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On Galois Extension of Hopf Algebras

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Abstract Let H be a cosemisimple Hopf algebra over a field k, and $\pi : A \to H$ be a surjective cocentral bialgebra homomorphism of bialgebras. The authors prove that if A is Galois over its coinvariants $B = \mathrm{LH}\ \mathrm{Ker}\,\pi$ and B is a sub-Hopf algebra of A, then A is itself a Hopf algebra. This generalizes a result of Cegarra [3] on group-graded algebras.

Keywords Hopf algebra, Galois extension
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1 Introduction

Let H be a bialgebra over a field k and (A, ρ) be a right H-comodule algebra. Then the coinvariant $A^{\mathrm{co}\,H}$ is defined as

$$A^{\mathrm{co}\,H} = \{a \in A \mid \rho(a) = a \otimes 1\}.$$

The algebra A is said to be a right H-Galois extension of its coinvariants $B = A^{\mathrm{co}\,H}$, if the Galois map $\beta : A \otimes_B A \to A \otimes H$, $a \otimes a' \mapsto \sum aa'_{\langle 0\rangle} \otimes a'_{\langle 1\rangle}$, is a bijection. This notion of Hopf-Galois extensions unifies the notions of classical Galois extensions over any field, strongly group-graded algebras, and affine algebraic principal homogeneous spaces.

The idea of using Galois extension to construct an antipode on a bialgebra is an old one. Schauenburg [9] proved that a bialgebra H is itself a Hopf algebra if H admits a Galois extension $A/A^{\mathrm{co}\,H}$. Another interesting result on this topic was given by Cegarra [3]. He showed that if A is a strongly group G-graded bialgebra over a field k such that each homogeneous component A_g of A is a sub-coalgebra of A, and specially A_1 is a finite-dimensional sub-Hopf algebra of A, then A is a Hopf algebra. His argument, without giving an explicit construction of the antipode, is tendinous and depends heavily on a theory of graded extensions (see [2]) of monoidal category.

Let us check Cegarra's assumption in detail. In his setting, A can be viewed as a natural right kG-comodule algebra via $\rho : A \to A \otimes kG$, $a_g \mapsto a_g \otimes g$ where $a_g \in A_g$. The coinvariant $A^{\mathrm{co}\,kG}$ is nothing else but A_1, and the condition that A is strongly G-graded is equivalent to the statement that A/A_1 is right kG-Galois. If we define $\pi : A \to kG$ via $\pi(a_g) = g$ $\forall a_g \in A_g$, then that each homogeneous component A_g is a sub-coalgebra of A implies that π is a Hopf algebra homomorphism. Moreover, the map $\pi : A \to kG$ satisfies an additional identity $\sum \pi(a_{(1)}) \otimes a_{(2)} = \sum \pi(a_{(2)}) \otimes a_{(1)}$ (termed cocentral in [1]) for any $a \in A$.

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In this note, we will generalize Cegarra's result to a slightly more general case. We prove that, for his special case, the antipode can be constructed explicitly as

$$S_A(a) = \sum_{i=1}^{n} \varepsilon_A(a_i) S_B(b_{i(1)}a) b_{i(2)},$$

where $a \in A_g$, $\sum\limits_{i=1}^{n} a_i \otimes b_i$ is chosen in $A_g \otimes_B A_{g^{-1}}$ and satisfies $\sum\limits_{i=1}^{n} a_i b_i = 1$.

2 The Main Theorem

Main Theorem *Let $(H, \Delta_H, \varepsilon_H, M_H, u_H, S_H)$ be a Hopf algebra over a field k and let $(A, \Delta_A, \varepsilon_A, M_A, u_A)$ be a bialgebra. Assume that $\pi : A \to H$ is a surjective cocentral bialgebra homomorphism. View A as a natural right H-comodule algebra via: $\rho : A \to A \otimes H$, sending $a \in A$ to $\sum a_{(1)} \otimes \pi(a_{(2)})$. Then*

(1) *$B = A^{\mathrm{co}\,H}$ is a sub-bialgebra of A,*

(2) *The extension $k \longrightarrow B \xrightarrow{\iota} A \xrightarrow{\pi} H \longrightarrow k$ is exact,*

(3) *Assume further that H is cosemisimple and A/B is Galois, then A is a Hopf algebra if and only if B is a Hopf algebra.*

Throughout this paper, k is a field. We use $\otimes$ to stand for $\otimes_k$. For a general theory of Hopf algebras, we refer to the standard books (see [8, 10]). We use Sweedler's "sigma" notation (see [10]): $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ for an element c in a coalgebra (C, Δ, ε), and $\rho(a) = \sum a_{\langle 0\rangle} \otimes a_{\langle 1\rangle}$ for an element a in a right C-comodule (A, ρ).

We first prove the first statement of the main theorem.

Lemma 2.1 *Let H, A, ρ be as in Main Theorem. Then $A^{\mathrm{co}\,H}$ is a sub-bialgebra of A.*

Proof Clearly $B = A^{\mathrm{co}\,H}$ is a subalgebra of A. It suffices to show that B is also a subcoalgebra of A.

For any $a \in A$,

$$\begin{aligned}(\rho\otimes\rho)\triangle_A(a) &= \sum a_{(1)} \otimes \pi(a_{(2)}) \otimes a_{(3)} \otimes \pi(a_{(4)})\\ &= \sum a_{(1)} \otimes \pi(a_{(3)}) \otimes a_{(2)} \otimes \pi(a_{(4)}) \quad (\pi \text{ is cocentral})\\ &= \sum (1\otimes\tau\otimes 1)(a_{(1)} \otimes a_{(2)} \otimes \pi(a_{(3)})_{(1)} \otimes \pi(a_{(3)})_2)\\ &= (1\otimes\tau\otimes 1)(\triangle_A \otimes \triangle_H)\rho(a),\end{aligned}$$

that is,

$$(\rho\otimes\rho)\triangle_A = (1\otimes\tau\otimes 1)(\triangle_A \otimes \triangle_H), \tag{2.1}$$

i.e., $\rho : A \to A \otimes H$ is a coalgebra homomorphism.

For any nonzero element $b \in B$, write $\Delta_A(b) = \sum\limits_{i=1}^{m} x_i \otimes y_i$ with m being chosen as small as possible. Then one easily check that $\{x_1, \cdots, x_m\}$ and $\{y_1, \cdots, y_m\}$ are linearly independent. For the finite set $\{x_1, \cdots, x_m, y_1, \cdots, y_m\}$, we have a finite-dimensional subspace L of H such that

$$\rho(\{x_1, \cdots, x_m, y_1, \cdots, y_m\}) \subseteq A \otimes L.$$

Take a basis $\{h_1 = 1, \cdots, h_n\}$ for L, and for each i $(1 \le i \le m)$ write

$$\rho(x_i) = \sum_{j=1}^{n} x_{ij} \otimes h_j \quad \text{and} \quad \rho(y_i) = \sum_{j=1}^{n} y_{ij} \otimes h_j.$$

Then

$$\begin{aligned}\sum_{i=1}^{m} \rho(x_i) \otimes \rho(y_i) &= (1 \otimes \tau \otimes 1)(\Delta_A \otimes \Delta_H)(b \otimes 1) \\ &= \sum_{i=1}^{m} x_i \otimes 1 \otimes y_i \otimes 1 \quad \text{by (2.1)},\end{aligned}$$

that is,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{n} x_{ij} \otimes h_j \otimes y_{il} \otimes h_l = \sum_{i=1}^{m} x_i \otimes h_1 \otimes y_i \otimes h_1.$$

By the linearly independence of $\{h_1, \cdots, h_n\}$,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \otimes h_j \otimes y_{il} = \delta_{l1} \sum_{i=1}^{m} x_i \otimes h_1 \otimes y_i,$$

and again,

$$\sum_{i=1}^{m} x_{ij} \otimes y_{il} = \delta_{j1} \delta_{l1} \sum_{i=1}^{m} x_i \otimes y_i \tag{2.2}$$

for any j, l $(1 \le j, l \le m)$. Taking $j = l = 1$, we have

$$\sum_{i=1}^{m} x_{i1} \otimes y_{i1} = \sum_{i=1}^{m} x_i \otimes y_i = \Delta_A(b).$$

By the minimality of m, we see that $\{x_{11}, \cdots, x_{m1}\}$ and $\{y_{11}, \cdots, y_{m1}\}$ are linearly independent as well.

Fix $j = 1$. For any l $(2 \le l \le m)$, by (2.2)

$$\sum_{i=1}^{m} x_{i1} \otimes y_{il} = \delta_{l1} \sum_{i=1}^{m} x_i \otimes y_i = 0.$$

Since $\{x_{11}, \cdots, x_{m1}\}$ is linearly independent, $y_{il} = 0$ for any i $(1 \le i \le m)$. Therefore

$$\rho(y_i) = \sum_{l=1}^{n} y_{il} \otimes h_l = y_{i1} \otimes h_1 = y_{i1} \otimes 1,$$

and by the counital property $y_{i1} = y_i$, i.e., $y_i \in B$ for all i $(1 \le i \le m)$. Similarly all $x_i \in B$.

Thus $\Delta_A(B) \subseteq B \otimes B$, and B is a subcoalgebra of A.

The second statement of Main Theorem can be checked by the definition of B.

For completing the proof of Main Theorem, we recall some properties of a cosemisimple Hopf algebra H.

A fundamental theorem (see [10, 14.0.3] and [8, 2.4.6]) is that H is cosemisimple if and only if there exists a two-sided integral $\lambda \in H^*$ such that $\langle \lambda, 1 \rangle \ne 0$. Moreover, if H is cosemisimple, then H has a bijective antipode (see [7]).

Lemma 2.2 (cf. [6]) *Let H be a cosemisimple Hopf algebra and $\lambda \in H^*$ be the integral with $\lambda(1) = 1$. Let A be a right H-comodule algebra. Then the map*

$$P : A \to A, \quad a \mapsto \sum a_{\langle 0\rangle}\lambda(a_{\langle 1\rangle})$$

is an $A^{\mathrm{co}\,H}$-$A^{\mathrm{co}\,H}$-bimodule projection of A onto $A^{\mathrm{co}\,H}$.

Proof It is clear that the map $\phi : H \to A$, $h \mapsto \lambda(h)1_A$ is a total integral. By the cosemisimplicity of H, $\lambda \circ S_H = \lambda$. So the given map P is the trace function in the sense of Doi [6], and the statement of the lemma is clear.

Lemma 2.3 (cf. [5]) *Let H be a Hopf algebra with bijective antipode S_H, A be a right H-comodule algebra, and $B = A^{\mathrm{co}\,H}$. Let*

$$\beta : A \otimes_B A \to A \otimes H,\ a \otimes a' \mapsto \sum aa'_{\langle 0\rangle} \otimes a'_{\langle 1\rangle},$$

$$\beta' : A \otimes_B A \to A \otimes H,\ a \otimes a' \mapsto \sum a_{\langle 0\rangle}a' \otimes a_{\langle 1\rangle}.$$

Then

(1) *β is injective if and only if β' is injective,*

(2) *β is surjective if and only if β' is surjective.*

Proof It is easily checked that $\beta' = (1 \otimes M_H) \circ (\rho \otimes S_H) \circ \beta$ and $\beta = (1 \otimes (M_H \circ \tau)) \circ (\rho \otimes S_H^{-1}) \circ \beta'$. The conclusion is obvious.

Proof of Main Theorem We only need to prove the third statement. The "only if" part is obvious, as one can see by the cocentralness of π that B is stable under S_A.

Now we prove the "if" part. Assume further that H is cosemisimple, A/B is Galois and that B is a Hopf algebra with the antipode S_B. Let $\lambda \in H^*$ be the integral of H^* such that $\lambda(1) = 1$, and $P : a \mapsto \sum a_{\langle 0\rangle}\lambda(a_{\langle 1\rangle}) = \sum a_{(1)}\lambda(\pi(a_{(2)}))$ be the projection onto B. Then, since π is cocentral,

$$\Delta(P(a)) = \Delta\Big(\sum a_{(1)}\lambda(\pi(a_{(2)}))\Big) = \sum \lambda(\pi(a_{(3)}))a_{(1)} \otimes a_{(2)} = \sum P(a_{(1)}) \otimes a_{(2)}.$$

Hence, we have

$$\sum P(a)_{(1)} \otimes P(a)_{(2)} = \sum P(a_{(1)}) \otimes a_{(2)} = \sum a_{(1)} \otimes P(a_{(2)}). \tag{2.3}$$

For any $c \in A$, define

$$\Phi_c : A \otimes_k A \to A \quad \text{via} \quad a \otimes a' \mapsto \sum \varepsilon_A(a)S_B(P(a'_{(1)}c))a'_{(2)}.$$

Then for any $b \in B$ and $a, a' \in A$,

$$\begin{aligned}
\Phi_c(a \otimes ba') &= \sum \varepsilon_A(a)S_B(P(b_{(1)}a'_{(1)}c))b_{(2)}a'_{(2)} \\
&= \sum \varepsilon_A(a)S_B(b_{(1)}P(a'_{(1)}c))b_{(2)}a'_{(2)} \quad (P \text{ is a } B\text{-}B\text{-bimodule map}) \\
&= \sum \varepsilon_A(a)S_B(P(a'_{(1)}c))S_B(b_{(1)})b_{(2)}a'_{(2)} \quad (S_B \text{ is the antipode}) \\
&= \sum \varepsilon_A(a)S_B(P(a'_{(1)}c))\varepsilon_A(b)a'_{(2)} \\
&= \Phi_c(ab \otimes a').
\end{aligned}$$

Thus Φ_c induces a homomorphism $\Phi : (A \otimes_B A) \otimes A \to A$ which assigns:

$$\Phi(a \otimes a' \otimes c) = \Phi_c(a \otimes a') = \sum \varepsilon_A(a) S_B(P(a'_{(1)}c))a'_{(2)}.$$

Define: $S^l : A \to A$,

$$S^l(a) = \sum \Phi(\beta^{-1}(1_A \otimes S_H^{-1}(a_{\langle 1 \rangle})) \otimes a_{\langle 0 \rangle}) = \sum \Phi(\beta^{-1}(1_A \otimes S_H^{-1}(\pi(a_{(2)}))) \otimes a_{(1)}).$$

Then by introducing the notation

$$\beta^{-1}(1_A \otimes S_H^{-1}(h)) = \sum_{(\beta,h)} a_i \otimes a'_i,$$

for simplification, we have

$$\sum_{(\beta,h)} a_i a'_{i(1)} \otimes \pi(a'_{i(2)}) = \sum_{(\beta,h)} a_i a'_{i\langle 0 \rangle} \otimes a'_{i\langle 1 \rangle} = \beta\Big(\sum_{(\beta,h)} a_i \otimes a'_i\Big) = 1_A \otimes S_H^{-1}(h).$$

By applying $\varepsilon_A \otimes 1$ to it,

$$\sum_{(\beta,h)} \varepsilon_A(a_i a'_{i(1)}) \pi(a'_{i(2)}) = S_H^{-1}(h) \tag{2.4}$$

for any $h \in H$.

Thus for any $a \in A$,

$$\begin{aligned}
\sum S^l(a_{(1)})a_{(2)} &= \sum \Phi(\beta^{-1}(1_A \otimes S_H^{-1}(\pi(a_{(2)}))) \otimes a_{(1)})a_{(3)} \\
&= \sum_{(\beta,\pi(a_{(2)}))} \Phi(a_i \otimes a'_i \otimes a_{(1)})a_{(3)} \\
&= \sum_{(\beta,\pi(a_{(2)}))} \varepsilon_A(a_i) S_B(P(a'_{i(1)}a_{(1)}))a'_{i(2)}a_{(3)} \\
&= \sum_{(\beta,\pi(a_{(1)}))} \varepsilon_A(a_i) S_B(P(a'_{i(1)}a_{(2)}))a'_{i(2)}a_{(3)} \quad (\pi \text{ is cocentral}) \\
&= \sum_{(\beta,\pi(a_{(1)}))} \varepsilon_A(a_i) S_B(P(a'_i a_{(2)})_{(1)}) P(a'_i a_{(2)})_{(2)} \quad (\text{by } (2.3)) \\
&= \sum_{(\beta,\pi(a_{(1)}))} \varepsilon_A(a_i) \varepsilon_A(P(a'_i a_{(2)}))1_A \\
&= \sum_{(\beta,\pi(a_{(1)}))} \varepsilon_A(a_i) \varepsilon_A(a'_{i(1)}a_{(2)}) \lambda(\pi(a'_{i(2)}a_{(3)}))1_A \\
&= \sum_{(\beta,\pi(a_{(2)}))} \varepsilon_A(a_i a'_{i(1)}) \varepsilon_A(a_{(1)}) \lambda(\pi(a'_{i(2)}a_{(3)}))1_A \\
&= \sum_{(\beta,\pi(a_{(2)}))} \varepsilon_A(a_{(1)}) \lambda(\varepsilon_A(a_i a'_{i(1)}) \pi(a'_{i(2)}) \pi(a_{(3)}))1_A \\
&= \sum_{(\beta,\pi(a_{(3)}))} \varepsilon_A(a_{(1)}) \lambda(\varepsilon_A(a_i a'_{i(1)}) \pi(a'_{i(2)}) \pi(a_{(2)}))1_A \\
&= \sum \varepsilon_A(a_{(1)}) \lambda[S_H^{-1}(\pi(a_{(3)}))\pi(a_{(2)})]1_A \quad (\text{by } (2.4)) \\
&= \varepsilon_A(a)\lambda(1)1_A \\
&= \varepsilon_A(a)1_A;
\end{aligned}$$

that is, $S^l * \mathrm{Id}_A = \varepsilon_A 1_A$, S^l is a left convolution-inverse of Id_A. Similarly, we have a well-defined homomorphism

$$\Psi : (A \otimes_B A) \otimes A \to A \quad \text{via} \quad a \otimes a' \otimes c \mapsto \sum a_{(1)} S_B(P(ca_{(2)}))\varepsilon_A(a'),$$

and the map $S^r : A \to A$,

$$S^r(a) = \sum \Psi(\beta'^{-1}(1 \otimes S_H(a_{\langle 1\rangle})) \otimes a_{\langle 0\rangle}),$$

is a right convolution-inverse of Id_A. Then $S^l = S^r$ is the required antipode of A.

Remark 2.1 Back to Cegarra's case, for any $a \in A_g$, $\beta^{-1}(1_A \otimes g^{-1})$ can be chosen as an element $\sum\limits_{i=1}^{n} a_i \otimes b_i \in A_g \otimes_B A_{g^{-1}}$ such that $\sum\limits_{i=1}^{n} a_i b_i = 1$. Thus as in the proof, the antipode on A is

$$S_A(a) = S^l(a) = \sum_{i=1}^{n} \varepsilon_A(a_i) S_B(b_{i(1)} a) b_{i(2)}.$$

Remark 2.2 The surjectivity and cocentralness of π in Main Theorem are sufficient for the Hopf algebra H being cocommutative.

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