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Ordering Trees with Nearly Perfect Matchings by Algebraic Connectivity

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Abstract Let $\mathcal{T}_{2k+1}$ be the set of trees on $2k+1$ vertices with nearly perfect matchings and $\alpha(T)$ be the algebraic connectivity of a tree T. The authors determine the largest twelve values of the algebraic connectivity of the trees in $\mathcal{T}_{2k+1}$. Specifically, 10 trees $T_2, T_3, \cdots, T_{11}$ and two classes of trees $T(1)$ and $T(12)$ in $\mathcal{T}_{2k+1}$ are introduced. It is shown in this paper that for each tree $T_1', T_1'' \in T(1)$ and $T_{12}', T_{12}'' \in T(12)$ and each i, j with $2 \le i < j \le 11$, $\alpha(T_1') = \alpha(T_1'') > \alpha(T_i) > \alpha(T_j) > \alpha(T_{12}') = \alpha(T_{12}'')$. It is also shown that for each tree T with $T \in \mathcal{T}_{2k+1} \setminus (T(1) \cup \{T_2, T_3, \cdots, T_{11}\} \cup T(12))$, $\alpha(T_{12}') > \alpha(T)$.

Keywords Laplacian eigenvalue, Tree, Nearly perfect matching, Algebraic connectivity
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1 Introduction

Unless stated otherwise, all graphs in this paper are finite, undirected and simple. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{e_1, e_2, \cdots, e_m\}$. Denote the order of G by $|G|$. We will abuse the language by writing $v \in G$ and $uv \in G$, rather than $v \in V$ and $uv \in E$, to indicate that v is a vertex of G and uv is an edge of G, respectively. Denote the degree of a vertex v_i by $d(v_i)$. The Laplacian matrix $L(G) = D(G) - A(G)$ is the difference of $D(G) = \mathrm{diag}(d(v_1), d(v_2), \cdots, d(v_n))$ and the adjacency matrix $A(G)$ of G. It is well known that $L(G)$ is positive semidefinite symmetric and singular. Denote its eigenvalues by

$$\mu_1(G) \ge \mu_2(G) \ge \cdots \ge \mu_{n-1}(G) \ge \mu_n(G) = 0,$$

$\mu_1(G)$ is called the Laplacian spectral radius of G, and $\mu_s(G)$ is called the s-th Laplacian eigenvalue of the graph G.

From the well-known Matrix-Tree Theorem, we deduce that $\mu_{n-1}(G) > 0$ if and only if G is connected. This observation led M. Fiedler to think of $\mu_{n-1}(G)$ as a quantitative measure of connectivity (cf. [2]) and thus $\mu_{n-1}(G)$ is called the algebraic connectivity of G, denoted by $\alpha(G)$. And if X is a unit eigenvector of G corresponding to $\alpha(G)$, we commonly call it a Fiedler vector of G. Let $\xi(G)$ be the set of all the Fiedler vectors of G throughout this paper.

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Let $X \in \xi(G)$ and $X(v)$ denote the coordinate of X corresponding to the vertex v. It is obvious that $X^T e = 0$, where $e = (1, 1, \cdots, 1)^T$ is an n dimensional column vector, and

$$\alpha(G) = X^T L(G) X = \sum_{v_i v_j \in E} (X(v_i) - X(v_j))^2 = \min_{\substack{Y \in R^n \setminus \{0\} \\ Y^T e = 0}} \frac{Y^T L(G) Y}{Y^T Y}.$$

Throughout this paper, we shall denote by $\Phi(B) = \Phi(B, x) = \det(xI - B)$ the characteristic polynomial of a square matrix B. If $v \in G$, let $L_v(G)$ be the principal submatrix of $L(G)$ obtained by deleting the row and column corresponding to the vertex v. We denote by $\tau(M)$ the smallest eigenvalue of a real symmetric matrix M. Let P_n denote a path of order n and $K_{1,n-1}$ denote a star of order n.

Two distinct edges in a graph G are independent if they are not incident with a common vertex in G. A set of pairwisely independent edges of G is called a matching of G. A matching of maximum cardinality is called a maximum matching of G. The cardinality of a maximum matching of G is called the matching number of G. A matching M of G is called a nearly perfect matching of G if it satisfies $2|M| = |V(G)| - 1$. For a fixed matching M, an edge which is in M is called a matched edge, and is called a free edge otherwise. A vertex on some matched edge of M is called a matched vertex, and is called a free vertex otherwise.

Let k be an integer and $k \geq 12$ throughout this paper.

Let $\mathcal{T}_{2k+1}$ be the set of trees on $2k+1$ vertices with nearly perfect matchings. In this paper, we determine the largest twelve values of the algebraic connectivity of the trees in $\mathcal{T}_{2k+1}$. Specifically, we introduce 10 trees $T_2, T_3, \cdots, T_{11}$ and two classes of trees $T(1)$ and $T(12)$ in $\mathcal{T}_{2k+1}$. We show in this paper that for each tree T_1', $T_1'' \in T(1)$ and T_{12}', $T_{12}'' \in T(12)$, we have $\alpha(T_1') = \alpha(T_1'') > \alpha(T_2) > \cdots > \alpha(T_{11}) > \alpha(T_{12}') = \alpha(T_{12}'')$. Also, we show that for each tree T with $T \in \mathcal{T}_{2k+1} \setminus (T(1) \cup \{T_2, T_3, \cdots, T_{11}\} \cup T(12))$, we have $\alpha(T_{12}') > \alpha(T)$.

2 Preliminaries

Let G be a graph and E_1 be a subset of $E(G)$ with $|E_1| = t$. Let G' be the spanning subgraph of G obtained from G by deleting all the edges in E_1. It follows by the well-known Courant-Weyl inequalities (cf. [1]) that the following is true.

Lemma 2.1 (cf. [1]) *The Laplacian eigenvalues of G and G' interlace, that is,*

$$\mu_i(G) \geq \mu_i(G') \geq \mu_{i+t}(G), \quad i = 1, 2, \cdots, n-t.$$

Corollary 2.1 *If T' is a subtree of a tree T, then $\alpha(T') \geq \alpha(T)$.*

Proof Let $|V(T)| = n$, $|V(T')| = n'$ and write $t = n - n'$. Then we have

$$|E(T)| - |E(T')| = |V(T)| - |V(T')| = n - n' = t.$$

Let $E_1 = E(T) \backslash E(T')$ and $T^* = T - E_1$. Then $|E_1| = |E(T)| - |E(T')| = t$ and $T^* = T' \cup (n-n')K_1$ is a spanning subgraph of T, where K_1 is an isolated vertex. Then from Lemma 2.1 we have

$$\mu_{n'-1}(T') = \mu_{n'-1}(T^*) \geq \mu_{n'-1+t}(T) = \mu_{n-1}(T).$$

So $\alpha(T') \geq \alpha(T)$ holds.

Lemma 2.2 (cf. [1]) *Let A be a Hermitian matrix of order n with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and B have a principal submatrix of order m. Let B be eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, 2, \cdots, m)$ hold.*

Lemma 2.3 (cf. [2, 7]) *Let T be a tree. Then $\alpha(T) \leq 1$ and equality holds if and only if T is a star.*

Lemma 2.4 (cf. [9]) *Let v_0, v_1, v_2 be vertices of a tree T with $d(v_1) = d(v_2) = 1$ and $v_0 v_i \in E(T)$ $(i = 1, 2)$. Let $T' = T - v_0 v_1 + v_1 v_2$ and $X \in \xi(T)$. If $X(v_0) \neq 0$, then $\alpha(T') < \alpha(T)$.*

Lemma 2.5 (cf. [3]) *Let T be a tree of order n with the vertex set $V = \{v_1, \cdots, v_n\}$. Suppose $X \in \xi(T)$. Then two cases can occur.*

Case I *If $\widetilde{V} = \{v_i \in V \mid X(v_i) = 0\} \neq \emptyset$, then the graph $\widetilde{T} = \{\widetilde{V}, \widetilde{E}\}$ induced by T on $\widetilde{V}$ is connected and there is exactly one vertex $v_j \in \widetilde{V}$ which is adjacent (in T) to a vertex not belonging to $\widetilde{V}$. Moreover, the values of X along any path starting at v_j are increasing, decreasing, or identically zero.*

Case II *If $X(v_i) \neq 0$ for all $v_i \in V$, then T contains exactly one edge $v_s v_t$ such that v_s and v_t have different signs, say $X(v_s) > 0$ and $X(v_t) < 0$. Moreover, the values of X along any path that starts at v_s and does not contain v_t increase while the values of X along any path that starts at v_t and does not contain v_s decrease.*

We refer to a tree in which Case I (resp. Case II) occurs as a type I (resp. type II) tree. In Case I, the vertex v_j is called the characteristic vertex (cf. [7]) of T; in Case II, the edge $v_s v_t$ is called the characteristic edge (cf. [8]) of T.

T is a tree, $v \in T$, and $T - v$ denotes the graph obtained by deleting v and all edges incident with it. A branch of T at v is a connected component of $T - v$. If B is a branch of T at v, we denote by $r(B)$ the vertex of B which is adjacent (in T) to v. We view B as a root tree and $r(B)$ as the root (cf. [10]).

Suppose that T is a rooted tree with vertex set $\{u_1, \cdots, u_m\}$. Denote by $\widehat{L}(T) = (b_{ij})$ the m-by-m matrix where

$$b_{ij} = \begin{cases} d(u_i) + 1, & \text{if } i = j \text{ and } u_i \text{ is the root,} \\ d(u_i), & \text{if } i = j \text{ and } u_i \text{ is not the root,} \\ -1, & \text{if } u_i u_j \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that all the branches of T at v are $B_1, \cdots, B_s$. We say that a branch B_j at v is a Perron branch at v if $\tau(\widehat{L}(B_j)) = \min\{\tau(\widehat{L}(B_i)) \mid i = 1, \cdots, s\}$ (cf. [5]).

Lemma 2.6 (cf. [5]) *A tree T is a type* I *tree if and only if there is a vertex v of T at which there are two or more Perron branches. In that instance, v is the characteristic vertex and $\alpha(T) = \tau(\widehat{L}(B))$, where B is the Perron branch of T at v.*

Lemma 2.7 (cf. [5]) *A tree T is a type* II *tree with the characteristic edge $v_i v_j$ if and only if the branch at v_i containing v_j is the unique Perron branch at v_i, while the branch at v_j containing v_i is the unique Perron branch at v_j.*

Lemma 2.8 (cf. [4]) *Let T be a tree with vertex set $\{v_1, \cdots, v_n\}$. Suppose $X \in \xi(T)$. If $X(v_i) = 0$, then either for any vertex $v \in N(v_i)$, $X(v) = 0$, or there are at least two vertices in $N(v_i)$ corresponding to nonzero coordinates in X.*

Lemma 2.9 (cf. [6]) *Let G be a connected graph. Let W be a set of vertices of G such that $G - W$ is disconnected. Let G_1, G_2 be two components of $G - W$ and let L_1 and L_2 be the principal submatrices of $L(G)$ corresponding to G_1 and G_2, respectively. If $\tau(L_1) < \tau(L_2)$, then $\alpha(G) < \tau(L_2)$.*

3 Classifying Trees in $\mathcal{T}_{2k+1}$ by Diameter

Let a, b, c, γ denote the smallest roots of the following equations

$$x^2 - 3x + 1 = 0, \tag{3.1}$$
$$x^3 - 5x^2 + 6x - 1 = 0, \tag{3.2}$$
$$x^4 - 7x^3 + 14x^2 - 8x + 1 = 0, \tag{3.3}$$
$$x^3 - 5x^2 + 5x - 1 = 0, \tag{3.4}$$

respectively.

By direct computations we can get

$$0.3819 < a < 0.3820, \quad 0.1980 < b < 0.1981, \quad 0.1729 < c < 0.1730, \quad 0.2679 < \gamma < 0.2680.$$

Let P_i^1 denote the rooted tree which is a path P_i with the root at a pendant vertex of P_i, and P_i^2 $(i = 3, 4)$ denote the rooted tree, which is a path P_i with the root at a non-pendant vertex of P_i. Specifically, we let P_1^0 denote the rooted tree which is an isolated vertex P_1 with the root at the only vertex of P_1.

Lemma 3.1 *Let $P_1^0, P_2^1, P_3^1, P_3^2$ and P_4^2 be rooted trees as defined above. Then*

$$\tau(\widehat{L}(P_1^0)) = 1 > \tau(\widehat{L}(P_2^1)) = a > \tau(\widehat{L}(P_3^2)) = \gamma > \tau(\widehat{L}(P_3^1)) = b > \tau(\widehat{L}(P_4^2)) = c.$$

Lemma 3.2 *For any tree T with $d(T) \geq 7$, we have $\alpha(T) < c$.*

Proof Any tree T with $d(T) \geq 7$ must contain P_8 as a subtree. By direct computations and Corollary 2.1, we have

$$\alpha(T) \leq \alpha(P_8) < 0.1523 < c.$$

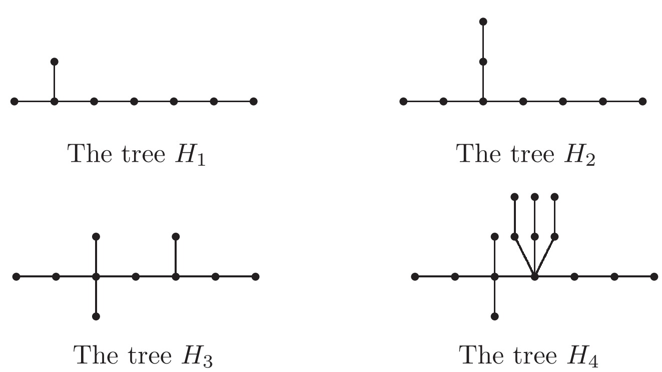


Figure 1

Lemma 3.3 *For any tree T containing one of H_1, H_2, H_3, H_4 (cf. Figure 1) as a subtree, we have $\alpha(T) < c$.*

Proof By direct computations, we get

$$\alpha(H_1) < 0.1668, \quad \alpha(H_2) < 0.1709, \quad \alpha(H_3) < 0.1627, \quad \alpha(H_4) < 0.1729.$$

So by Corollary 2.1, we have

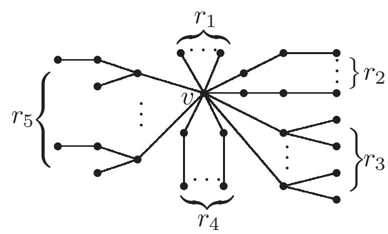
$$\alpha(T) \le \max\{\alpha(H_1),\ \alpha(H_2),\ \alpha(H_3),\ \alpha(H_4)\} < 0.1729 < c.$$

In the following we give two definitions.

Definition 3.1 *Let $r_1,\ r_2,\ \cdots,\ r_5$ be nonnegative integers. Let $G(r_1, r_2, r_3, r_4, r_5)$ (cf. Figure 2) be the tree which contains a vertex v such that*

$$G(r_1, r_2, r_3, r_4, r_5) - v = r_1 P_1^0 \ \dot{\bigcup}\ r_2 P_3^1 \ \dot{\bigcup}\ r_3 P_3^2 \ \dot{\bigcup}\ r_4 P_2^1 \ \dot{\bigcup}\ r_5 P_4^2,$$

where $P_1^0,\ P_2^1,\ P_3^1,\ P_3^2,\ P_4^2$ are viewed as rooted trees as defined above.



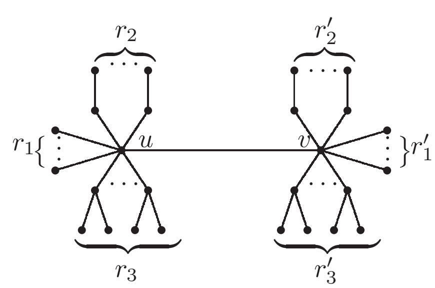
The tree $G(r_1, r_2, r_3, r_4, r_5)$

Figure 2

Definition 3.2 *Let $r_1,\ r_2,\ r_3,\ r_1',\ r_2',\ r_3'$ be nonnegative integers, and $F(r_1, r_2, r_3, r_1', r_2', r_3')$ (cf. Figure 3) be the tree which contains an edge uv such that*

$$F(r_1, r_2, r_3, r_1', r_2', r_3') - \{u, v\} = r_1 P_1^0 \ \dot{\bigcup}\ r_2 P_2^1 \ \dot{\bigcup}\ r_3 P_3^2 \ \dot{\bigcup}\ r_1' P_1^0 \ \dot{\bigcup}\ r_2' P_2^1 \ \dot{\bigcup}\ r_3' P_3^2,$$

where $r_1 P_1^0,\ r_2 P_2^1$ and $r_3 P_3^2$ are branches (viewed as rooted trees) of $F(r_1, r_2, r_3, r_1', r_2', r_3')$ at u not containing v and $r_1' P_1^0,\ r_2' P_2^1$ and $r_3' P_3^2$ are branches (viewed as rooted trees) of $F(r_1, r_2, r_3, r_1', r_2', r_3')$ at v not containing u.



The tree $F(r_1, r_2, r_3, r_1', r_2', r_3')$

Figure 3

Note that $F(r_1', r_2', r_3', r_1, r_2, r_3) \cong F(r_1, r_2, r_3, r_1', r_2', r_3')$.

Now let $T(1)$ be the following set of two trees:

$$T(1) = \{G(2,0,0,k-1,0),\ G(0,0,0,k,0)\},$$

and let

$$\begin{aligned} T_2 &= G(1,0,1,k-2,0), & T_3 &= F(1,k-2,0,0,1,0),\\ T_4 &= F(0,k-3,1,0,1,0), & T_5 &= G(0,2,0,k-3,0),\\ T_6 &= F(2,k-3,0,1,1,0), & T_7 &= F(1,1,0,0,k-2,0),\\ T_8 &= F(1,k-4,1,1,1,0), & T_9 &= G(1,1,0,k-4,1),\\ T_{10} &= G(0,1,1,k-5,1), & T_{11} &= G(0,2,0,k-5,1). \end{aligned}$$

Let $T(12)$ be the following set of trees:

$$T(12) = \{T \in \mathcal{T}_{2k+1} \mid T = G(r_1,r_2,r_3,r_4,r_5) \text{ with } r_5 \geq 2\}.$$

Let

$$\mathcal{F}_1 = \{T \mid T = F(1,r_2,0,0,r_2',0) \text{ with } r_2 \geq 1, r_2' \geq 1 \text{ and } r_2 + r_2' = k-1\},$$

$$\mathcal{F}_2 = \{T \mid T = F(2,r_2,0,1,r_2',0) \text{ with } r_2 \geq 1, r_2' \geq 1 \text{ and } r_2 + r_2' = k-2\},$$

$$\mathcal{F}_3 = \{T \mid T = F(0,r_2,1,0,r_2',0) \text{ with } r_2 \geq 0, r_2' \geq 1 \text{ and } r_2 + r_2' = k-2\},$$

$$\mathcal{F}_4 = \{T \mid T = F(1,r_2,1,1,r_2',0) \text{ with } r_2 \geq 0, r_2' \geq 1 \text{ and } r_2 + r_2' = k-3\}.$$

And let

$$F_1 = F(1,2,0,0,4,0),\quad F_2 = F(1,5,0,0,2,0),\quad F_3 = F(2,1,0,1,9,0),\quad F_4 = F(0,0,1,0,4,0).$$

Lemma 3.4 *For any tree T containing one of $F_1,\cdots,F_4$ as a subtree, we have $\alpha(T) < c$.*

Proof By direct computations, we have

$$\alpha(F_1) < 0.1726,\quad \alpha(F_2) < 0.1727,\quad \alpha(F_3) < 0.1724,\quad \alpha(F_4) < 0.1727.$$

By Corollary 2.1, we have

$$\alpha(T) \leq \max_{1\leq i\leq 4} \alpha(F_i) < 0.1727 < c.$$

Let Q_5 denote the tree of order 5, which is obtained from a star $K_{1,3}$ by joining one of its degree 1 vertices to a new vertex by an edge, and let Q_5^3 denote the rooted tree, which is Q_5 with the root at the vertex of degree 3.

Lemma 3.5 *The following two sets of conditions for a tree T are equivalent:*

(1) T satisfies the following three conditions:

(1.1) $T \in \mathcal{T}_{2k+1}$ $(k \geq 12)$,

(1.2) $d(T) = 6$,

(1.3) T contains none of H_1, H_2, H_3, H_4 as a subtree;

(2) $T = G(r_1,r_2,r_3,r_4,r_5)$ for nonnegative integers r_1, r_2, r_3, r_4 and r_5 satisfying the following three conditions:

(2.1) $r_3 \leq 1$, $r_1 + r_2 + r_3 = 0$ or 2,

(2.2) $r_2 + r_5 \geq 2$,

(2.3) $1 + r_1 + 3r_2 + 3r_3 + 2r_4 + 4r_5 = 2k + 1$.

Proof (2) $\Rightarrow$ (1) Let $\mathcal{G} = \{T \mid T$ satisfies Lemma 3.5(2)$\}$. It is not difficult to get $\mathcal{G} = \{T_5, T_9, T_{10}, T_{11}\} \cup T(12)$ and every tree in $\{T_5, T_9, T_{10}, T_{11}\} \cup T(12)$ satisfies Lemma 3.5(1).

(1) $\Rightarrow$ (2) $T \in \mathcal{T}_{2k+1}$, so for any vertex $u \in T$ there are at most $2P_1^0$ at u.

Since $d(T) = 6$, there exists a vertex v of T such that for any vertex $u \in T$ the distance between v and u is at most 3.

Let $W = \{w \in T \mid$ the distance between v and w is $2\}$. Since T does not contain H_1 as a subtree, for every vertex $w \in W$, we have $d(w) \leq 2$. At the same time, T does not contain H_2 as a subtree, so the branches of T at v does not contain P_5 as a subtree. Therefore all the possible branches of T at v are $r_1P_1^0,\ r_4P_2^1,\ r_2P_3^1,\ r_3P_3^2,\ r_5P_4^2,\ xQ_5^3$. From $T \in \mathcal{T}_{2k+1}$ we get $r_1 \leq 2,\ r_2 \leq 2,\ r_3 \leq 1,\ x \leq 1$.

We now prove $x = 0$.

Suppose $x = 1$. If $r_5 \geq 1$, then T contains H_3 as a subtree, a contradiction. If $r_5 = 0$, in this case $r_4 > 3$ since $k \geq 12,\ r_1 \leq 2,\ r_2 \leq 2,\ r_3 \leq 1$, so T contains H_4 as a subtree, a contradiction.

We now get $T = G(r_1, r_2, r_3, r_4, r_5)$ with $r_3 \leq 1$.

It is easy to get (2.2) and (2.3). We now only need to prove $r_1 + r_2 + r_3 = 0$ or 2.

It is not difficult to get $r_1 + r_2 + r_3$ to be an even number. For any matching M of T, there are at least $r_1 + r_2 + r_3 - 1$ free vertices in $r_1P_1^0 \cup r_2P_3^1 \cup r_3P_3^2$. But $T \in \mathcal{T}_{2k+1}$ means that for any a nearly perfect matching of T, there is only one free vertex, so we have $r_1 + r_2 + r_3 - 1 \leq 1$. Thus we get $r_1 + r_2 + r_3 \leq 2$.

So $r_1 + r_2 + r_3 = 0$ or 2.

Lemma 3.6 *The following two sets of conditions for a tree T are equivalent:*

(1) *T satisfies the following two conditions:*

(1.1) $T \in \mathcal{T}_{2k+1}$,

(1.2) $d(T) = 4$;

(2) $T = G(r_1, 0, r_3, r_4, 0)$ *for nonnegative integers r_1, r_3, r_4 satisfying the following conditions:*

(2.1) $r_3 \leq 1,\ \ r_1 + r_3 = 0$ *or* 2,

(2.2) $r_3 + r_4 \geq 2$,

(2.3) $1 + r_1 + 3r_3 + 2r_4 = 2k + 1$.

Proof (2) $\Rightarrow$ (1) Let $\mathcal{G}' = \{T \mid T$ satisfies Lemma 3.6(2)$\}$. It is not difficult to get $\mathcal{G}' = T(1) \cup \{T_2\}$. For any tree T in $T(1) \cup \{T_2\}$, we can easily see that T satisfies Lemma 3.6(1).

Using the methods similar to those used in proving Lemma 3.5, we can prove (1) $\Rightarrow$ (2).

Lemma 3.7 *The following two sets of conditions for a tree T are equivalent:*

(1) *T satisfies the following two conditions:*

(1.1) $T \in \mathcal{T}_{2k+1}$,

(1.2) $d(T) = 5$;

(2) $T = F(r_1, r_2, r_3, r_1', r_2', r_3')$ *for nonnegative integers r_1, r_2, r_3, r_1', r_2' and r_3' satisfying the following three conditions:*

(2.1) $r_3 + r_3' \leq 1,\ \ r_3 + r_3' + \max\{r_1, r_1'\} \leq 2,\ \ r_1 + r_3 + r_1' + r_3' = 1$ *or* 3,

(2.2) $r_2 + r_3 \geq 1,\ r_2' + r_3' \geq 1$,

(2.3) $2 + r_1 + 2r_2 + 3r_3 + r_1' + 2r_2' + 3r_3' = 2k + 1$.

Proof (2) ⇒ (1) Let $\mathcal{F} = \{T \mid T$ satisfies Lemma 3.7(2)$\}$. It is easy to get $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. For each tree T in $\big(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4\big)$, we can easily prove that T satisfies Lemma 3.7(1).

(1) ⇒ (2) $T \in \mathcal{T}_{2k+1}$, so for any vertex $u \in T$ there are at most $2P_1^0$ at u.

Since $d(T) = 5$, there exists an edge $e = uv$ of T such that $T - e = G_1 \dot{\bigcup} G_2$ with $u \in G_1$, $v \in G_2$ and the distance between any vertex of G_1 (resp. G_2) and u (resp. v) is at most 2. So $T = F(r_1, r_2, r_3, r_1', r_2', r_3')$.

It is easy to get (2.2) and (2.3). From $T \in \mathcal{T}_{2k+1}$, we get $r_3+r_3' \leq 1$, $r_3+r_3'+\max\{r_1, r_1'\} \leq 2$. We now only need to prove $r_1 + r_3 + r_1' + r_3' = 1$ or 3.

It is easy to know that $r_1 + r_3 + r_1' + r_3'$ is an odd number. For any matching M of T, there are at least $r_1 + r_3 + r_1' + r_3' - 2$ free vertices in $r_1P_1^0 \cup r_3P_3^2 \cup r_1'P_1^0 \cup r_3'P_3^2$. But $T \in \mathcal{T}_{2k+1}$ means that for any a nearly perfect matching of T, there is only one free vertex, so we have $r_1 + r_3 + r_1' + r_3' - 2 \leq 1$. Thus we get $r_1 + r_3 + r_1' + r_3' \leq 3$.

So $r_1 + r_3 + r_1' + r_3' = 1$ or 3.

By the structures of trees, we can find that each tree in $(\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4) \backslash \{T_3, T_4,\ T_6,\ T_7,\ T_8\}$ must contain one of $F_1, \cdots, F_4$ as a subtree. So we get

$$\begin{aligned} &\{T_3,\ T_4,\ T_6,\ T_7,\ T_8\} \\ &= \{T \mid T \in \mathcal{T}_{2k+1} \text{ with } d(T) = 5 \text{ and containing none of } F_1, \cdots, F_4 \text{ as a subtree}\}. \end{aligned}$$

Write $T(i) = \{T_i\}$ for $i = 2, \cdots, 11$. Then we get twelve classes of trees $T(1), \cdots, T(12)$. By Lemmas 3.5–3.7, we know that all the trees of $\bigcup\limits_{i=1}^{12} T(i)$ have nearly perfect matchings.

Since $k \geq 12$, for any tree $T \in \mathcal{T}_{2k+1}$ we have $d(T) \geq 4$.

By the above analysis and Lemmas 3.2–3.7, we can get the main result of the section.

Theorem 3.1 *For any tree $T \in \mathcal{T}_{2k+1} \setminus \Big(\bigcup\limits_{i=1}^{12} T(i)\Big)$, we have $\alpha(T) < c$.*

4 Ordering Trees of $T(1)$–$T(12)$ by Algebraic Connectivity

Theorem 4.1 (i) *For any tree $T \in T(12)$, we have $\alpha(T) = c$;*

(ii) $\alpha(T_5) = b$;

(iii) *For each tree $T \in T(1)$, we have $\alpha(T) = a$.*

Proof (i) The branches of T ($\in T(12)$) at v are $r_1P_1^0$, $r_2P_3^1$, $r_3P_3^2$, $r_4P_2^1$, $r_5P_4^2$ (see Figure 2). By Lemma 3.1,

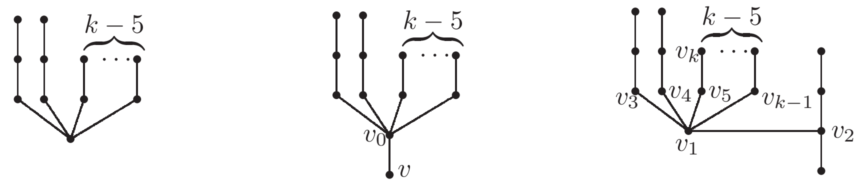
$$\tau(\widehat{L}(P_4^2)) < \min\{\tau(\widehat{L}(P_3^1)),\ \tau(\widehat{L}(P_3^2)),\ \tau(\widehat{L}(P_2^1)),\ \tau(\widehat{L}(P_1^0))\},$$

so P_4^2 is the Perron branch of T at v, since $r_5 \geq 2$. By Lemma 2.6, we have

$$\alpha(T) = \tau(\widehat{L}(P_4^2)) = c.$$

Using the similar methods we can get (ii) and (iii).

Lemma 4.1 *For $H_5^+ = G(1, 2, 0, k-5, 0)$, we have $\tau(L_v(H_5^+)) < a$ (cf. Figure 4).*



The tree H_5 The tree H_5^+ The tree T_{11}

$H_5 = G(0,2,0,k-5,0)$ $H_5^+ = G(1,2,0,k-5,0)$ $T_{11} = G(0,2,0,k-5,1)$

Figure 4

Proof The branches of H_5^+ at v_0 are P_1^0, $2P_3^1$, $(k-5)P_2^1$. By Lemma 3.1,

$$\tau(\widehat{L}(P_3^1)) < \min\{\tau(\widehat{L}(P_1^0)), \tau(\widehat{L}(P_2^1))\},$$

so P_3^1 is the Perron branch of H_5^+ at v_0. By Lemmas 2.6 and 2.2, we have

$$\tau(L_v(H_5^+)) \le \alpha(H_5^+) = \tau(\widehat{L}(P_3^1)) = b < a.$$

Lemma 4.2 *Let T be a type* II *tree of order n with characteristic edge $e = v_1v_2$. v_i, v_j and v_s are vertices of T such that v_1 is on the path from v_2 to v_j, v_s is not on the path from v_2 to v_j and the path from v_1 to v_s contains neither v_2 nor v_j, v_i is on the path from v_1 to v_s (including $v_i = v_1$) and $v_iv_s \in E(T)$. Let $T' = T - v_iv_s + v_jv_s$, $X \in \xi(T)$. We have*

(1) *If $|X(v_j)| > |X(v_i)|$, then $\alpha(T') < \alpha(T)$;*

(2) *If $X(v_j) = X(v_i)$, then $\alpha(T') \le \alpha(T)$.*

Proof By Lemma 2.5, we can suppose $X(v_1) > 0$, so $X(v_s) > X(v_i) > 0$, and $X(v_j) > 0$.

(1) If $X(v_j) > X(v_i)$, the proof is similar to [8, Theorem 7].

$$\begin{aligned} X^T L(T')X &= X^T L(T)X - (X(v_i) - X(v_s))^2 + (X(v_j) - X(v_s))^2 \\ &= X^T L(T)X + [X(v_j) - X(v_i)][X(v_i) + X(v_j) - 2X(v_s)]. \end{aligned}$$

If $X(v_s) > \frac{X(v_i)+X(v_j)}{2}$, then $\alpha(T') \le X^T L(T')X < X^T L(T)X = \alpha(T)$.

If $X(v_s) = \frac{X(v_i)+X(v_j)}{2}$, then $X^T L(T')X = X^T L(T)X = \alpha(T)$. But (cf. Lemma 2.5), $X \notin \xi(T')$ because, if it were, the path $v_1 \longrightarrow \cdots \longrightarrow v_j \longrightarrow v_s$ would be increasing in X, meaning $X(v_j) < X(v_s)$. This contradicts $X(v_s) = \frac{X(v_i)+X(v_j)}{2} < X(v_j)$. So $\alpha(T') < \alpha(T)$.

If $X(v_s) < \frac{X(v_i)+X(v_j)}{2}$, we may suppose $t = [X(v_j) - X(v_s)] - [X(v_s) - X(v_i)] > 0$. Suppose that the branch of T at v_i containing v_s is S and $|S| = p$. Form a new vector Y by adding t to each of the p coordinates of X corresponding to a vertex of S and let $Z = Y - (pt/n)e$. It is not difficult to get $Z^T L(T')Z = X^T L(T)X = \alpha(T)$, but $Z^T Z = 1 + 2t \sum\limits_{v \in S} X(v) + \frac{p(n-p)t^2}{n} > 1$ because $X(v) > 0$ for every vertex v of S.

So $\alpha(T') \le \frac{Z^T L(T')Z}{Z^T Z} = \frac{\alpha(T)}{Z^T Z} < \alpha(T)$.

(2) If $X(v_j) = X(v_i)$, we have

$$\begin{aligned} \alpha(T') &\le X^T L(T')X \\ &= X^T L(T)X - (X(v_i) - X(v_s))^2 + (X(v_j) - X(v_s))^2 \\ &= X^T L(T)X = \alpha(T). \end{aligned}$$

Theorem 4.2 $\alpha(T_{11}) > c$.

Proof Through the following two steps we can get our conclusion.

(1) We first prove that T_{11} (cf. Figure 4) is a type II tree with the characteristic edge v_1v_2.

The branches of T_{11} at v_1 are $2P_3^1$, $(k-5)P_2^1$ and P_4^2. By Lemma 3.1, we have $\tau(\widehat{L}(P_4^2)) < \tau(\widehat{L}(P_3^1)) < \tau(\widehat{L}(P_2^1))$. So P_4^2 is the unique Perron branch of T_{11} at v_1 containing v_2.

The branches of T_{11} at v_2 are P_1^0, P_2^1, H_5. Since $\tau(\widehat{L}(H_5)) = \tau(L_v(H_5^+))$, by Lemma 4.1 and Lemma 3.1, we have $\tau(\widehat{L}(H_5)) < \tau(\widehat{L}(P_2^1)) < \tau(\widehat{L}(P_1^0))$. So H_5 is the unique Perron branch of T_{11} at v_2 containing v_1.

By Lemma 2.7, we see that T_{11} is a type II tree with the characteristic edge v_1v_2.

(2) We now prove $\alpha(T_{11}) > c$.

Let $X \in \xi(T_{11})$. By Lemma 2.5, we can suppose $X(v_1) > 0, X(v_2) < 0$.

Let $V_1 = \{v_3, v_4\}$, $V_2 = \{v_5, \cdots, v_{k-1}\}$. Then two cases can occur.

(2.1) There exist two vertices, one in V_1 and the other in V_2, such that they have different coordinates in X. Without loss of generality, suppose $X(v_3) > X(v_5)$ $(X(v_3) < X(v_5)$ is the same). We know that $T_{11} - v_5v_k + v_3v_k = G(1,1,0,k-6,2) \in T(12)$.

By Lemma 4.2 and Theorem 4.1, we have

$$\alpha(T_{11}) > \alpha(G(1,1,0,k-6,2)) = c.$$

(2.2) $X(v_3) = X(v_4) = \cdots = X(v_{k-1})$.

$$\begin{aligned}\alpha(T_{11}) &= X^T L(T_{11})X = \sum_{v_iv_j \in E(T_{11})} (X(v_i) - X(v_j))^2 \\ &= X^T L(G(1,1,0,k-6,2))X \geq \alpha(G(1,1,0,k-6,2)) = c. \end{aligned} \tag{4.1}$$

If $\alpha(T_{11}) = \alpha(G(1,1,0,k-6,2)) = c$, by (4.1) we know

$$L(G(1,1,0,k-6,2))X = cX.$$

So

$$X(v_1) = (1-c)X(v_5). \tag{4.2}$$

By $L(T_{11})X = cX$, we can get

$$X(v_5) = (1-c)X(v_k), \quad X(v_1) = (c^2 - 3c + 1)X(v_k).$$

So

$$X(v_1) = \Big(\frac{c^2 - 3c + 1}{1-c}\Big)X(v_5). \tag{4.3}$$

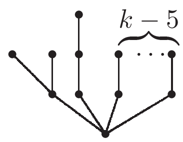
By (4.2) and (4.3), we get

$$1 - c = \frac{c^2 - 3c + 1}{1-c},$$

so $c = 0$, a contradiction.

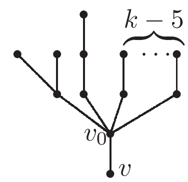
So

$$\alpha(T_{11}) > \alpha(G(1,1,0,k-6,2)) = c.$$



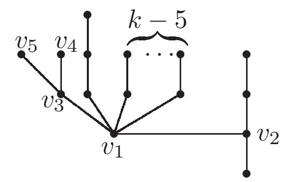
The tree H_6

$H_6 = G(0,1,1,k-5,0)$



The tree H_6^+

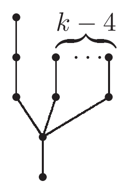
$H_6^+ = G(1,1,1,k-5,0)$



The tree T_{10}

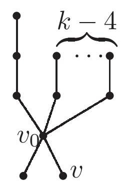
$T_{10} = G(0,1,1,k-5,1)$

Figure 5



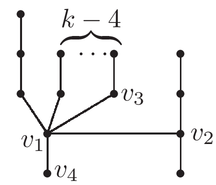
The tree H_7

$H_7 = G(1,1,0,k-4,0)$



The tree H_7^+

$H_7^+ = G(2,1,0,k-4,0)$



The tree T_9

$T_9 = G(1,1,0,k-4,1)$

Figure 6

Lemma 4.3 *For $H_6^+ = G(1,1,1,k-5,0)$ and $H_7^+ = G(2,1,0,k-4,0)$, we have $\tau(L_v(H_i^+)) < a$ (cf. Figures 5, 6), where $i = 6, 7$.*

Proof We first prove $\tau(L_v(H_6^+)) < a$. P_3^1 and P_2^1 are two of the branches of H_6^+ at v_0. By Lemma 3.1, we know $\tau(\widehat{L}(P_3^1)) < \tau(\widehat{L}(P_2^1))$.

So by Lemmas 2.2 and 2.9, we have

$$\tau(L_v(H_6^+)) \leq \alpha(H_6^+) < \tau(\widehat{L}(P_2^1)) = a.$$

Using the similar methods, we can get $\tau(L_v(H_7^+)) < a$.

Theorem 4.3 $\alpha(T_{10}) > \alpha(T_{11})$.

Proof Using the similar methods to Theorem 4.2(1), we can prove that T_{10} (cf. Figure 5) is a type II tree with the characteristic edge v_1v_2. So for any $X \in \xi(T_{10})$, $X(v_3) \neq 0$.

From Figures 4 and 5, we know that $T_{10} - v_3v_5 + v_4v_5 = T_{11}$.

By Lemma 2.4, we have

$$\alpha(T_{10}) > \alpha(T_{11}).$$

Theorem 4.4 $\alpha(T_9) > \alpha(T_{10})$.

Proof Using the methods similar to those used in proving Theorem 4.2(1), we can prove that T_9 (cf. Figure 6) is a type II tree with the characteristic edge v_1v_2.

From Figures 5 and 6, we know that $T_9 - v_1v_4 + v_3v_4 = T_{10}$.

By Lemma 4.2,

$$\alpha(T_9) > \alpha(T_{10}).$$

Lemma 4.4 *Let v_1, v_2 be vertices of a graph G with $d(v_1) = 1$ and $v_1v_2 \in E(G)$. Let $X \in \xi(G)$.*

(1) *Suppose $\alpha(G) < 1$. If one of $X(v_1), X(v_2)$ is 0, then*

$$X(v_1) = X(v_2) = 0.$$

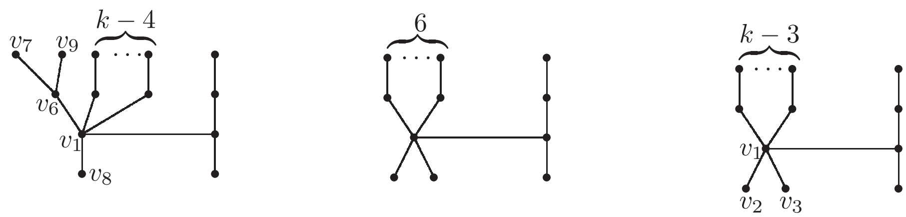
(2) *Suppose $\alpha(G) < \frac{3-\sqrt{5}}{2}$, $v_3 \in V(G)$, $v_2v_3 \in E(G)$, $d(v_2) = 2$. If one of $X(v_1), X(v_2)$, $X(v_3)$ is 0, then*

$$X(v_1) = X(v_2) = X(v_3) = 0.$$

Proof Let $\alpha(G) = \alpha$. By $L(G)X = \alpha X$, we can complete the proof as follows.

If G satisfies the conditions of (1), then $X(v_2) = f_1(\alpha)X(v_1)$, where $f_1(\alpha) = 1 - \alpha$. So (1) holds.

If G satisfies the conditions of (2), then $X(v_2) = f_1(\alpha)X(v_1)$, $X(v_3) = f_2(\alpha)X(v_1)$, where $f_2(\alpha) = \alpha^2 - 3\alpha + 1$. Since $\alpha(G) < \frac{3-\sqrt{5}}{2}$, $f_1(\alpha) > 0$ and $f_2(\alpha) > 0$. So (2) holds.



The tree T_8 The tree H_8 The tree T_6

$T_8 = F(1, k-4, 1, 1, 1, 0)$ $H_8 = F(2, 6, 0, 1, 1, 0)$ $T_6 = F(2, k-3, 0, 1, 1, 0)$

Figure 7

Theorem 4.5 $\alpha(T_8) > \alpha(T_9)$, $\alpha(T_6) > \alpha(T_7)$, $\alpha(T_4) > \alpha(T_5)$, $\alpha(T_2) > \alpha(T_3)$.

Proof We first prove $\alpha(T_8) > \alpha(T_9)$.

P_2^1 and P_4^2 are two of the branches of T_8 at v_1 (cf. Figure 7). By Lemma 3.1, we know that

$$\tau(\widehat{L}(P_4^2)) < \tau(\widehat{L}(P_2^1)) = a = \frac{3-\sqrt{5}}{2}.$$

So by Lemma 2.9, we have

$$\alpha(T_8) < \tau(\widehat{L}(P_2^1)) = a = \frac{3-\sqrt{5}}{2}.$$

Let $X \in \xi(T_8)$. If $X(v_6) = 0$, using Lemmas 2.8 and 4.4 repeatedly, we can get for any $v \in T_8$, $X(v) = 0$, a contradiction. So $X(v_6) \neq 0$.

From Figures 6 and 7, we know $T_8 - v_6v_9 + v_7v_9 = T_9$.

By Lemma 2.4, we have

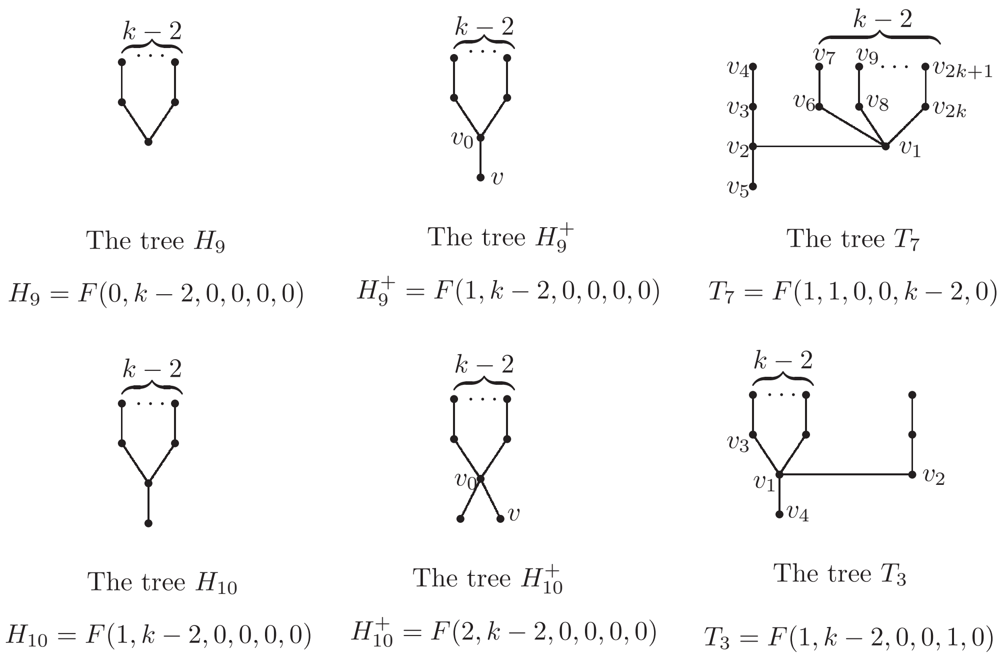
$$\alpha(T_8) > \alpha(T_9).$$

Using the similar methods, we can get $\alpha(T_6) > \alpha(T_7)$, $\alpha(T_4) > \alpha(T_5)$, $\alpha(T_2) > \alpha(T_3)$.

Theorem 4.6 $\alpha(T_5) > \alpha(T_6)$.

Proof Since $k \geq 12$, T_6 must contain H_8 (cf. Figure 7) as a subtree. By direct computations, we have $\alpha(H_8) < 0.1963$. By Corollary 2.1 and Theorem 4.1, we have

$$\alpha(T_6) \leq \alpha(H_8) < 0.1963 < b = \alpha(T_5).$$



The tree H_9 — $H_9 = F(0, k-2, 0, 0, 0, 0)$

The tree H_9^+ — $H_9^+ = F(1, k-2, 0, 0, 0, 0)$

The tree T_7 — $T_7 = F(1, 1, 0, 0, k-2, 0)$

The tree H_{10} — $H_{10} = F(1, k-2, 0, 0, 0, 0)$

The tree H_{10}^+ — $H_{10}^+ = F(2, k-2, 0, 0, 0, 0)$

The tree T_3 — $T_3 = F(1, k-2, 0, 0, 1, 0)$

Figure 8

Lemma 4.5 *For $H_9^+ = F(1, k-2, 0, 0, 0, 0)$ and $H_{10}^+ = F(2, k-2, 0, 0, 0, 0)$, we have $\tau(L_v(H_i^+)) < a$ (cf. Figure 8), where $i = 9, 10$.*

Proof We first prove $\tau(L_v(H_9^+)) < a$.

$$\begin{aligned}\Phi(L_v(H_9^+)) &= \Phi(L(H_9)) - (x^2-3x+1)^{k-2}\\ &= x(x^2-3x+1)^{k-3}\big(x^2-(k+1)x+2k-3\big) - (x^2-3x+1)^{k-2}\\ &= (x^2-3x+1)^{k-3} g_1(x),\end{aligned}$$

where $g_1(x) = x^3 - (k+2)x^2 + 2kx - 1$.

Since $g_1(\frac{3}{10}) = \frac{510k-1153}{1000} > 0$, we have $\tau(L_v(H_9^+)) < 0.3 < a$.

We now prove $\tau(L_v(H_{10}^+)) < a$.

$$\begin{aligned}&\Phi(L_v(H_{10}^+))\\ &= \Phi(L(H_{10})) - (x-1)(x^2-3x+1)^{k-2}\\ &= x(x^2-3x+1)^{k-3}(x^3-(k+3)x^2+(3k+1)x-(2k-2)) - (x-1)(x^2-3x+1)^{k-2}\\ &= (x^2-3x+1)^{k-3} g_2(x),\end{aligned}$$

where $g_2(x) = x^4 - (k+4)x^3 + (3k+5)x^2 - (2k+2)x + 1$.

Since $g_2(\frac{3}{10}) = \frac{-3570k+7501}{10000} < 0$, we have $\tau(L_v(H_{10}^+)) < 0.3 < a$.

Theorem 4.7 $\alpha(T_7) > \alpha(T_8)$, $\alpha(T_3) > \alpha(T_4)$.

Proof Using the methods similar to those used in proving Theorems 4.2 and 4.4, respectively, we can complete the proof.

Theorem 4.8 *For any tree* $T \in T(1)$, *we have* $\alpha(T) > \alpha(T_2)$.

Proof Since $T_2 = G(1, 0, 1, k-2, 0)$ and $k \geq 12$, T_2 has only one vertex u such that $d(u) = \Delta(T_2)$. P_2^1 and P_3^2 are two of the branches of T_2 at u. By Lemma 3.1, $\tau(\widehat{L}(P_3^2)) < \tau(\widehat{L}(P_2^1)) = a$.

By Lemma 2.9, we have

$$\alpha(T_2) < \tau(\widehat{L}(P_2^1)) = a = \alpha(T).$$

From the above theorems, we can get the main result of this paper:

Theorem 4.9 *For any tree* $T \in \mathcal{T}_{2k+1} \setminus (T(1) \cup \{T_2, \cdots, T_{11}\} \cup T(12))$, *any tree* $T_1 \in T(1)$ *and any tree* $T_{12} \in T(12)$, *we have*

$$\begin{aligned} &\alpha(T_1) = a > \alpha(T_2) > \alpha(T_3) > \alpha(T_4) > \alpha(T_5) = b \\ &> \alpha(T_6) > \alpha(T_7) > \alpha(T_8) > \alpha(T_9) > \alpha(T_{10}) > \alpha(T_{11}) \\ &> \alpha(T_{12}) = c > \alpha(T). \end{aligned}$$

References

[1] Cvetković, D. M., Doob, M. and Sachs, H., Spectra of Graphs-Theory and Applications, VEB Deutscher Verlag d. Wiss./Academic Press, Berlin/New York, 1979.

[2] Fiedler, M., Algebraic connectivity graphs, *Czech. Math. J.*, **23**(98), 1973, 298–305.

[3] Fiedler, M., A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, *Czech. Math. J.*, **25**(100), 1975, 619–633.

[4] Fiedler, M., Eigenvectors of acyclic matrices, *Czech. Math. J.*, **25**(100), 1975, 607–618.

[5] Kikland, S., Neumann, M. and Shader, B. L., Characteristic vertices of weighted trees via Perron values, *Linear Multilinear Algebra*, **40**, 1996, 311–325.

[6] Bapat, R. B. and Pati, S., Algebraic connectivity and the characteristic set of a graph, *Linear Multilinear Algebra*, **45**, 1998, 247–273.

[7] Merris, R., Characteristic vertices of trees, *Linear Multilinear Algebra*, **22**, 1987, 115–131.

[8] Grone, R. and Merris, R., Ordering trees by algebraic connectivity, *Graphs Combi.*, **6**, 1990, 229–237.

[9] Yin, S.-H., Shu, J.-L. and Wu, Y.-R., Graft and algebraic connectivity of trees (in Chinese), *J. East China Norm. Univ.* (*Natur. Sci.*), **2**, 2005, 6–15.

[10] Grone, R. and Merris, R., Algebraic connectivity of trees, *Czech. Math. J.*, **37**(112), 1987, 660–670.