S^{k-1} \subset S^{k+1}$.*

Proof of Theorem 2.1(a) Consider decomposition

$$S^{k+2} = D^{k+2}_+ \bigcup_{\partial D^{k+2}_\pm = S^{k+1}} D^{k+2}_-.$$

Let $B^k_\pm := D^{k+2}_\pm \cap S^k$ be the cone over S^{k-1} (see Figure 1(a)). Denote $N_\pm := R_{D^{k+2}_\pm}(B^k_\pm)$ so that $N_\pm \cap \partial D^{k+2}_\pm = R_{S^{k+1}}(S^{k-1})$ and $\partial_\pm N := \partial N_\pm - \mathrm{Int} R_{\partial N_\pm}(S^{k-1})$ so that $N = N_+ \cup N_-$ and $\partial N = \partial_{+N} \cup \partial_{-N}$.

Since $B^k_\pm$ is a cone, it follows that $D^{k+2}_\pm$ collapses to $B^k_\pm$. Therefore $D^{k+2}_\pm$ is also a regular neighborhood of $B^k_\pm$ in $D^{k+2}_\pm$. Hence by the uniqueness of regular neighborhoods,

$$(\partial N_\pm, S^{k-1}) \cong (S^{k+1}, S^{k-1}),$$

so $\pi_1(\partial_\pm N) \cong G$.

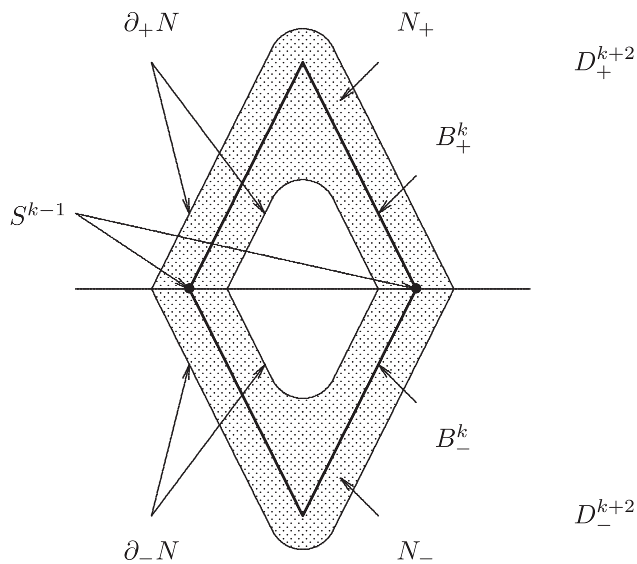

Figure 1(a)

The composition $\partial_+ N \xrightarrow{i} \partial_+ N \cup \partial_- N \xrightarrow{r} \partial_+ N$ of the inclusion i and the 'symmetric' retraction r is a homeomorphism. So the induced composition $G \xrightarrow{i_*} \pi_1(\partial N) \xrightarrow{r_*} G$ is an isomorphism. Hence i_* is a monomorphism, and Theorem 2.1(a) is proved.

Proof of Theorem 2.1(b) Let x be the isolated non-locally flat point. Then the singularity $S^{k-1} \subset S^{k+1}$ is locally flat. Consider the decomposition of S^{k+2} as above such that $x \in \mathrm{Int} D^{k+2}_+$. Let $B^k_+ := D^{k+2}_+ \cap S^k$ be the cone over S^{k-1} with the vertex x (see Figure 1(b)). Let

$$B^k_- := D^{k+2}_- \cap S^k \quad \text{and} \quad N_+ := D^{k+2}_+.$$

Define N_- and $\partial_\pm N$ as in the proof of Theorem 2.1(a). Then $N \cong N_+ \cup N_-$ and $\partial N = \partial_+ N \cup \partial_- N$. We have

$$\pi_1(\partial_+ N) \cong G \quad \text{and} \quad \pi_1(\partial_+ N \cap \partial_- N) \cong \pi_1(S^1 \times \partial D^k) \cong \mathbb{Z} \quad \text{for } k \geq 3.$$

First consider the simple case when x is the only non-locally flat point (formally, the proof of the general case does not use the simpler case). Then B^k_- is locally flat. Since $N_- = R_{D^{k+2}_-}(B^k_-)$, it follows that the pair (N_-, B^k_-) is standard (see [30]). Therefore $\partial_- N \cong S^1 \times D^k$. Hence $\pi_1(\partial N) \cong G$ by the van Kampen Theorem.

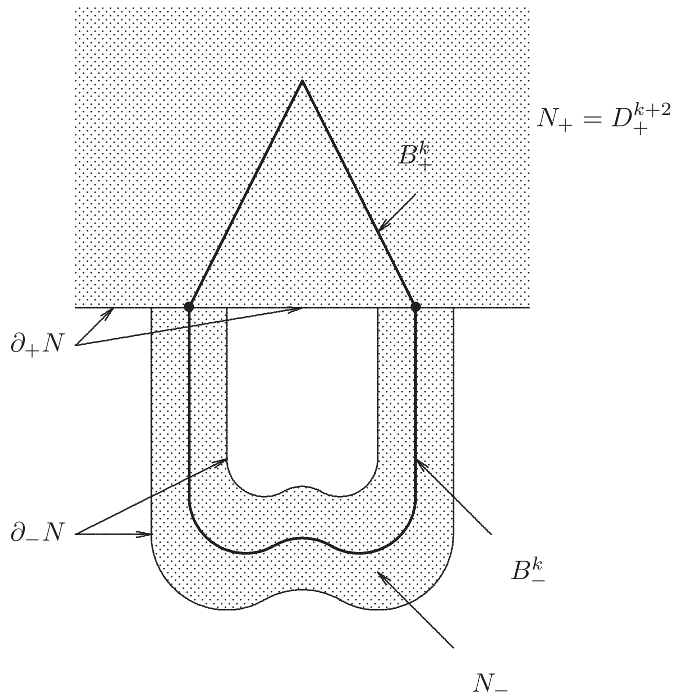

Figure 1(b)

In the general case, denote by a the generator of $\pi_1(\partial_+ N \cap \partial_- N) \cong H_1(\partial_+ N \cap \partial_- N; \mathbb{Z}) \cong \mathbb{Z}$. Observe that a is represented by a small circle in $\partial R_{\partial N_\pm}(S^{k-1})$, bounding a small 2-disk in $R_{\partial N_\pm}(S^{k-1})$, transversal to S^{k-1}. By [13, p. 57], $\partial N_- \cong S^{k+1}$. Hence the class of a is the generator of $H_1(\partial_\pm N, \mathbb{Z}) \cong \mathbb{Z}$. Thus the maps of $H_1(\cdot; \mathbb{Z})$ induced by the inclusions $\partial_+ N \cap \partial_- N \to \partial_\pm N$ are injective. Therefore the maps of π_1 induced by the inclusions are also injective. Hence by the van Kampen Theorem (see [26, pp. 370–371]) the inclusion-induced map $\pi_1(\partial_+ N) \to \pi_1(\partial N)$ is also injective and also Theorem 2.1(b) is proved.

Observe that under the assumptions of Theorem 2.1 the group $\pi_1(\partial N)$ maps onto $G := \pi_1(S^{k+1} - S^{k-1})$. We remark that for $k \geq 3$ there is a PL embedding $S^k \subset S^{k+2}$ which is locally flat except at one point and such that $R_{S^{k+2}}(S^k) \not\cong S^k \times D^2$. Indeed, there exist slice knots $S^{k-1} \subset S^{k+1}$ having non-cyclic $\pi_1(S^{k+1} - S^{k-1})$ (see [15]), and for each slice knot $S^{k-1} \subset S^{k+1}$ there is an embedding $S^k \subset S^{k+2}$, having only one non locally-flat point with the singularity $S^{k-1} \subset S^{k+1}$ (see [6]). Now the remark follows by (the simple case of) Theorem 2.1(b). The remark also follows from Theorem 2.1(a) because the regular neighborhood of the suspension over a knot L is homeomorphic to the regular neighborhood of certain knot with the only singular point (having singularity $L\#(-L)$). Or else see [7, 8, Remark after Corollary 5].

Theorem 1.1 was announced in [22] and at the International Conference on Knots, Links and Manifold (Siegen, January 2001).

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