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On Representations Associated with Completely n-Positive Linear Maps on Pro-C^*-Algebras**

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Abstract It is shown that an $n \times n$ matrix of continuous linear maps from a pro-C^*-algebra A to $L(H)$, which verifies the condition of complete positivity, is of the form $[V^*T_{ij}\Phi(\cdot)V]_{i,j=1}^n$, where Φ is a representation of A on a Hilbert space K, V is a bounded linear operator from H to K, and $[T_{ij}]_{i,j=1}^n$ is a positive element in the C^*-algebra of all $n \times n$ matrices over the commutant of $\Phi(A)$ in $L(K)$. This generalizes a result of C. Y. Suen in Proc. Amer. Math. Soc., **112**(3), 1991, 709–712. Also, a covariant version of this construction is given.

Keywords Pro-C^*-Algebra, Completely n-positive linear maps, Covariant completely n-positive linear maps
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1 Introduction and Preliminaries

Pro-C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology on a pro-C^*-algebra is defined by a directed family of C^*-seminorms. In fact, a pro-C^*-algebra is a projective limit of C^*-algebras. A pro-C^*-algebra A is a complete Hausdorff topological $*$-algebra over $\mathbb{C}$ whose topology is determined by its continuous C^*-seminorms in the sense that the net $\{a_i\}_{i\in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for any continuous C^*-seminorm p on A. The set $S(A)$ of all continuous C^*-seminorms on A is directed ($p \geq q$ if $p(a) \geq q(a)$ for all a in A). For each $p \in S(A)$, $\ker p = \{a \in A; p(a) = 0\}$ is a closed two-sided ideal in A and the quotient $*$-algebra $A/\ker p$, denoted by A_p, is a C^*-algebra in the C^*-norm induced by p. The canonical map from A to A_p is denoted by π_p. For p and q in $S(A)$ with $p \geq q$, there is a canonical morphism $\pi_{pq} : A_p \to A_q$ of C^*-algebras such that $\pi_{pq}(a + \ker p) = a + \ker q$ for all $a \in A$. Moreover, $\{A_p, \pi_{pq}\}_{p\geq q},\ p, q \in S(A)$ is an inverse system of C^*-algebras and $\lim\limits_{\substack{\leftarrow \\ p}} A_p$ is a pro-C^*-algebra which is algebraically and topologically isomorphic with A. In the literature, pro-C^*-algebras have been given by different name such as b^*-algebras (by C. Apostol), LMC^*-algebras (by G. Lessner, K. Schmüdgen) or locally C^*-algebras (by A. Inoue, M. Fragoulopoulou, etc.). Besides an intrinsic interest in pro-C^*-algebras as topological algebras comes from the fact that they provide an important tool in investigation of certain aspect of C^*-algebras (like multipliers of Pedersen ideal; tangent algebra of a C^*-algebra) and quantum field theory.

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A continuous $*$-morphism from a pro-C^*-algebra A to another pro-C^*-algebra B is called a morphism of pro-C^*-algebras. An isomorphism of pro-C^*-algebras is a bijective map $\Phi : A \to B$ such that Φ and Φ^{-1} are morphisms of pro-C^*-algebras.

A representation of a pro-C^*-algebra A on a Hilbert space H is a continuous $*$-morphism φ from A to $L(H)$, the C^*-algebra of all bounded linear operators on H. A representation (φ, H) of A is nondegenerate if $\varphi(A)H$ is dense in H (see [3]).

A continuous action of a locally compact group G on a pro-C^*-algebra A is a morphism of groups $\alpha : G \to \mathrm{Aut}(A)$. Here $\mathrm{Aut}(A)$ is the group of all isomorphisms of pro-C^*-algebras from A onto A, such that the map $(g, a) \mapsto \alpha_g(a)$ from $G \times A$ to A is jointly continuous. The action α is an inverse limit action if we can write A as an inverse limit $\lim\limits_{\overleftarrow{\delta\in\Delta}} A_\delta$ of C^*-algebras in such a way that there are continuous actions $\alpha^{(\delta)}$ of G on A_δ, $\delta \in \Delta$ such that $\alpha_g = \lim\limits_{\overleftarrow{\delta\in\Delta}} \alpha_g^{(\delta)}$ for all $g \in G$ (see [15]). If G is a compact group, then any continuous action of G on A is an inverse limit action (see [15]).

A covariant representation of a dynamical system (A, G, α) is a triple (φ, u, H), where (φ, H) is a representation of A and (u, H) is a unitary representation of G, such that

$$\varphi(\alpha_g(a)) = u_g \varphi(a)(u_g)^*$$

for all $a \in A$ and for all $g \in G$. A covariant representation (φ, u, H) is nondegenerate if (φ, H) is nondegenerate.

A pro-C^*-dynamical system is a triple (A, G, α), where A is a pro-C^*-algebra, G is a locally compact group and α is a continuous inverse limit action of G on A.

Let (A, G, α) be a pro-C^*-dynamical system. The set $C_c(G, A)$ of continuous functions from G to A with compact support is a $*$-algebra with multiplication of two elements defined by $(f, h) \mapsto f \times h$,

$$(f \times h)(s) = \int_G f(t)\alpha_t(h(t^{-1}s))\mathrm{d}t,$$

and involution $f \mapsto f^\#$,

$$f^\#(s) = \gamma(s)^{-1}\alpha_s(f(s^{-1})^*),$$

where γ is the modular function on G. The Hausdorff completion of $C_c(G, A)$ with respect to the topology defined by the family of submultiplicative $*$-seminorms $\{N_p\}_{p\in S(A)}$ (N_p is defined by

$$N_p(f) = \int_G p(f(t))\mathrm{d}t,$$

$f \in C_c(G, A)$) is a complete locally m-convex $*$-algebra $L^1(A, G, \alpha)$ with bounded approximate unit. The enveloping algebra of $L^1(A, G, \alpha)$ is a pro-C^*-algebra, denoted by $A \times_\alpha G$ and called the crossed product of A by α (see [7]).

If A is a pro-C^*-algebra, then $M_n(A)$, the set of all $n \times n$ matrices over A with the algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A is a pro-C^*-algebra. The concept of matricial order plays an important role to understand the infinite-dimensional noncommutative structure of operator algebras. Completely positive linear maps as the natural ordering attached to this structure have been extensively studied in [1, 4–6, 8–12, 16, 17].

A completely n-positive linear map from A to $L(H)$ is an $n\times n$ matrix $[\rho_{ij}]_{i,j=1}^n$ of continuous linear maps from A to $L(H)$ such that the map $\rho : M_n(A) \to M_n(L(H))$ defined by

$$\rho([a_{ij}]_{i,j=1}^n) = [\rho_{ij}(a_{ij})]_{i,j=1}^n$$

is completely positive. We say that a completely n-positive linear map $[\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ is nondegenerate if for some approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ of A, the nets $\{\rho_{ii}(e_\lambda)\}_{\lambda\in\Lambda}$, $i = 1, 2, \cdots, n$, converge strictly to the identity operator on H (see [6]).

In [17], Suen showed that each unital completely n-positive linear map $[\rho_{ij}]_{i,j=1}^n$ from a unital C^*-algebra A to $L(H)$ is of the form $[V^*T_{ij}\Phi(\cdot)V]_{i,j=1}^n$ where Φ is a unital representation of A on a Hilbert space K, V is a partial isometry from H to K, and $[T_{ij}]_{i,j=1}^n$ is a positive element in $M_n(\Phi(A)')$. $\Phi(A)'$ denotes the commutant of $\Phi(A)$ in $L(K)$.

In this paper, using a Radon-Nikodym type theorem for completely positive linear maps from a pro-C^*-algebra A to $L(H)$, we extend the result of Suen in the context of pro-C^*-algebras (see Theorem 2.1). Moreover, we prove that the representation associated with a completely n-positive linear map is unique up to unitary equivalence and give a necessary and sufficient criterion of irreducibility for this representation (see Corollary 2.1). In Section 3, we prove a covariant version of Theorem 2.1. Also we prove that a u-covariant, nondegenerate completely n-positive linear map $[\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ induces a nondegenerate completely n-positive linear map $[\theta^\rho{}_{ij}]_{i,j=1}^n$ from $A\times_\alpha G$ to $L(H)$ such that the representation of $A\times_\alpha G$ induced by $[\theta^\rho{}_{ij}]_{i,j=1}^n$ is unitarily equivalent with the representation of $A\times_\alpha G$ associated with the covariant representation of (A, G, α) induced by $[\rho_{ij}]_{i,j=1}^n$ (see Proposition 3.1 and Remark 3.2).

2 Representations Associated with Completely n-Positive Linear Maps

Remark 2.1 Let A be a C^*-algebra. If $\rho = [\rho_{ij}]_{i,j=1}^n$ is a completely n-positive linear map from A to $L(H)$, then for each $i = 1, \cdots, n$, the map ρ_{ii} is completely positive and for each $i, j \in \{1, \cdots, n\}$ with $i \neq j$, $\rho_{ji} = \rho_{ij}^*$ (ρ_{ij}^* is a linear map from A to $L(H)$ defined by $\rho_{ij}^*(a) = (\rho_{ij}(a^*))^*$ for all $a \in A$). Moreover, the linear maps $(\rho_{ii} + \rho_{jj}) \pm 2\mathrm{Re}\,\rho_{ij}$ and $(\rho_{ii}+\rho_{jj})\pm 2\mathrm{Im}\,\rho_{ij}$ are completely positive for each $i, j \in \{1, \cdots, n\}$ with $i \neq j$ (see, for example, [13]). Then the linear maps $\sum_{k=1}^n \rho_{kk}$, $\sum_{k=1}^n \rho_{kk} \pm 2\mathrm{Re}\,\rho_{ij}$ and $\sum_{k=1}^n \rho_{kk} \pm 2\mathrm{Im}\,\rho_{ij}$, $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$ are completely positive.

Remark 2.2 If $\rho = [\rho_{ij}]_{i,j=1}^n$ is a continuous completely n-positive linear map from a pro-C^*-algebra A to $L(H)$, then there is $p \in S(A)$ and a completely n-positive linear map $\rho^p = [\rho_{ij}^p]_{i,j=1}^n$ from A_p to $L(H)$ such that $[\rho_{ij}]_{i,j=1}^n = [\rho_{ij}^p \circ \pi_p]_{i,j=1}^n$ (see [5]). From this fact and Remark 2.1 we deduce that for each $i = 1, \cdots, n$, the continuous linear map ρ_{ii} is completely positive and for each $i, j \in \{1, \cdots, n\}$ with $i \neq j$, the continuous linear maps $\sum_{k=1}^n \rho_{kk}$, $\sum_{k=1}^n \rho_{kk} \pm 2\mathrm{Re}\,\rho_{ij}$ and $\sum_{k=1}^n \rho_{kk} \pm 2\mathrm{Im}\,\rho_{ij}$, are completely positive.

The following theorem is a generalization of [17, Proposition 2.7].

Theorem 2.1 *Let A be a pro-C^*-algebra, let H be a Hilbert space, and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a completely n-positive linear map from A to $L(H)$.*

(1) *Then there is a representation* Φ_ρ *of* A *on a Hilbert space* H_ρ*, a bounded linear operator* $V : H \to H_\rho$*, and a positive element* $[T_{ij}^\rho]_{i,j=1}^n \in M_n(\Phi(A)')$ *with* $\sum_{i=1}^n T_{ii}^\rho = n\,\mathrm{id}_{H_\rho}$ *such that*

(a) $\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho$ *for all* $a \in A$ *and* $i, j = 1, \cdots, n$;

(b) $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\}$ *is dense in* H_ρ.

(2) *If* Φ *is another representation of* A *on a Hilbert space* K, $V : H \to K$ *is a bounded linear operator and* $[S_{ij}]_{i,j=1}^n$ *is a positive element in* $M_n(\Phi(A)')$ *with* $\sum_{i=1}^n S_{ii} = n\,\mathrm{id}_K$ *such that*

(a) $\rho_{ij}(a) = V^* S_{ij} \Phi(a) V$ *for all* $a \in A$ *and* $i, j = 1, 2, \cdots, n$;

(b) $\{\Phi(a)V\xi; a \in A, \xi \in E\}$ *is dense in* K;

then there is a unitary operator $U : H_\rho \to K$ *such that*

(i) $\Phi(a) = U\Phi_\rho(a)U^*$ *for all* $a \in A$;

(ii) $V = UV_\rho$;

(iii) $S_{ij} = UT_{ij}^\rho U^*$ *for all* $i, j = 1, 2, \cdots, n$.

Proof (1) Let $\widetilde{\rho} = \frac{1}{n}\sum_{k=1}^n \rho_{kk}$. By Remark 2.2, $\widetilde{\rho}$ is completely positive. Let $(\Phi_\rho, V_\rho, H_\rho)$ be the Stinespring representation associated with $\widetilde{\rho}$ (see [9, Theorem 2.2]). Then $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\}$ generates a dense subspace in H_ρ.

Let $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$. Since $\widetilde{\rho} - \frac{1}{2}(\widetilde{\rho} + \frac{2}{n}\mathrm{Re}\,\rho_{ij}) = \frac{1}{2}(\widetilde{\rho} - \frac{2}{n}\mathrm{Re}\,\rho_{ij})$ and $\widetilde{\rho} - \frac{1}{2}(\widetilde{\rho} + \frac{2}{n}\mathrm{Im}\,\rho_{ij}) = \frac{1}{2}(\widetilde{\rho} - \frac{2}{n}\mathrm{Im}\,\rho_{ij})$ and since the linear maps $\widetilde{\rho} - \frac{2}{n}\mathrm{Re}\,\rho_{ij}$ and $\widetilde{\rho} - \frac{2}{n}\mathrm{Im}\,\rho_{ij}$ are completely positive (see Remark 2.2), by Radon Nikodym type theorem for completely positive linear maps [9, Theorem 3.5], there are two positive operators $T_{ij}^{(1)}, T_{ij}^{(2)} \in \Phi_\rho(A)'$ such that

$$(\mathrm{Re}\,\rho_{ij})(a) = V_\rho^*\Big(nT_{ij}^{(1)} - \frac{n}{2}\mathrm{id}_{H_\rho}\Big)\Phi_\rho(a)V_\rho,$$
$$(\mathrm{Im}\,\rho_{ij})(a) = V_\rho^*\Big(nT_{ij}^{(2)} - \frac{n}{2}\mathrm{id}_{H_\rho}\Big)\Phi_\rho(a)V_\rho$$

for all $a \in A$. Moreover, the positive bounded linear operators $T_{ij}^{(1)}, T_{ij}^{(2)} \in \Phi_\rho(A)'$ are unique with the above properties. Let $T_{ij}^\rho = (nT_{ij}^{(1)} - \frac{n}{2}\mathrm{id}_{H_\rho}) + i(nT_{ij}^{(2)} - \frac{n}{2}\mathrm{id}_{H_\rho})$. Clearly, $T_{ij}^\rho \in \Phi_\rho(A)'$ and

$$\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho$$

for all $a \in A$. It is not difficult to check that T_{ij}^ρ is unique with the above property. Moreover, $(T_{ij}^\rho)^* = T_{ji}^\rho$.

Let $i \in \{1, \cdots, n\}$. Since $\frac{1}{n}\rho_{ii} \leq \widetilde{\rho}$, by Radon Nikodym type theorem for completely positive linear maps (see [9, Theorem 3.5]), there is a unique positive element $T_{ii}^\rho \in \Phi_\rho(A)'$ such that

$$\rho_{ii}(a) = V_\rho^* T_{ii}^\rho \Phi_\rho(a) V_\rho$$

for all $a \in A$. From

$$\widetilde{\rho}(a) = \frac{1}{n}\sum_{i=1}^n \rho_{ii}(a) = \frac{1}{n}\sum_{i=1}^n V_\rho^* T_{ii}^\rho \Phi_\rho(a) V_\rho = V_\rho^*\Big(\frac{1}{n}\sum_{i=1}^n T_{ii}^\rho\Big)\Phi_\rho(a)V_\rho$$

and [9, Theorem 3.5], we conclude that $\sum_{i=1}^n T_{ii}^\rho = n\,\mathrm{id}_{H_\rho}$.

From

$$\begin{aligned}
&\langle [T^{\rho}_{ij}]^n_{i,j=1}(\Phi_\rho(a_k)V_\rho\xi_k)^n_{k=1}, (\Phi_\rho(a_k)V_\rho\xi_k)^n_{k=1}\rangle \\
&= \sum_{i,j=1}^n \langle T^{\rho}_{ij}\Phi_\rho(a_j)V_\rho\xi_j, \Phi_\rho(a_i)V_\rho\xi_i\rangle \\
&= \sum_{i,j=1}^n \langle V^*_\rho T^{\rho}_{ij}\Phi_\rho(a^*_i a_j)V_\rho\xi_j, \xi_i\rangle \\
&= \sum_{i,j=1}^n \langle \rho_{ij}(a^*_i a_j)\xi_j, \xi_i\rangle
\end{aligned}$$

for all $\xi_1, \cdots, \xi_n \in H$ and for all $a_1, \cdots, a_n \in A$, and taking into account that $\rho = [\rho_{ij}]^n_{i,j=1}$ is completely positive and $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in E\}$ generates H_ρ, we conclude that $[T^{\rho}_{ij}]^n_{i,j=1}$ is a positive element in $M_n(\Phi_\rho(A)')$.

(2) We consider the linear map $U_0 : \mathrm{Sp}\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\} \to \mathrm{Sp}\{\Phi(a)V\xi;\ a \in A, \xi \in E\}$ defined by

$$U(\Phi_\rho(a)V_\rho\xi) = \Phi(a)V\xi.$$

Since

$$\begin{aligned}
\langle U_0(\Phi_\rho(a)V_\rho\xi), U_0(\Phi_\rho(a)V_\rho\xi)\rangle &= \langle \Phi(a)V\xi, \Phi(a)V\xi\rangle = \langle V^*\Phi(a^*a)V\xi, \xi\rangle \\
&= \frac{1}{n}\Big\langle V^*\Big(\sum_{i=1}^n S_{ii}\Big)\Phi(a^*a)V\xi, \xi\Big\rangle = \frac{1}{n}\sum_{i=1}^n \langle \rho_{ii}(a^*a)\xi, \xi\rangle \\
&= \frac{1}{n}\sum_{i=1}^n \langle V^*_\rho T^{\rho}_{ii}\Phi_\rho(a^*a)V_\rho\xi, \xi\rangle = \langle V^*_\rho\Phi_\rho(a^*a)V_\rho\xi, \xi\rangle \\
&= \langle \Phi_\rho(a)V_\rho\xi, \Phi_\rho(a)V_\rho\xi\rangle
\end{aligned}$$

for all $a \in A$ and for all $\xi \in E$, U_0 extends to a unitary operator U from H_ρ to K. It is easy to verify that $U\Phi_\rho(a) = \Phi(a)U$ for all $a \in A$ and $UV_\rho = V$. Let $i, j \in \{1, 2, \cdots, n\}$ and $a \in A$. Clearly, $U^*S_{ij}U \in \Phi_\rho(A)'$. From

$$\rho_{ij}(a) = V^*S_{ij}\Phi(a)V = V^*_\rho U^*S_{ij}\Phi(a)UV_\rho = V^*_\rho U^*S_{ij}U\Phi_\rho(a)V_\rho$$

and taking into account that T^{ρ}_{ij} is the unique element in $\Phi_\rho(A)'$ such that

$$\rho_{ij}(a) = V^*_\rho T^{\rho}_{ij}\Phi_\rho(a)V_\rho$$

for all $a \in A$, we deduce that $U^*S_{ij}U = T^{\rho}_{ij}$ and the theorem is proved.

Remark 2.3 If $\rho = [\rho_{ij}]^n_{i,j=1}$ is nondegenerate, then V_ρ is an isometry, since for some approximate unit $\{e_\lambda\}_{\lambda\in\Lambda}$ of A, we have

$$\begin{aligned}
\xi &= \frac{1}{n}\sum_{i=1}^n \lim_\lambda \rho_{ii}(e_\lambda)\xi = \frac{1}{n}\sum_{i=1}^n \lim_\lambda V^*_\rho T^{\rho}_{ii}\Phi_\rho(e_\lambda)V_\rho\xi = \lim_\lambda V^*_\rho\Phi_\rho(e_\lambda)V_\rho\xi \\
&= V^*_\rho V_\rho\xi \quad \text{(by [5, Proposition 4.2])}
\end{aligned}$$

for all $\xi \in H$.

Corollary 2.1 *Let A be a pro-C^*-algebra, let H be a Hilbert space, and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a continuous completely n-positive linear map from A to $L(H)$. The representation (Φ_ρ, H_ρ) of A associated with ρ is irreducible if and only if there is a pure completely positive linear map θ from A to $L(H)$ and a positive matrix $[\lambda_{ij}]_{i,j=1}^n$ in $M_n(\mathbb{C})$ with $\sum\limits_{k=1}^n \lambda_{kk} = n$ such that $\rho = [\lambda_{ij}\theta]_{i,j=1}^n$.*

Proof First we suppose that (Φ_ρ, H_ρ) is irreducible. Then $\widetilde{\rho}$ is pure and for each $i, j \in \{1, 2, \cdots, n\}$ there is $\lambda_{ij} \in \mathbb{C}$ such that $T_{ij}^\rho = \lambda_{ij}\mathrm{id}_{H_\rho}$ (see [9, Corollary 3.6]). Moreover, $[\lambda_{ij}]_{i,j=1}^n$ is a positive matrix in $M_n(\mathbb{C})$ with $\sum\limits_{k=1}^n \lambda_{kk} = n$, and since

$$\rho_{ij}(a) = \lambda_{ij}V_\rho^*\Phi_\rho(a)V_\rho = \lambda_{ij}\widetilde{\rho}(a)$$

for all $a \in A$ and for all $i, j = 1, 2, \cdots, n$,

$$\rho = [\lambda_{ij}\widetilde{\rho}]_{i,j=1}^n.$$

Conversely, if $\rho = [\lambda_{ij}\theta]_{i,j=1}^n$ and $\sum\limits_{k=1}^n \lambda_{kk} = n$, then $\widetilde{\rho} = \theta$ and since θ is pure, the representation of A associated with $\widetilde{\rho}$ is irreducible (see [9, Corollary 3.6]). Therefore the representation (Φ_ρ, H_ρ) of A associated with ρ is irreducible.

3 Covariant Representations Associated with Covariant Completely n-Positive Linear Maps

Definition 3.1 *Let A be a pro-C^*-algebra, let (G, A, α) be a pro-C^*-dynamical system and let u be a unitary representation of G on a Hilbert space H. We say that a completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ is u-covariant with respect to the pro-C^*-dynamical system (G, A, α) if*

$$\rho_{ij}(\alpha_g(a)) = u_g\rho_{ij}(a)(u_g)^*$$

for all $a \in A$ and for all $g \in G$.

The following theorem is a covariant version of Theorem 2.1.

Theorem 3.1 *Let A be a pro-C^*-algebra, let (G, A, α) be a pro-C^*-dynamical system, let u be a unitary representation of G on a Hilbert space H, and let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a u-covariant nondegenerate completely n-positive linear map from A to $L(H)$.*

(1) *Then there is a covariant representation $(\Phi_\rho, v^\rho, H_\rho)$ of (G, A, α), an isometry V_ρ in $L(H, H_\rho)$ and a positive element $[T_{ij}^\rho]_{i,j=1}^n$ in $M_n(\Phi_\rho(A)' \cap v^\rho(G)')$ with $\sum\limits_{i=1}^n T_{ii}^\rho = n\,\mathrm{id}_{H_\rho}$ such that*

(a) *$\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho$ for all $a \in A$ and for all $i, j = 1, 2, \cdots, n$;*
(b) *$\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\}$ spans a dense subspace of H_ρ;*
(c) *$v_g^\rho V_\rho = V_\rho u_g$ for all $g \in G$.*

(2) *If (Φ, v, K) is a covariant representation of (G, A, α), V is an isometry in $L(H, K)$, and $[S_{ij}]_{i,j=1}^n$ is a positive element in $M_n(\Phi(A)' \cap v(G)')$ with $\sum\limits_{i=1}^n S_{ii} = n\,\mathrm{id}_K$ such that*

(a) $\rho(a) = V^* S_{ij} \Phi(a) V$ *for all* $a \in A$ *and for all* $i, j = 1, 2, \cdots, n$;

(b) $\{\Phi(a)V\xi; a \in A, \xi \in H\}$ *spans a dense subspace of* K;

(c) $v_g V = V u_g$ *for all* $g \in G$,

then there is a unitary operator U *in* $L(H_\rho, K)$ *such that*

(i) $\Phi(a) = U \Phi_\rho(a) U^*$ *for all* $a \in A$;

(ii) $v_g = U v_g^\rho U^*$ *for all* $g \in G$;

(iii) $V = U V_\rho$;

(iv) $S_{ij} = U T_{ij}^\rho U^*$ *for all* $i, j = 1, 2, \cdots, n$.

Proof (1) Let $\widetilde{\rho} = \frac{1}{n} \sum\limits_{i=1}^{n} \rho_{ii}$. Clearly, $\widetilde{\rho}$ is a u-covariant nondegenerate continuous completely positive linear map from A to $L(H)$. Let $(\Phi_\rho, v^\rho, V_\rho, H_\rho)$ be the covariant Stinespring construction associated with $\widetilde{\rho}$ (see, for example, [8, Theorem 3.6]). Moreover, the triple $(\Phi_\rho, V_\rho, H_\rho)$ is the Stinespring representation associated with $\widetilde{\rho}$. Therefore the quadruple $(\Phi_\rho, v^\rho, V_\rho, H_\rho)$ verifies the relations Theorem 3.1(1)(a) and Theorem 3.1(1)(c) and by the proof of Theorem 2.1, there is a positive element $[T_{ij}^\rho]_{i,j=1}^n$ in $M_n(\Phi_\rho(A)')$ with $\sum\limits_{i=1}^{n} T_{ii}^\rho = n \, \mathrm{id}_{H_\rho}$, such that

$$\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho \quad \text{for all } a \in A \text{ and for all } i, j = 1, 2, \cdots, n.$$

Let $i, j \in \{1, 2, \cdots, n\}$. To show that $T_{ij}^\rho \in v^\rho(G)'$, let $a \in A$. From

$$\begin{aligned} \rho_{ij}(a) &= u_g^* \rho_{ij}(\alpha_g(a)) u_g = u_g^* V_\rho^* T_{ij}^\rho \Phi_\rho(\alpha_g(a)) V_\rho u_g \\ &= V_\rho^* (v_g^\rho)^* T_{ij}^\rho v_g^\rho \Phi_\rho(a) (v_g^\rho)^* v_g^\rho V_\rho \\ &= V_\rho^* (v_g^\rho)^* T_{ij}^\rho v_g^\rho \Phi_\rho(a) V_\rho \end{aligned}$$

for all $g \in G$ and the uniqueness of T_{ij}^ρ such that $\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho$, we deduce that $T_{ij}^\rho = (v_g^\rho)^* T_{ij}^\rho v_g^\rho$ for all $g \in G$ and so $T_{ij}^\rho \in v^\rho(G)'$.

(2) Since

$$\widetilde{\rho}(a) = \frac{1}{n} \sum_{i=1}^{n} \rho_{ii}(a) = \frac{1}{n} \sum_{i=1}^{n} V^* S_{ii} \Phi(a) V = V^* \Big(\frac{1}{n} \sum_{i=1}^{n} S_{ii} \Big) \Phi(a) V = V^* \Phi(a) V$$

for all $a \in A$, $\{\Phi(a)V\xi; a \in A, \xi \in H\}$ spans a dense subspace of K and since $v_g V = V u_g$ for all $g \in G$, (Φ, v, K) is a covariant representation of (A, G, α) associated with $\widetilde{\rho}$ and then there is a unitary operator $U : H_\rho \to K$ (see [8, Theorem 3.6]) such that

(a) $\Phi(a) = U \Phi_\rho(a) U^*$ for all $a \in A$;

(b) $v_g = U v_g^\rho U^*$ for all $g \in G$;

(c) $V = U V_\rho$.

Let $i, j \in \{1, 2, \cdots, n\}$. From

$$\rho_{ij}(a) = V^* S_{ij} \Phi(a) V = V_\rho^* U^* S_{ij} \Phi(a) U V_\rho = V_\rho^* (U^* S_{ij} U) \Phi_\rho(a) V_\rho$$

for all $a \in A$ and the uniqueness of the bounded linear operator $T_{ij}^\rho \in \Phi_\rho(A)'$ such that $\rho_{ij}(a) = V_\rho^* T_{ij}^\rho \Phi_\rho(a) V_\rho$ for all $a \in A$, we deduce that $T_{ij}^\rho = U^* S_{ij} U$ and the theorem is proved.

Remark 3.1 Any u-covariant completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ with respect the pro-C^*-dynamical system (G, A, α) induces a nondegenerate covariant representation $(\Phi_\rho, v^\rho, H_\rho)$ of (G, A, α) and so a nondegenerate representation $(\Phi_\rho \times v^\rho, H_\rho)$ of $A \times_\alpha G$ (see [7]).

Proposition 3.1 *Let (G, A, α) be a pro-C^*-dynamical system, let H be a Hilbert space, and let u be a unitary representation of G on H. If $\rho = [\rho_{ij}]_{i,j=1}^n$ is a u-covariant nondegenerate completely n-positive linear map from A to $L(H)$, then there is a unique completely n-positive linear map $\theta^\rho = [\theta_{ij}^\rho]_{i,j=1}^n$ from $A \times_\alpha G$ to $L(H)$ such that*

$$\theta_{ij}^\rho(f) = \int_G \rho_{ij}(f(g))u_g \mathrm{d}g$$

for all $f \in C_c(G, A)$ and for all $i, j \in \{1, 2, \cdots, n\}$. Moreover, θ^ρ is nondegenerate.

Proof Let $(\Phi_\rho, v^\rho, V_\rho, H_\rho, [T_{ij}^\rho]_{i.j=1}^n)$ be the construction associated with ρ by Theorem 3.1. Since $T_{ij}^\rho \in \Phi_\rho(A)' \cap v^\rho(G)'$, it is not difficult to verify that $T_{ij}^\rho \in (\Phi_\rho \times v^\rho)(A \times_\alpha G)'$ for all $i, j \in \{1, 2, \cdots, n\}$.

For each $i, j \in \{1, 2, \cdots, n\}$, we consider the linear map $\theta_{ij}^\rho : A \times_\alpha G \to L(H)$ defined by

$$\theta_{ij}^\rho(x) = V_\rho^* T_{ij}^\rho (\Phi_\rho \times v^\rho)(x) V_\rho.$$

Clearly, θ_{ij}^ρ is continuous. To show that $\theta^\rho = [\theta_{ij}^\rho]_{i,j=1}^n$ is completely n-positive, it is sufficient to show that the map $S(\theta^\rho) : A \times_\alpha G \to M_n(L(H))$ defined by

$$S(\theta^\rho)(x) = [\theta_{ij}^\rho(x)]_{i,j=1}^n$$

is completely positive (see [9, Remark 2.1] and [4, Theorem 1.4]). Let $x_1, \cdots, x_m \in A \times_\alpha G$ and $(\xi_{1i})_{i=1}^n, \cdots, (\xi_{mi})_{i=1}^n \in \bigoplus_{i=1}^n H$. Then

$$\begin{aligned}
&\sum_{l,k=1}^m \langle S(\theta^\rho)(x_l^* x_k)(\xi_{ki})_{i=1}^n, (\xi_{li})_{i=1}^n \rangle \\
&= \sum_{l,k=1}^m \langle [\theta_{ij}^\rho(x_l^* x_k)]_{i,j=1}^n (\xi_{ki})_{i=1}^n, (\xi_{li})_{i=1}^n \rangle \\
&= \sum_{l,k=1}^m \sum_{i,j=1}^n \langle \theta_{ij}^\rho(x_l^* x_k)\xi_{ki}, \xi_{lj} \rangle \\
&= \sum_{l,k=1}^m \sum_{i,j=1}^n \langle V_\rho^* T_{ij}^\rho (\Phi_\rho \times v^\rho)(x_l^* x_k) V_\rho \xi_{ki}, \xi_{lj} \rangle \\
&= \sum_{l,k=1}^m \sum_{i,j=1}^n \langle T_{ij}^\rho (\Phi_\rho \times v^\rho)(x_k) V_\rho \xi_{ki}, (\Phi_\rho \times v^\rho)(x_l) V_\rho \xi_{lj} \rangle \\
&= \sum_{i,j=1}^n \Big\langle T_{ij}^\rho \sum_{k=1}^m (\Phi_\rho \times v^\rho)(x_k) V_\rho \xi_{ki}, \sum_{l=1}^m (\Phi_\rho \times v^\rho)(x_l) V_\rho \xi_{lj} \Big\rangle.
\end{aligned}$$

From this fact and taking into account that $[T_{ij}^\rho]_{i,j=1}^n$ is a positive element in $M_n(\Phi_\rho(A)')$, we conclude that θ^ρ is completely n-positive.

Let $i, j \in \{1, 2, \cdots, n\}$ and $f \in C_c(G, A)$. Then

$$\theta^\rho_{ij}(f) = V^*_\rho T^\rho_{ij}(\Phi_\rho \times v^\rho)(f)V_\rho = V^*_\rho T^\rho_{ij} \int_G \Phi_\rho(f(g))v^\rho_g V_\rho \mathrm{d}g$$
$$= \int_G V^*_\rho T^\rho_{ij} \Phi_\rho(f(g))V_\rho u_g \mathrm{d}g = \int_G \rho_{ij}(f(g))u_g \mathrm{d}g.$$

From this fact and taking into account that $C_c(G, A)$ is dense in $A \times_\alpha G$, we conclude that θ^ρ is unique such that

$$\theta^\rho_{ij}(f) = \int_G \rho_{ij}(f(g))u_g \mathrm{d}g$$

for all $f \in C_c(G, A)$ and for all $i, j \in \{1, 2, \cdots, n\}$.

Let $\{f_\delta\}_{\delta\in\Delta}$ be an approximate unit of $A \times_\alpha G$, $\{e_\lambda\}_{\lambda\in\Lambda}$ an approximate unit for A such that the nets $\{\rho_{ii}(e_\lambda)\}_{\lambda\in\Lambda}$, $i = 1, 2, \cdots, n$, converge strictly to the identity operator on H, and $\xi \in H$. Then

$$\lim_\delta \theta^\rho_{ii}(f_\delta)\xi = \lim_\delta V^*_\rho T^\rho_{ii}(\Phi_\rho \times v^\rho)(f_\delta)V_\rho\xi = V^*_\rho T^\rho_{ii} V_\rho\xi$$
$$= \lim_\lambda V^*_\rho T^\rho_{ii} \Phi_\rho(e_\lambda)V_\rho\xi = \lim_\lambda \rho_{ii}(e_\lambda)\xi = \xi$$

for all $i = 1, 2, \cdots, n$. Therefore, θ^ρ is nondegenerate.

Remark 3.2 Let (G, A, α) be a pro-C^*-dynamical system, let $\rho = [\rho_{ij}]^n_{i,j=1}$ be a u-covariant nondegenerate completely n-positive linear map from A to $L(H)$, and let $(\Phi_\rho, v^\rho, V_\rho, H_\rho, [T^\rho_{ij}]^n_{i,j=1})$ be the construction associated with ρ by Theorem 3.1. Then $\Phi_\rho \times v^\rho$ is a nondegenerate representation of $A \times_\alpha G$ on H_ρ (see [7]). Moreover, $[T^\rho_{ij}]^n_{i,j=1}$ is a positive element in $M_n((\Phi_\rho \times v^\rho)(A \times_\alpha G)')$ such that

$$\theta^\rho_{ij}(x) = V^*_\rho T^\rho_{ij}(\Phi_\rho \times v^\rho)(x)V_\rho$$

for all $x \in A \times_\alpha G$ and for all $i, j \in \{1, 2, \cdots, n\}$ and $\{(\Phi_\rho \times v^\rho)(x)V_\rho\xi; \xi \in H, x \in A \times_\alpha G\}$ spans a dense subspace of H_ρ, since

$$(\Phi_\rho \times v^\rho)(a \times f)V_\rho\xi = \int_G \Phi_\rho(af(g))v^\rho_g V_\rho\xi \mathrm{d}g = \Phi_\rho(a)V_\rho \int_G f(g)u_g\xi \mathrm{d}g$$

for all $a \in A$ for all $f \in C_c(G, A)$ and for all $\xi \in H$, and since $\{\Phi_\rho(a)V_\rho\xi; a \in A, \xi \in H\}$ spans a dense subspace of H_ρ. From these facts and Theorem 2.1, we conclude that there is a unitary operator $U : H_\rho \to H_{\theta^\rho}$ such that

(1) $(\Phi_\rho \times v^\rho)(x) = U\Phi_{\theta^\rho}(x)U^*$ for all $x \in A \times_\alpha G$;
(2) $V_\rho = UV_{\theta^\rho}$;
(3) $T^\rho_{ij} = UT^{\theta^\rho}_{ij}U^*$ for all $i, j \in \{1, 2, \cdots, n\}$.

Therefore the representation of $A \times_\alpha G$ induced by θ^ρ is unitarily equivalent to the representation $\Phi_\rho \times v^\rho$ induced by the covariant nondegenerate completely n-positive linear map $\rho = [\rho_{ij}]^n_{i,j=1}$.

Suppose that (G, A, τ) is a trivial pro-C^*-dynamical system (that is, $\tau_g = \mathrm{id}_A$ for all $g \in G$). Then the pro-C^*-algebras $A \times_\tau G$ and $A \otimes_{\max} C^*(G)$, where $C^*(G)$ is the universal C^*-algebra associated with G, are isomorphic (see [7, 2, 3, 12]). By Theorem 3.1, any u-covariant nondegenerate completely n-positive linear map $\rho = [\rho_{ij}]_{i,j=1}^n$ from A to $L(H)$ with respect to (G, A, τ) induces a nondegenrate covariant representation of (G, A, τ) and so it induces a nondegenerate representation of $A \otimes_{\max} C^*(G)$ on a Hilbert space K.

Corollary 3.1 *Let $\rho = [\rho_{ij}]_{i,j=1}^n$ be a u-covariant nondegenerate completely n-positive linear map from A to $L(H)$ with respect to the trivial pro-C^*-dynamical system (G, A, τ). Then, ρ induces a completely n-positive linear map $\theta^\rho = [\theta_{ij}^\rho]_{i,j=1}^n$ from $A \otimes_{\max} C^*(G)$ to $L(H)$. Moreover, the representation of $A \otimes_{\max} C^*(G)$ induced by ρ is unitarily equivalent to the representation induced by θ^ρ.*

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