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Sign-Changing Solutions for a Class of Nonlinear Elliptic Problems**

Aixia QIAN*

Abstract In the present paper, the author studies the existence of sign-changing solutions for nonlinear elliptic equations, which have jumping nonlinearities, and may or may not be resonant with respect to Fučik spectrum, via linking methods under Cerami condition.

Keywords Sign-changing solutions, Fučik spectrum, Jumping nonlinearities
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1 Introduction

We consider the following semilinear elliptic boundary value problem

$$\begin{cases} -\triangle u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $f(x,t)$ is a Carethéodory function on $\overline{\Omega} \times \mathbb{R}$ such that

$$\begin{cases} \dfrac{f(x,t)}{t} \to a, & \text{a.e. } x \in \Omega, \text{ as } t \to -\infty, \\ \dfrac{f(x,t)}{t} \to b, & \text{a.e. } x \in \Omega, \text{ as } t \to \infty. \end{cases} \tag{1.2}$$

The existence of solutions of (1.1) is closely related to the equation

$$-\triangle u = bu^+ - au^-, \quad \text{where } u^\pm = \max\{\pm u, 0\}. \tag{1.3}$$

Conventionally, the set

$$\Sigma := \{(a,b) \in \mathbb{R}^2 : -\triangle u = bu^+ - au^- \text{ has nontrivial solutions}\}$$

is called the Fučik spectrum of $-\triangle$.

In [1, 2, 4, 5, 8, 9] and so on, the authors studied the above problem and obtained an existence theory of one solution. Of course, their results are closely related to the Fučik spectrum and no information concerning nodal structure of solutions is obtained. The results in [10] allowed

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*Department of mathematics, Qufu Normal University, Qufu 273165, Shandong, China.
E-mail: qianax@163.com
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(a,b) to be independent of the Fučik spectrum and gave a positive answer for the existence of sign-changing solutions for various resonant elliptic equations; furthermore, there is no need to involve the Fučik spectrum in dealing with problems with jumping nonlinearities. Suggested by the results in [10], we extend the new linking theorem, which may provide the location of the critical point, to such a case that the functional only satisfies Cerami condition, and obtain the same results under weaker conditions. Furthermore, we will study nonautonomous second order quasilinear hyperbolic systems as in [11].

Let $0 < \lambda_1 < \cdots < \lambda_k < \cdots$ denote the distinct Dirichlet eigenvalues of $-\triangle$ on Ω. We make the following assumptions. The letter c will be indiscriminately used to denote various constants when the exact values are irrelevant.

(e_0) $f \in C(\overline{\Omega} \times \mathbb{R}), \exists\, c > 0$ and $1 < p < 2^* = \frac{2N}{N-2}$ (for $N = 1, 2$ we take $2^* = \infty$) such that

$$|f(x,t)| \le c(1 + |u|^{p-1}), \quad \forall\, (x,u) \in \Omega \times \mathbb{R}.$$

And $\exists L > 0$ such that $f(x,u) + Lu$ is increasing in u.

(e_1) $a, b > \lambda_k$ for some $k \ge 2$.

(e_2) $2F(x,t) \ge \lambda_{k-1} t^2,\ \forall\, (x,u) \in \Omega \times \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)ds$.

(e_3) $f(x,0) = 0$ and $\forall\, x \in \Omega, |t| \le r_0, 2F(x,t) \le \beta_0 t^2$ with $r_0 > 0,\ \beta_0 \in (\lambda_{k-1}, \lambda_k)$.

(e_4) Either $f(x,t)t - 2F(x,t) \ge H(x) \in L^1(\Omega)$ and

$$f(x,t)t - 2F(x,t) \to +\infty, \quad \text{a.e. } x \in \Omega, \text{ as } |t| \to \infty,$$

or $f(x,t)t - 2F(x,t) \le H(x) \in L^1(\Omega)$ and

$$f(x,t)t - 2F(x,t) \to -\infty, \quad \text{a.e. } x \in \Omega, \text{ as } |t| \to \infty.$$

Remark 1.1 By assumption (e_1), the point a or b may be situated across multiple eigenvalues λ_l $(l > k)$. Particularly, $a = b = \lambda_l$ (for any $\lambda_l \ge k+1$) is permitted, which means that resonance case at infinite occurs at λ_l (for any $\lambda_l \ge k+1$). Assumptions (e_2) and (e_3) contain the case when $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = \lambda_{k-1}$, a resonant case at the origin.

Let E_l denote the eigenspace corresponding to λ_l $(l \ge 1)$ and $N_k = E_1 \cup \cdots \cup E_k$. Define

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx, \quad u \in H_0^1(\Omega).$$

Our results are the following.

Theorem 1.1 *Assume that* (e_0)–(e_4) *hold. Then problem* (1.1) *has a sign-changing solution* u^* *with* $J(u^*) > 0$.

Next, we consider another case of equation (1.1) which includes double resonant case, oscillating and jumping nonlinearities.

$$\frac{f(x,t)}{t} \to b_\pm(x), \quad \text{a.e. } x \in \Omega, \text{ as } t \to \pm\infty. \tag{1.4}$$

where $\lambda_k \le b_\pm(x) \le \lambda_{k+1}$ $(k \ge 2)$.

Theorem 1.2 *Assume that* (e_0), (e_2), (e_3) *and* (1.4) *hold. Suppose*

(e_5) $\min\{b_+(x), b_-(x)\} \not\equiv \lambda_k$.

(e_6) $\exists\, \alpha > 0$ *such that*

$$\lim_{|t|\to\infty} \frac{f(x,t)t - 2F(x,t)}{|t|^\alpha} = \beta(x), \quad a.e.\ x \in \Omega,$$

where $\int_\Omega \beta(x)|w(x)|^\alpha dx \neq 0$ *and* $0 \neq w \in H_0^1(\Omega)$.

Then problem (1.1) *has a sign-changing solution* u^* *with* $J(u^*) > 0$.

Remark 1.2 In Theorems 1.1 and 1.2, for a special case, we may get a more precise position of u^*. That is, if

$$J(u^*) = \inf_{u\in N_{k-1}^\perp, \|u\|=\rho_0} J \quad \text{for some } \rho_0 > 0,$$

then $u^* \in \{u \in H_0^1(\Omega) : u \in N_{k-1}^\perp,\ \|u\| = \rho_0\}$.

Remark 1.3 Obviously, the results of Theorems 1.1 and 1.2 are independent of the Fučik spectrum. Under our assumptions, it is not necessary to involve the idea of Fučik spectrum in dealing with problems with jumping nonlinearities.

2 A Linking Type Theorem with Cerami Condition

Define a class of contractions of a Hilbert space E as follows:

$\Phi := \{\Gamma(\cdot\,,\cdot) \in C([0,1]\times E, E) : \Gamma(0,\cdot) =$id; for each $t \in [0,1)$, $\Gamma(t,\cdot)$ is a homeomorphism of E onto itself and $\Gamma^{-1}(\cdot\,,\cdot)$ is continuous on $[0,1) \times E$; there exists a $x_0 \in E$ such that $\Gamma(1,x) = x_0$ for each $x \in E$ and that $\Gamma(t,x) \to x_0$ as $t \to 1$ uniformly on bounded subsets of $E\}$.

Obviously, $\Gamma(t,u) = (1-t)u \in \Phi$.

Definition 2.1 *A subset* A *of* E *links a closed subset* B *of* E *if* $A \cap B = \emptyset$ *and, for every* $\Gamma \in \Phi$*, there is a* $t \in [0,1]$ *such that* $\Gamma(t,A) \cap B \neq \emptyset$.

There are some typical examples as those in papers [9, 10]; we only write out one of them for our own sake.

Example 2.1 *Let* $E = M \oplus N$*, where* M, N *are closed subspaces with* $\dim N < \infty$*. If* $y_0 \in M\backslash\{0\}$ *and* $0 < \rho < R$*, then the sets*

$$A := \{u = v + sy_0 : v \in N, s \geq 0, \|u\| = R\} \cup \{N \cap \overline{B}_R\},$$
$$B := M \cap \partial B_\rho$$

link each other in the sense of Definition 2.1.

Let E be a Hilbert space and $X \subset E$ be a Banach space densely embedded in E. Assume that E has a closed convex cone P_E and $P := P_E \cap X$ has interior points in X, i.e., $P = \overset{\circ}{P} \cup \partial P$ in X. We use $\|\cdot\|$ to denote the norm in E. We also use $\text{dist}_E(\cdot\,,\cdot)$ and $\text{dist}_X(\cdot\,,\cdot)$ to denote the distances in E and X, respectively.

Let $J \in C^1(E,\mathbb{R})$. We use the following notations: for $a,b,c \in \mathbb{R}$, $K = K(J) = \{x \in E : J'(x) = 0\}$, $J^b = \{x \in E : J(x) \le b\}$, $K_c = \{x \in E : J(x) = c, J'(x) = 0\}$, $K([a,b]) = \{x \in E : J(x) \in [a,b],\ J'(x) = 0\}$.

In [5], a pseudo-gradient vector field W for functional $J \in C^1(E,\mathbb{R})$ is constructed. Since $\frac{\|W(u)\|}{1+\|W(u)\|} \le 1$, the following normalized negative flow σ for J is well defined for $(t,u) \in \mathbb{R}\times E$:

$$\frac{d}{dt}\sigma(t,u) = -\frac{W(u)}{1+\|W(u)\|}, \quad \sigma(0,u) = u. \tag{2.1}$$

We shall assume that

(J) $K(J) \subset X$ and for all $u \in E$, $J'(u) = u - A(u)$, $A : X \to X$ is continuous.

Under this assumption, we have $\sigma(t,x) \in X$ for $x \in X$ and is continuous in $(t,u) \in \mathbb{R}\times X$. With the flow σ, we call a subset $A \subset E$ an invariant set if $\sigma(t,M) \subset M$ for $t \ge 0$.

We use the following concept given in [3].

Definition 2.2 *Let $M \subset X$ be an invariant set under σ. M is said to be an admissible invariant set for J, if* (a) *M is the closure of an open set in X, i.e., $M = \overset{\circ}{M} \cup \partial M$*; (b) *if $u_n = \sigma(t_n, v) \to u$ in E as $t_n \to \infty$ for some $v \notin M$ and $u \in K$, then $u_n \to u$ in X*; (c) *if $u_n \in K \cap M$ is such that $u_n \to u$ in E, then $u_n \to u$ in X*; (d) *for any $u \in \partial M \backslash K$, we have $\sigma(t,u) \in \overset{\circ}{M}$ for $t > 0$.*

Now, let $A \subset X$ be a compact set in the X-topology such that $A \cap S \ne \emptyset$, where $S = X\backslash W$, $W = P \cup (-P)$. Let $B \subset E\backslash S$ be closed, and define

$$\Phi^* := \{\Gamma \in \Phi : \Gamma(t,x) : [0,1]\times X \to X \text{ is continuous in the } X_- \text{ topology and } \Gamma(t,W) \subset W\}.$$

Then $\Gamma(t,u) = (1-t)u \in \Phi^*$.

Definition 2.3 *We say that J satisfies Cerami condition, if for any sequence $\{u_n\}$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$, we have either $\{u_n\}$ is bounded and has a convergent subsequence or $\|J'(u_n)\|\|u_n\| \to \infty$.*

The following theorem is our main result in this section, where A and B are given as above.

Theorem 2.1 *Let A link B, and let J satisfy the Cerami condition. Assume that W is an admissible invariant set of J and that*

$$a_0 := \sup_A J \le b_0 := \inf_B J.$$

Then a^* defined below is a critical value of J:

$$a^* = \inf_{\Gamma\in\Phi^*} \sup_{\Gamma([0,1],A)\cap S} J(u).$$

Furthermore, assume $0 \notin K_{a^*}$. Then $K_{a^*} \cap S \ne \emptyset$ if $a^* > b_0$, and $K_{a^*} \cap B \ne \emptyset$ if $a^* = b_0$.

Proof Evidently $a^* \ge b_0$ since A links B. In fact, for any $\Gamma \in \Phi^*$ we have that $\Gamma([0,1],A) \cap B \ne \emptyset$, $\Gamma([0,1],A) \subset X$, $B \cap X \subset S$ and $\Gamma([0,1],A) \cap S \ne \emptyset$. Then $\Gamma([0,1],A) \cap B \cap S \ne \emptyset$.

Therefore

$$\sup_{\Gamma([0,1],A)\cap S} J \geq \sup_{\Gamma([0,1],A)\cap S\cap B} J \geq \inf_{\Gamma([0,1],A)\cap S\cap B} J \geq \inf_{\Gamma([0,1],A)\cap B} J \geq \inf_{B} J = b_0.$$

We first consider the case of $a^* > b_0$. By a contradiction, we assume that $K_{a^*} \cap S = \emptyset$. Note that for any $u \in W\backslash\{0\}$, the vector $-J'(u)$ points toward the interior of W. If J has no critical point in the boundary of $W\backslash\{0\}$, then $K_{a^*} \subset \overset{\circ}{W}$. By the Cerami condition, there are $\varepsilon_0 > 0, \delta_0 > 0$ such that

$$\frac{\|J'(u)\|^2}{1+2\|J'(u)\|} \geq \frac{6\varepsilon_0}{\delta_0} \quad \text{for } u \in J^{-1}[a^* - \varepsilon_0, a^* + \varepsilon_0]\backslash(K_{a^*})_{\delta_0},$$

where $(K_{a^*})_{\delta_0} := \{u \in E : \text{dist}_E(u, K_{a^*}) \leq \delta_0\}$. By decreasing ε_0, we may assume that $\varepsilon_0 < a^* - b_0$, and $K[a^* - \varepsilon_0, a^* + \varepsilon_0] \cap S = \emptyset$ (otherwise, we are done). Let $Y(u) = \frac{W(u)}{1+\|W(u)\|}$. Then $\langle J'(u), W(u)\rangle \geq 0$ for all u and $\langle J'(u), W(u)\rangle \geq \frac{3\varepsilon_0}{\delta_0}$ for any $u \in J^{-1}[a^* - \varepsilon_0, a^* + \varepsilon_0]\backslash(K_{a^*})_{\delta_0}$. Let $X_0 = \{u \in E : |J(u) - a^*| \leq 3\varepsilon_0\},\ X_1 = \{u \in E : |J(u) - a^*| \leq 2\varepsilon_0\}$ and consider

$$\eta(u) = \frac{\text{dist}_E(u, X_2)}{\text{dist}_E(u, X_1) + \text{dist}_E(u, X_2)},$$

where $X_2 = E\backslash X_0$. Then $\eta(u)Y(u)$ is a locally Lipschitz vector field on both E and X, since X is embedded continuously in E. We consider the following Cauchy initial value problem:

$$\frac{d\sigma(t,u)}{dt} = -\eta(\sigma(t,u))Y(\sigma(t,u)), \quad \sigma(0,u) = u \in X,$$

which has a unique continuous solution $\sigma(t,u)$ in both X and E. Evidently, $\frac{dJ(\sigma(t,u))}{dt} \leq 0$. By the definition of a^*, there exists a $\Gamma \in \Phi^*$, such that $\Gamma([0,1], A) \cap S \subset J^{a^*+\varepsilon_0}$. Therefore $\Gamma([0,1], A)$ is a subset of $J^{a^*+\varepsilon_0} \cup W$. Denote $A^* = \Gamma([0,1], A)$. We claim that there exists a $T_0 > 0$ such $\sigma(T_0, A^*) \subset J^{a^*+\varepsilon_0} \cup W$. This follows from the deformation lemma with Cerami condition obtained in [7].

Next, for the case of $a^* = b_0$, please see the second part of Theorem 2.1 in [10], where we may assume that $\varepsilon_2 < \frac{\varepsilon_1^2\varepsilon_3}{4(1+2\varepsilon_1)}$. We omit its proof here.

3 Proofs of Theorems 1.1 and 1.2

Let $E := H_0^1(\Omega)$ be the usual Sobolev space endowed with the inner product

$$\langle u, v\rangle = \int_\Omega (\nabla u \nabla v + Luv)dx, \quad u, v \in E,$$

and the associated norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$, which is equivalent to the usual norm $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$. Here L is the constant given in condition (e_0). Let $X := C_0^1(\overline{\Omega})$ be the usual Banach space which is densely embedded in E. The solution of (1.1) is associated with the critical points of the following functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx, \quad u \in H_0^1(\Omega).$$

By the theory of elliptic equations, $K = K(J) \subset X$. The positive cones in E and X are given, respectively, by

$$P_E := \{u \in E : u(x) \geq 0 \text{ for a.e. } x \in \Omega\},$$
$$P := \{u \in X : u(x) \geq 0 \text{ for a.e. } x \in \Omega\}.$$

Clearly, P_E has an empty interior in E and P has a nonempty interior $\overset{\circ}{P}$ in X. Therefore, $P = \overset{\circ}{P} \cup \partial P$.

We rewrite the functional J as following:

$$J(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \Big(\frac{1}{2}Lu^2 + F(x,u)\Big)dx.$$

It is known that $J \in C^1(E, \mathbb{R})$ and that the gradient of J at u is given by

$$J'(u) = u - (-\triangle + L)^{-1}(f(x,u) + Lu) := u - Au,$$

where the operator $A : E \to E$ is continuous and compact, and $A(X) \subset X$. Particularly, by the strong maximum principle, $A|_X$, the restriction of A to X, is strongly order preserving, that is $u > v \Rightarrow A(u) \gg A(v)$. Therefore, condition (J) is satisfied.

In papers [6, 7], we have proved that $W = P \cup (-P)$ is an admissible invariant set of the pseudo gradient flow σ, by assumption (e_0) and the (PS) or Cerami condition.

To prove Theorem 1.1, we need some lemmas.

Lemma 3.1 *Assume that* (e_0), (e_4) *and* (1.3) *hold. Then J satisfies the Cerami condition.*

Proof The proof is standard. Assume that $\{u_n\}$ is a sequence such that $J(u_n) \to c$, $J'(u_n) \to 0$. Without loss of generality, we suppose that $\{\|J'(u_n)\|\|u_n\|\}$ is bounded (otherwise, we are done by the definition of Cerami condition). It suffices to prove that $\{u_n\}$ is bounded.

If $\{u_n\}$ is unbounded, we note that, as n is large enough,

$$\frac{1}{2}\|u_n\|^2 \leq c + \int_\Omega F(x,u_n) \leq c + c\int_\Omega |u_n|^p.$$

Then for a renamed subsequence

$$1 \leq c \lim_{n\to\infty} \int_\Omega \frac{|u_n|^p}{\|u_n\|^2}dx.$$

It follows that $\lim\limits_{n\to\infty} |u_n|^p = \infty$, and $\lim\limits_{n\to\infty} |u_n| = \infty$ in turn, on a subset of Ω with positive measure. Combing this with (e_4), we have

$$\Big|\int_\Omega \Big(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\Big)dx\Big| \longrightarrow \infty.$$

However,

$$\Big|J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle\Big| = \Big|\int_\Omega \Big(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\Big)dx\Big| < c,$$

a contradiction. Thus, we see that $\{u_n\}$ is bounded.

We rewrite J as

$$J(u)=\frac{1}{2}\|u\|^2-\frac{1}{2}a\|u^-\|_2^2-\frac{1}{2}b\|u^+\|_2^2-\int_\Omega P(x,u)dx,\quad u\in H_0^1(\Omega),$$

where $P(x,u)=\int_0^u p(x,t)dt;\ p(x,t)=f(x,t)-(bt^+-at^-)$, and $t^{\pm}=\max\{\pm t,0\}$.

Let E_l denote the eigenspace corresponding to λ_l ($l\geq 1$) and $N_k=E_l\cup\cdots\cup E_k$. Then we have

Lemma 3.2 *Assume* (e$_1$). *Then* $J(u)\to-\infty$ *as* $\|u\|\to\infty,\ u\in N_k$.

Lemma 3.3 *Assume* (e$_2$). *Then* $J(u)\leq 0$ *for all* $u\in N_{k-1}$.

Proof This is an immediate consequence of condition (e$_2$).

Lemma 3.4 *Assume* (1.2) *or* (1.4) *and* (e$_3$). *Then there exist* $\rho_0>0,\ c_0>0$ *such that* $J(u)\geq c_0$ *for* $\|u\|=\rho_0>0,\ u\in N_{k-1}^{\perp}$.

Proof of Theorem 1.1 By Lemmas 3.1–3.4, there exists $R_0>\rho_0>0$ such that

$$a_0:=\sup_A J(u)\leq 0<c_0\leq b_0:=\inf_B J(u),$$

where $A:=\{u=v+sy_0:v\in N_{k-1},s\geq 0,\|u\|=R_0\}\cup\{N_{k-1}\cap\overline{B}_{R_0}\}$, with $y_0\in E_k$ satisfying $\|y_0\|=1$; $B:=\{u\in N_{k-1}^{\perp}:\|u\|=\rho_0\}$.

Theorem 2.1 and Example 2.1 imply that there is a critical point u^* satisfying $J'(u^*)=0,\ J(u^*)=a^*\geq b_0>0$. Obviously, $u^*\neq 0$ and either $u^*\in S$ or $u^*\in B$. The second alternative occurs when $J(u^*)=b_0$ (see Remark 1.2). Both cases imply that u^* is sign-changing.

Remark 3.1 Our condition (e$_4$) is weaker than that of [10], where that is applied to prove the (PS) condition. Under our condition (e$_4$) here, J only satisfies Carami condition.

To prove Theorem 1.2, we need the following lemmas.

Lemma 3.5 *Under assumptions* (e$_0$) *and* (e$_5$), $J(u)\to-\infty$ *for* $u\in N_k$ *as* $\|u\|\to\infty$.

Please see [10] for the proofs of above Lemmas 3.2, 3.4 and 3.5.

Lemma 3.6 *Assume condition* (e$_0$), (e$_6$) *and* (1.4) *hold. Then* J *satisfies the Cearmi condition.*

Proof Assume that $\{u_n\}$ is a sequence such that $J(u_n)\to c,\ J'(u_n)\to 0$ and $\{\|J'(u_n)\|\|u_n\|\}$ is bounded. We are going to prove that $\{u_n\}$ is bounded. By a contradiction, we assume that $\|u_n\|\to\infty$ as $n\to\infty$. Let $w_n=\frac{u_n}{\|u_n\|}$. Then $\|w_n\|=1$ and, up to a subsequence, $w_n\to w$ weakly in E, strongly in $L^2(\Omega)$ and a.e. in Ω. Moreover,

$$\langle J'(u_n),v\rangle=\langle u_n,v\rangle-\int_\Omega f(x,u_n)vdx\to 0,$$
$$\langle w_n,v\rangle-\int_\Omega\frac{f(x,u_n)v}{\|u_n\|}dx\to 0.$$

By (1.4), we see that

$$-\triangle w=b_+w^+-b_-w^-.$$

Since

$$\frac{J(u_n)}{\|u_n\|^2} = \frac{1}{2} - \frac{\int_\Omega F(x,u_n)dx}{\|u_n\|^2} \to 0,$$

we see that

$$\int_\Omega (b_+(w^+)^2 + b_-(w^-)^2)dx = 1.$$

It implies that $w \neq 0$. By the Lebesgue Theorem, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle}{\|u_n\|^\alpha} \\ &= \lim_{n\to\infty} \frac{\int_\Omega (f(x,u_n)u_n - 2F(x,u_n))dx}{\|u_n\|^\alpha} \\ &= \lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)u_n - 2F(x,u_n)}{|u_n|^\alpha} \cdot \frac{|u_n|^\alpha}{\|u_n\|^\alpha} dx \\ &= \int_\Omega \beta(x)|w(x)|^\alpha dx \neq 0, \end{aligned}$$

a contradiction.

Proof of Theorem 1.2 By Lemmas 3.3–3.5, there exist $R_0 > \rho_0 > 0$ such that

$$a_0 := \sup_A J(u) \leq 0 < c_0 \leq b_0 := \inf_B J(u),$$

where A and B are defined as in the proof of Theorem 1.1. By Lemma 3.6, we get the conclusions of Theorem 1.2 and Remark 1.2 as in the proof of Theorem 1.1.

Remark 3.2 Our condition (e_6) here is different from that of [10], which is used to prove the Cerami condition; furthermore, it is more intuitive.

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