

Remarks on Thurston's Construction of Pseudo-Anosov Maps of Riemann Surfaces

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Abstract It is well known that certain isotopy classes of pseudo-Anosov maps on a Riemann surface \tilde{S} of non-excluded type can be defined through Dehn twists $t_{\tilde{\alpha}}$ and $t_{\tilde{\beta}}$ along simple closed geodesics $\tilde{\alpha}$ and $\tilde{\beta}$ on \tilde{S} , respectively. Let G be the corresponding Fuchsian group acting on the hyperbolic plane \mathbb{H} so that $\mathbb{H}/G \cong \tilde{S}$. For any point $a \in \tilde{S}$, define $S = \tilde{S} \setminus \{a\}$. In this article, the author gives explicit parabolic elements of G from which he constructs pseudo-Anosov classes on S that can be projected to a given pseudo-Anosov class on \tilde{S} obtained from Thurston's construction.

Keywords Quasiconformal mappings, Riemann surfaces, Teichmüller spaces, Mapping classes, Dehn twists, Pseudo-Anosov, Bers fiber spaces

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1 Introduction and Statement of Results

Let \tilde{S} be a Riemann surface of genus p and n punctures. We assume that $3p - 3 + n > 0$ and $n \geq 1$. Let $a \in \tilde{S}$ be an arbitrary point and let $S = \tilde{S} \setminus \{a\}$. The relationship between the mapping class groups Mod_S and $\text{Mod}_{\tilde{S}}$ of S and \tilde{S} was extensively studied in [5, 8] and literature given there.

In [3], Bers developed a complex analytic method to investigate the two groups and their relationships. We denote by Mod_S^a the subgroup of Mod_S consisting of mapping classes fixing a . There is a natural group epimorphism $i : \text{Mod}_S^a \rightarrow \text{Mod}_{\tilde{S}}$ defined by filling in the puncture a . Bers [3] used his method to show that the kernel of i is isomorphic to a Fuchsian group G that uniformizes \tilde{S} (the result is also known in [5]). Later in [9], Kra classified all pseudo-Anosov mapping classes that project to the trivial class on \tilde{S} as a is filled in. Kra also considered more general questions of finding pseudo-Anosov classes on the Riemann surface S that fix a and can be projected to classes with given types, and obtained important results by using methods of [3, 4].

In this paper, we continue to study this problem. We consider a pair $\{\tilde{\alpha}, \tilde{\beta}\}$ of simple closed geodesics on \tilde{S} that fills \tilde{S} in the sense that $\tilde{S} \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ consists of disks and once punctured disks. It was shown by Thurston in [13] (see also [7, 8]) that $t_{\tilde{\beta}}^n \circ t_{\tilde{\alpha}}^{-m}$ represents a pseudo-Anosov class for any positive integers n and m . More generally, the following theorem is well known:

Theorem 1.1 (see [10, 12, 13]) *Let \tilde{S} be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. Let $(\tilde{\alpha}, \tilde{\beta})$ be a pair of simple closed geodesics that fills \tilde{S} , and let $\{(n_i, m_i)\}$ be a finite number*

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of pairs of non-negative integers such that at least one n_i and one m_i are non-zero. Then the finite product

$$\prod_i (t_{\tilde{\beta}}^{n_i} \circ t_{\tilde{\alpha}}^{-m_i}) \quad (1.1)$$

represents a pseudo-Anosov class on \tilde{S} .

For general information of constructions of pseudo-Anosov class, see, for example, [7, 8, 10, 12, 13].

A question arises as to whether there is a pseudo-Anosov map on S that projects to a map isotopic to (1.1).

If \tilde{S} is compact, then every map of S that projects to a map isotopic to (1.1) is pseudo-Anosov (see [14]). Let us consider the case that \tilde{S} is non-compact, that is, S contains some punctures. Note that $\tilde{\alpha}$ and $\tilde{\beta}$ can be thought of as closed curves α and β on S . If there is a disk component D of $\tilde{S} \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ and a is positioned in D , then $D \setminus \{a\}$ is a newly created once punctured disk among the components of $S \setminus \{\alpha, \beta\}$, and it is obvious that $\{\alpha, \beta\}$ still fills S . Thus (1.1) intimately represents a pseudo-Anosov class on S that projects to the class represented by (1.1) in a trivial way when a is filled in. But this case does not always occur. For example, if $\tilde{S} \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ only consists of once punctured disks, then any positioning of a that is off the curves will provide a twice punctured disk. In this case, the naturally defined map (1.1) represents a reducible class on S yet it still projects to the pseudo-Anosov class on \tilde{S} represented by (1.1).

Another way of producing pseudo-Anosov classes on S based on an existing pseudo-Anosov class on \tilde{S} is due to Kra [9]. Since (1.1) represents a pseudo-Anosov class, we can find a surface \tilde{S} so that \tilde{S} is minimal for the class (see [4] for the definition and terminology). Let $w : \tilde{S} \rightarrow \tilde{S}$ be an extremal Teichmüller mapping that is isotopic to a map of form (1.1).

Let ϕ be the corresponding quadratic differential. If $p \geq 2$, ϕ must have zeros invariant under w . We assume, by taking a suitable power if necessary, that w fixes a zero on \tilde{S} denoted by a . It turns out that the restriction $w|_S = w|_{\tilde{S} \setminus \{\alpha\}}$ is pseudo-Anosov and has a required property. This construction is, however, contingent upon the assumption that ϕ has a zero on \tilde{S} fixed by w . It was shown in [17] that if $\tilde{S} \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ only consists of once punctured disks, then all zeros of ϕ are punctures of \tilde{S} . Thus the construction of Kra [9] does not apply in the current situation.

The question remains as to whether or not there exist pseudo-Anosov classes on S projecting to a class represented by (1.1) under any circumstances. In this paper, we answer this question in a positive way by proving the following result:

Theorem 1.2 *Let \tilde{S} be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. There are (infinitely many) pseudo-Anosov classes on S that project to the pseudo-Anosov class represented by (1.1).*

The idea of the proof is the following. We first lift $\tilde{\alpha}$ and $\tilde{\beta}$ in a natural way to curves α and β on S . Note that the pair (α, β) may not fill S . But based on β one constructs a non-trivial curve δ that becomes trivial on \tilde{S} so that (α, δ) fills S . Furthermore, we show that the image curve β' of β under the positive Dehn twist along δ has the properties: (1) (α, β') fills S ; (2) β' is isotopic to β on \tilde{S} . Finally we prove the theorem by invoking Theorem 1.1 on the Riemann surface S .

The same method can also be used to study pseudo-Anosov classes on S projecting to a given Dehn twist. Some results are discussed in Section 4.

2 Preliminaries

In this section, we review some definitions and basic facts on Teichmüller theory. For general information about the structure of Teichmüller space, the reader is referred to [1, 3, 4, 9].

Let R be a compact Riemann surface of genus p with $n \geq 1$ points removed. Assume that $3p - 3 + n > 0$. The Teichmüller space $T(R)$ is defined as a space of conformal structures μ on R modulo an equivalence relation, where two structures μ, ν are called equivalent, if there is an isometry h between them such that h is isotopic to the identity as a self-map of the underlying surface. The equivalence class of μ is denoted by $[\mu]$.

Let $\mathbb{H} = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ denote the hyperbolic plane and $\varrho : \mathbb{H} \rightarrow \tilde{S}$ the universal covering with covering group G . Associate to each μ is a quasiconformal map w^μ of \mathbb{C} (see [2]) that fixes $0, 1$; is conformal on $\mathbb{H}^* = \{z \in \mathbb{C}, \operatorname{Im} z < 0\}$; and has Beltrami coefficient μ on \mathbb{H} . It was shown in [3] that the domain $w^\mu(\mathbb{H})$ depends only on $[\mu]$. Therefore, we can form a total space $F(R) = \{([\mu], z), [\mu] \in T(R), \text{ and } z \in w^\mu(\mathbb{H})\}$, which is called a Bers fiber space over $T(R)$.

The group of isotopy classes of self-maps (mapping classes) of R forms a mapping class group and is denoted by Mod_R . An element χ in Mod_R acts on $T(R)$ in the form $[\mu] \mapsto [\mu \cdot w^{-1}]$, where $w : R \rightarrow R$ is a representative of χ . w is said to represent a pseudo-Anosov class χ if there is a point $x_0 \in T(R)$ such that $\langle x_0, \chi(x_0) \rangle$ is a positive minimum for $\langle x, \chi(x) \rangle$ when x ranges over all points in $T(R)$, where $\langle \cdot, \cdot \rangle$ denotes the Teichmüller metric on $T(R)$.

Denote by $i(c, c')$ the infimum of the geometric intersections of two simple closed curves c and c' on R running over their homotopy classes. Let t_c denote the positive Dehn twist along the geodesic representative homotopic to c . Assume that $w : R \rightarrow R$ represents a pseudo-Anosov class. It was shown in [10] that $t_c^n \circ w$ represents a pseudo-Anosov class except for only a finite number of values of n . This result was refined by Fathi [6]. To be more precise, we first note that the set $\{c, w(c), \dots, w^k(c), \dots\}$ of closed curves always fills R . Let $k > 0$ be an integer such that $\mu_k := i(c, w^k(c)) > 0$ and $i(c, w^l(c)) = 0$ for $l = 0, 1, \dots, k-1$. From [6, Proposition 5.1], there are real numbers z_j and $y \geq 0$ such that for any real number λ ,

$$i(t_c^\lambda \circ w^k(c), w^{-k}(c)) = y + \sum_{j=1}^{\mu_k} |z_j - \lambda \mu_k|,$$

where t_c^λ denotes the interpolated twist along c by the factor λ . See [6] for more information. Let

$$\lambda_0 = \left(\sum_{j=1}^{\mu_k} z_j \right) (\mu_k \mu_{-k})^{-1},$$

and let I be the interval with center λ_0 and length $2 + \frac{4}{\mu_k}$. [6, Theorem 5.1] tells us that if $n \in \mathbb{Z} \setminus I$, then the map $t_c^n \circ w$ also represents a pseudo-Anosov class. In particular, since the length of I is ≤ 6 , $t_c^n \circ w$ represents a pseudo-Anosov class except for 7 consecutive values of n .

Let \tilde{S} and S be defined as in the introduction. Let $\hat{w}, \hat{w}' : \mathbb{H} \rightarrow \mathbb{H}$ be two maps that project to self-maps w and w' of \tilde{S} , respectively. \hat{w} and \hat{w}' are called equivalent (the equivalence class is denoted by $[\hat{w}]$) if

$$\hat{w} \circ g \circ (\hat{w})^{-1} = \hat{w}' \circ g \circ (\hat{w}')^{-1} \quad \text{for every } g \in G.$$

The group $\text{mod } \tilde{S}$ is the collection of $[\hat{w}]$ for all $w : \tilde{S} \rightarrow \tilde{S}$. Elements in the set of restrictions $\hat{w}|_{\mathbb{H}}$ are one-to-one correspondent with elements in $\text{mod } \tilde{S}$. Every element $[\hat{w}]$ of $\text{mod } \tilde{S}$ acts on $F(\tilde{S})$ by the formula:

$$[\hat{w}]([\mu], z) = ([\nu], w^\nu \circ \hat{w} \circ (w^\mu)^{-1}(z)),$$

where ν is the Beltrami coefficient of $w^\mu \circ \hat{w}^{-1}$. We see that $[\hat{w}]$ acts on $F(\tilde{S})$ as a fiber-preserving biholomorphism. By [3, Theorem 9], there is a biholomorphic map $\varphi : F(\tilde{S}) \rightarrow T(S)$ that is called a Bers isomorphism in the literature. From [3, Theorem 10], the isomorphism φ induces an isomorphism sending elements $[\hat{w}] \in \text{mod } \tilde{S}$ onto the elements $[\hat{w}]^* \in \text{Mod}_S^a$.

Example 2.1 Take a simple closed geodesic $\tilde{\alpha} \subset \tilde{S}$ that avoids a . The Dehn twist $t_{\tilde{\alpha}}$ can be lifted to a map $\tilde{t}_{\tilde{\alpha}}$ of \mathbb{H} along a geodesic $\hat{\alpha} \subset \mathbb{H}$ with $\varrho(\hat{\alpha}) = \tilde{\alpha}$. It was shown in [16] that $[\tilde{t}_{\tilde{\alpha}}]^*$ is represented by a Dehn twist t_α , where α is a simple closed curve on S that descends to $\tilde{\alpha}$ as a is filled in. Among the lifts of $t_{\tilde{\alpha}}$ there is a lift (also called $\tilde{t}_{\tilde{\alpha}}$) such that $[\tilde{t}_{\tilde{\alpha}}]^*$ is represented by t_α that is obtained just from $t_{\tilde{\alpha}}$ by deleting a . In what follows, we use the same notation to denote a self-map of a Riemann surface as well as its mapping class. In this convention, we simply write $[\tilde{t}_{\tilde{\alpha}}]^* = t_\alpha$.

Every element $g \in G$ naturally acts on $F(\tilde{S})$ by

$$g([\mu], z) = ([\mu], w^\mu \circ g \circ (w^\mu)^{-1}(z)).$$

In this way, $G \cong \pi_1(\tilde{S}, a)$ can be regarded as a normal subgroup of $\text{mod } \tilde{S}$. We use the letter G to denote the Fuchsian group as well as the normal subgroup of $\text{mod } \tilde{S}$. Note that g^* is an element of Mod_S^a that projects to the trivial class of Mod_S^a . Theorem 2 of [9, 11] states that g^* is represented by a Dehn twist along a boundary curve c of a twice punctured disk on S enclosing a (write as $g^* = t_c$) if and only if $g \in G$ is parabolic.

To proceed, we let b_1, \dots, b_n denote the punctures on \tilde{S} . Then, of course, S has punctures a, b_1, \dots, b_n .

Lemma 2.1 *Let D_i be an arbitrary twice punctured disk on S enclosing a and b_i and let $c = \partial D_i$. Let $\{T_i\}$ denote the conjugacy class in G corresponding to the puncture b_i . Then there is a parabolic element $g_i \in G$ in $\{T_i\}$ such that g_i^* is represented by t_c .*

Proof Since c becomes a trivial loop as a is filled in, $i(t_c)$ is a trivial mapping class. This means that t_c lies in the kernel of the homomorphism $i : \text{Mod}_S^a \rightarrow \text{Mod}_S^a$. By [3, Theorem 10], there is an element $g \in G$ such that $g^* = t_c$. It follows from Theorem 2 of [9, 11] that g is a parabolic element. Since c shrinks to the puncture b_i as a is filled in, g is in $\{T_i\}$.

3 Filling Curves and Pseudo-Anosov Maps

Let S^* be the compactification $\tilde{S} \cup \{b_1, \dots, b_n\}$ with marked points b_1, \dots, b_n , and let S_a^* be S^* with one more marked point a added. Let \mathcal{P}_a denote the set of equivalence classes of paths P_i on S_a^* connecting from a to another marked point b_i and no any other marked points are on P_i , where two paths P_i and P'_i are considered equivalent if both paths connect a and b_i and P_i is homotopic to P'_i by a homotopy fixing a and b_i without interfering with any other b_j , $j \neq i$.

Let $\alpha \subset S$ be defined as in the example in Section 2. Let $b = b_i$ be an arbitrary puncture. We have

Lemma 3.1 *Assume that there is a simple closed geodesic $\tilde{\beta}$ such that $\{\tilde{\alpha}, \tilde{\beta}\}$ fills \tilde{S} . There is an element Γ in \mathcal{P}_a connecting a and b such that each component of $S_a^* \setminus \{\Gamma, \tilde{\alpha}\}$ is either a disk or a disk with only one marked point enclosed.*

Proof By hypothesis, $\{\tilde{\alpha}, \tilde{\beta}\}$ fills $S^* = \tilde{S} \cup \{b_1, \dots, b_n\}$. For any marked point $b = b_i$, let Ω_1 be the disk component of $S^* \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ that includes b .

There is a subarc Γ_0 of $\tilde{\beta}$ that serves as a boundary portion of Ω_1 . Let $\{A, B\}$ be the endpoints of Γ_0 , i.e., $\Gamma_0 \cap \tilde{\alpha} = \{A, B\}$.

Cut off Γ_0 in the middle point, and denote by C, D the endpoints resulted from the cutting. Γ_0 breaks down to two smaller arcs, one of which connects from D to B , denoted by Γ_1 ; and the other connects from A to C and is denoted by Γ_3 . We parametrize Γ_1, Γ_3 by $\Gamma_1 = \Gamma_1(s)$ and $\Gamma_3(s)$, respectively, where $0 \leq s \leq 1$. Similarly, the subarc $\Gamma_2 = \tilde{\beta} \setminus \Gamma_0$ can be parametrized by $\Gamma_2(s)$, $0 \leq s \leq 1$, such that $\Gamma_2(0) = B$, and $\Gamma_2(1) = A$. Finally, we draw a small arc Γ_4 that connects C and b without leaving Ω_1 . Γ_4 can be parametrized as $\Gamma_4(s)$, $0 \leq s \leq 1$.

Now we construct a path Γ originating at D and terminating at b by the formula:

$$\Gamma(s) = \begin{cases} \Gamma_1(4s), & 0 \leq s \leq \frac{1}{4}, \\ \Gamma_2(4s - 1), & \frac{1}{4} < s \leq \frac{1}{2}, \\ \Gamma_3(4s - 2), & \frac{1}{2} < s \leq \frac{3}{4}, \\ \Gamma_4(4s - 3), & \frac{3}{4} < s \leq 1. \end{cases}$$

Obviously, $\Gamma = \Gamma(s)$ is a path on S^* connecting from D to b , and no marked points other than b lie on Γ . If necessary, a quasiconformal mapping of S^* onto S_a^* can be made to send b_i to b_i and D to a . One may assume that $D = a$. Since $\Gamma(0) = a$ and $\Gamma(1) = b$, we obtain a path Γ in \mathcal{P}_a .

To see that $S_a^* \setminus \{\Gamma, \tilde{\alpha}\}$ consists of disks and disks with only one marked point enclosed, we note that Γ_0 is a common boundary segment of two components Ω_1 and Ω_2 of $S_a^* \setminus \{\tilde{\alpha}, \tilde{\beta}\}$, and no other components of $S_a^* \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ take Γ_0 as a boundary segment. Observe that Ω_2 is either a disk or a disk with one marked point enclosed. Let Ω denote the interior of $\overline{\Omega_1 \cup \Omega_2}$. If Ω_2 is a disk with one marked point, then $\Omega \setminus \Gamma$ is also a topological disk with the same marked point. If Ω_2 is a disk, so is $\Omega \setminus \Gamma$.

Finally, let Ω' be any component of $S_a^* \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ other than Ω_1 and Ω_2 . In this case, all boundary segments of Ω' are portions of $\tilde{\alpha}$ and $\tilde{\beta}$, which means that Ω' is either a disk or a disk with one marked point enclosed. Since components of $S_a^* \setminus \{\tilde{\alpha}, \tilde{\beta}\}$ other than Ω_1 and Ω_2 are one-to-one correspondent with components of $S_a^* \setminus \{\Gamma, \tilde{\alpha}\}$ away from Ω , we conclude that every component of $S_a^* \setminus \{\Gamma, \tilde{\alpha}\}$ is either a disk or a disk with one marked point enclosed. So Γ is as required.

Let \mathcal{E}_a denote the set of equivalence classes of twice punctured disks on S enclosing a and another puncture, where two such disks are considered equivalent if both enclose the same punctures and both boundaries are homotopic to each other. There is a bijection

$$j : \mathcal{P}_a \rightarrow \mathcal{E}_a$$

defined as follows. For any path $P_i \in \mathcal{P}_a$ connecting a and b_i , fattening P_i and then deleting a and all b_i , $1 \leq i \leq n$. We thus obtain a twice punctured disk on S . Conversely, any twice punctured disk $\Delta_i \in \mathcal{E}_a$ defines a disk $\tilde{\Delta}_i$ on S_a^* with two marked points a and b_i enclosed if all points a and b_i , $1 \leq i \leq n$, are filled in. One obtains a path P_i in \mathcal{P}_a defined to be a deformation retract of $\tilde{\Delta}_i$ on S_a^* .

Let $\alpha, \beta \subset S$ be natural lifts of $\tilde{\alpha}$ and $\tilde{\beta}$ that are obtained from deleting the point a .

Lemma 3.2 *For each puncture $b = b_i$ of S , there is a twice punctured disk Δ on S that satisfies:*

- (1) Δ encloses a and b ,
- (2) $\delta = \partial\Delta$ together with α fills S .

Proof Let Γ be given as in Lemma 3.1, and let $\Delta = j(\Gamma)$. Then $\Delta \subset S$ is a twice punctured disk enclosing a and b . A schematic diagram is drawn in Figure 1.

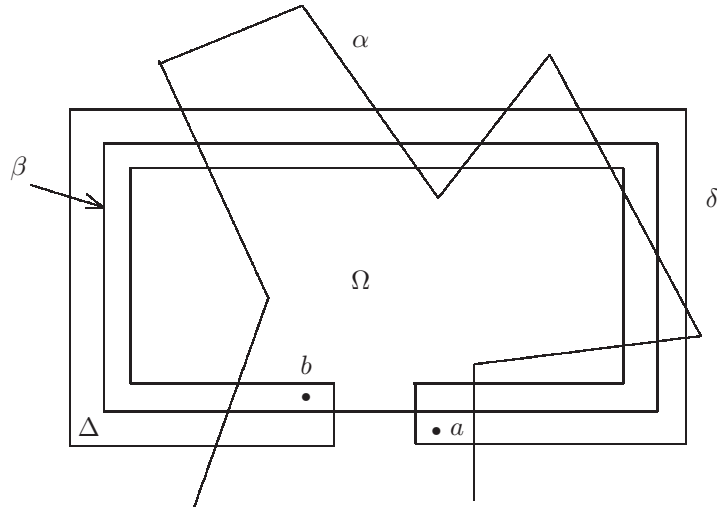


Figure 1

Clearly, $\delta \subset S$ is a non-peripheral simple closed curve. We need to show that $\alpha \cup \delta$ fills S . By definition of Δ , $\tilde{\alpha}$ crosses Γ at a point x if and only if α cuts Δ transversally along a segment passing through x . Note that α cuts Δ into several regions in order that they are all quadrilaterals except for the first one H_1 , which is an a -punctured disk; and the last one H_2 , which is a b -punctured disk. Let Ω, Ω' be defined as in Lemma 3.1. Each component of $S \setminus \{\alpha, \delta\}$ falls into one of the following collections:

- (I) H_1 and H_2 ;
- (II) components of $\Delta \setminus \{\alpha\}$ that are not H_1 and H_2 ;
- (III) components of $S \setminus \{\alpha, \Delta\}$ corresponding to Ω' ;
- (IV) the (unique) component of $S \setminus \{\alpha, \Delta\}$ that corresponds to the interior of $\Omega \setminus \{H_1, H_2\}$.

As noted above, a type I component is a once punctured disk, and a type II component is a quadrilateral that is isotopic to a disk of course. Every type III component can be deformed into an Ω' , which is a disk or a once punctured disk. Finally, let $\Omega^* \subset S$ denote the (unique) component of type IV. Then Ω^* can be deformed to $\Omega \setminus \Gamma$ as defined in Lemma 3.1. It follows

from Lemma 3.1 that Ω^* is either a disk or a once punctured disk. Overall, we conclude that each component of $S \setminus \{\alpha, \delta\}$ is either a disk or a once punctured disk, which means that $\{\alpha, \delta\}$ fills S .

Lemma 3.3 *Let δ and Δ be given as in Lemma 3.2, and let $t_\delta(\beta)$ be the image loop of β under the Dehn twist along the boundary curve δ of Δ . Then $t_\delta(\beta)$ is a non-peripheral simple closed curve with the properties: (1) $t_\delta(\beta)$ is homotopic to β on \tilde{S} ; (2) $\{t_\delta(\beta), \alpha\}$ fills S .*

Proof Without loss of generality, we assume that a and b lie in different sides of β as shown in Figure 1. The curve $t_\delta(\beta)$ is drawn in Figure 2. Following the same notations as in Lemmas 3.1 and 3.2, let $S' = S \setminus \Omega$. It is clear that $S' \cap \{t_\delta(\beta)\}$ is a disjoint union of simple arcs. These subarcs are almost parallel and placed in Δ in a nice position after a suitable homotopy is performed.

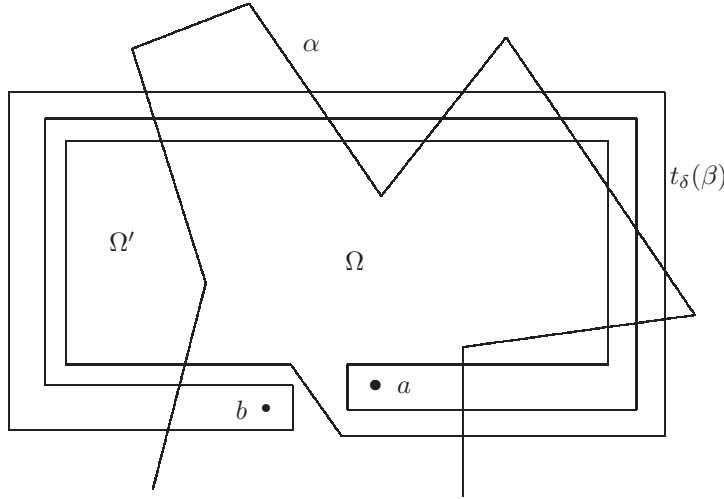


Figure 2

We need to examine each component of $S \setminus \{\alpha, t_\delta(\beta)\}$. Any type III component (defined in Lemma 3.2) can be deformed to a component of $S \setminus \{\alpha, t_\delta(\beta)\}$. So these components are either disks or once punctured disks. Any type II component is divided by $S' \cap \{t_\delta(\beta)\}$ into two smaller quadrilaterals, each of which is still a topological disk.

Finally, observe that $\{t_\delta(\beta)\} \cap \Omega$ divides Ω into four regions, among which one is an a -punctured disk; one is a b -punctured disk; and the rest two components are homeomorphic to disks (see Figure 2).

Since $t_\delta(\beta)$ is an image of a simple non-peripheral closed curve under a homeomorphism, it is simple and non-peripheral as well. Finally, notice that δ is peripheral on \tilde{S} , t_δ is trivial on \tilde{S} , and thus $t_\delta(\beta)$ is homotopic to β on \tilde{S} .

Proof of Theorem 1.2 By Lemma 3.3, the two simple closed curves $\sigma = t_\delta(\beta)$ and α fills S . Therefore by Theorem 1.1, the finite product

$$\prod_i (t_\sigma^{n_i} \circ t_\alpha^{-m_i}) \quad (3.1)$$

represent a pseudo-Anosov class. But $t_\sigma = t_\delta \circ t_\beta \circ t_\delta^{-1}$. By Lemma 2.1, there is a parabolic

element $g \in G$ such that $g^* = t_\delta$. Together with (3.1), we see that

$$\prod_i (g^* \circ t_\beta^{n_i} \circ (g^*)^{-1} \circ t_\alpha^{-m_i}) \quad (3.2)$$

represents a pseudo-Anosov class on S . By [3, Theorem 10], $i(g^*) = \text{id}$. Hence the class (3.2) projects to the class represented by (1.1) under the epimorphism $i : \text{Mod}_S^a \rightarrow \text{Mod}_{\tilde{S}}$. This completes the proof of Theorem 1.2.

4 Pseudo-Anosov Maps Obtained from Parabolic Elements

In this section, we use the parabolic elements g of G obtained from Section 3 to study pseudo-Anosov classes in Mod_S^a that project to a simple Dehn twist. We also discuss some properties of these classes. For convenience, $\tilde{\alpha} \subset \tilde{S}$ is called to pair with a geodesic $\tilde{\beta} \subset \tilde{S}$ if $\{\tilde{\alpha}, \tilde{\beta}\}$ fills \tilde{S} .

Associate to each simple geodesic $\alpha \subset S$ with $i(t_\alpha) = t_{\tilde{\alpha}}$ is a primary simple hyperbolic element g_α of G such that g_α^* commutes with t_α . We denote by A_g the axis of a hyperbolic element g of G in \mathbb{H} . It was shown in [15, 16] that if g has the property that $\varrho(A_g)$ fills \tilde{S} , then $t_\alpha \circ g^* \in \text{Mod}_S^a$ represents a pseudo-Anosov class. If g is simple hyperbolic and there is a fundamental region $F \subset \mathbb{H}$ of G such that F catches portions of A_g and A_{g_α} , then $t_\alpha \circ g^*$ is pseudo-Anosov if and only if $\{\tilde{\alpha}, \varrho(A_g)\}$ fills \tilde{S} . On the other hand, if g is parabolic with fixed point being a vertex of F that catches a portion of A_{g_α} , then t_α can be defined so that $t_\alpha \circ g^*$ is not pseudo-Anosov.

Theorem 4.1 *Let \tilde{S} be a Riemann surface of type (p, n) with $3p - 3 + n > 0$ and $n \geq 1$. Let $\tilde{\alpha}$ be a simple closed geodesic on \tilde{S} that pairs with a simple closed geodesic. Then for each puncture b_i , there are elements $g_i \in G$ in the conjugacy class of b_i such that $t_\alpha \circ g_i^* \in \text{Mod}_S^a$ are pseudo-Anosov and project to the class represented by the Dehn twist $t_{\tilde{\alpha}}$.*

Proof By Lemma 3.2, there is a non-peripheral closed curve δ on S that satisfies the conditions: (1) δ becomes a trivial loop around b_i on \tilde{S} if a is filled in; (2) δ together with α fills S . By Thurston's construction of pseudo-Anosov maps (see also [7]), the product $t_\alpha \circ t_\delta^{-1}$ represents a pseudo-Anosov class in Mod_S^a . By Lemma 2.1, we can find a parabolic element g of G so that g^* is represented by t_δ^{-1} , which tells us that $t_\alpha \circ g^*$ represents a pseudo-Anosov class in Mod_S^a , while it still projects to the Dehn twist $t_{\tilde{\alpha}}$. This completes the proof of Theorem 4.1.

Theorem 4.1 can be used to discuss some properties of mapping classes in Mod_S^a .

Theorem 4.2 *There are infinitely many pseudo-Anosov classes $\theta_n \in \text{Mod}_S^a$ and a fixed parabolic element g of G such that $i(\theta_n \circ g^*) \in \text{Mod}_{\tilde{S}}$ and $\theta_n \circ g^* \in \text{Mod}_S^a$ are all distinct pseudo-Anosov classes.*

Proof Let $\tilde{\alpha} \subset \tilde{S}$ be a curve that pairs with a curve $\tilde{\beta}$. Let α and $\beta \subset S$ be natural lifts of $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. By Theorem 4.1, we can find a parabolic element $g \in G$ so that for any positive integer m , $t_\alpha^{-m} \circ g^*$ is a pseudo-Anosov class in Mod_S^a . Since β is a non-peripheral closed curve on S , from [6, Theorem 5.1] (see also Section 2 for an exposition), $t_\alpha^{-m} \circ g^* \circ t_\beta^n$ is pseudo-Anosov except for at most 7 consecutive positive values of n . This particularly implies that $t_\alpha^{-m} \circ g^* \circ t_\beta^n = (t_\alpha^{-m} \circ g^* \circ t_\beta^n) \circ (g^*)^{-1} \circ g^*$ is pseudo-Anosov for infinitely many values of n .

Let $\theta_n = t_\alpha^{-m} \circ g^* \circ t_\beta^n \circ (g^*)^{-1}$. By Lemma 3.3 and the same argument of Theorem 1.2, we see that θ_n are pseudo-Anosov for infinitely many values of n . It is also easy to see that $i(\theta_n \circ g^*) = t_\alpha^{-m} \circ t_\beta^n$. By Theorem 1.1, $i(\theta_n \circ g^*)$ is pseudo-Anosov. Note also that for $n \neq n'$, $t_\alpha^{-m} \circ t_\beta^n$ is not isotopic to $t_\alpha^{-m} \circ t_\beta^{n'}$ on \tilde{S} , that is, $i(\theta_n \circ g^*)$ is not isotopic to $i(\theta_{n'} \circ g^*)$. It follows that $\theta_n \circ g^*$ and $\theta_{n'} \circ g^*$ lie in different fibers. In particular, this implies that $\theta_n \circ g^* \neq \theta_{n'} \circ g^*$. We conclude that all $\theta_n \circ g^*$ for different values of n are distinct. This completes the proof of Theorem 4.2.

As another application of Theorem 4.1, we intend to represent some pseudo-Anosov maps by virtue of products of Dehn twists along a pair of filling geodesics on S . Let $\Sigma = \{(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$ denote a non-empty finite set of pairs of simple closed geodesics on S such that each pair fills S . From Theorem 4.1, there is an integer $N > 0$ (depending on a fixed integer m) such that for any $n > N$, the map

$$f_n = t_\beta^n \circ t_\alpha^{-m} \circ g^* \quad (4.1)$$

represents a pseudo-Anosov class in Mod_S^a , where g^* is defined in Theorem 4.2. We ask if there is such a non-empty finite set Σ so that for each $n > N$, f_n is isotopic to

$$t_{\beta_i}^{c_n} \circ t_{\alpha_i}^{d_n} \quad (4.2)$$

for $(\alpha_i, \beta_i) \in \Sigma$ and certain integers $c_n, d_n \in \mathbb{Z}$. We will give a negative answer to this question by proving:

Theorem 4.3 *There does not exist any non-empty finite set Σ of pairs of simple closed geodesics on S such that any pseudo-Anosov map f_n in (4.1) are isotopic to a map of form (4.2).*

Proof By taking a subsequence if needed, we may assume that

$$f_n = t_{\beta'}^{c_n} \circ t_{\alpha'}^{d_n}, \quad (4.3)$$

where (α', β') is a fixed pair in Σ . Since f_n projects to $t_\beta^n \circ t_\alpha^{-m}$, α' and β' are isotopic on \tilde{S} to $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. It follows that $c_n = n$, $d_n = -m$, and there are maps f_1 and f_2 of S , isotopic to the identity on \tilde{S} , such that $f_1(\beta) = \beta'$ and $f_2(\alpha) = \alpha'$. We write $f_1 = u^*$ and $f_2 = v^*$ for some elements $u, v \in G$. Then

$$u^*(\beta) = \beta' \quad \text{and} \quad v^*(\alpha) = \alpha'. \quad (4.4)$$

Hence (4.3) can be written as $f_n = t_{u^*(\beta)}^n \circ t_{v^*(\alpha)}^{-m}$, or

$$f_n = (u^* \circ t_\beta^n \circ (u^*)^{-1}) \circ (v^* \circ t_\alpha^{-m} \circ (v^*)^{-1}). \quad (4.5)$$

As we remarked in Section 2, geodesics $\hat{\alpha}, \hat{\beta} \subset \mathbb{H}$ can be chosen with the properties: (1) $\varrho(\hat{\alpha}) = \tilde{\alpha}$ and $\varrho(\hat{\beta}) = \tilde{\beta}$; (2) there are lifts $\tilde{t}_{\hat{\alpha}}$ and $\tilde{t}_{\hat{\beta}}$ along $\hat{\alpha}$ and $\hat{\beta}$, respectively, such that $t_\alpha = [\tilde{t}_{\hat{\alpha}}]^*$ and $t_\beta = [\tilde{t}_{\hat{\beta}}]^*$. Therefore, using Bers isomorphism, together with (4.3) and (4.1) we obtain

$$[\tilde{t}_{\hat{\beta}}]^n \circ [\tilde{t}_{\hat{\alpha}}]^{-m} \circ g = (u \circ [\tilde{t}_{\hat{\beta}}]^n \circ u^{-1}) \circ (v \circ [\tilde{t}_{\hat{\alpha}}]^{-m} \circ v^{-1}), \quad (4.6)$$

which is equivalent to

$$\tilde{t}_{\beta}^{-n} \circ u^{-1} \circ \tilde{t}_{\beta}^n = (u^{-1} \circ v) \circ (\tilde{t}_{\alpha}^{-m} \circ v^{-1} \circ g^{-1} \circ \tilde{t}_{\alpha}^m) \quad \text{on } \mathbb{R}. \quad (4.7)$$

Note that $u_n = \tilde{t}_{\beta}^{-n} \circ u^{-1} \circ \tilde{t}_{\beta}^n$, $w = u^{-1} \circ v$, and $w' = \tilde{t}_{\alpha}^{-m} \circ v^{-1} \circ g^{-1} \circ \tilde{t}_{\alpha}^m$ are all elements of G . From (4.7), we get $u_n = w \circ w'$. Denote $u' = w \circ w'$. This says that u_n is a fixed element u' of G that is independent of n .

Let x be a fixed point of u . Then for all n , $\tilde{t}_{\beta}^{-n}(x)$ is fixed by $\tilde{t}_{\beta}^{-n} \circ u^{-1} \circ \tilde{t}_{\beta}^n$, and thus it is fixed by u' also. But this is impossible unless $\tilde{t}_{\beta}^{-n}(x) = x$. This implies that u commutes with \tilde{t}_{β} . It follows that $u^*(\beta) = \beta$. By (4.4) we get $\beta = \beta'$.

Now (4.1) together with (4.3) yields $t_{\alpha}^{-m} \circ g^* = t_{\alpha'}^{-m}$. By Theorem 4.1, $t_{\alpha}^{-m} \circ g^*$ is pseudo-Anosov, while $t_{\alpha'}^{-m}$ is a power of a Dehn twist. This contradiction proves Theorem 4.3.

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References

- [1] Abikoff, W., The Real Analytic Theory of Teichmüller Spaces, Lecture Notes in Mathematics, **820**, Springer-Verlag, New York, 1980.
- [2] Ahlfors, L. V. and Bers, L., Riemann's mapping theorem for variable metrics, *Ann. of Math.* (2), **72**, 1960, 385–404.
- [3] Bers, L., Fiber spaces over Teichmüller spaces, *Acta Math.*, **130**, 1973, 89–126.
- [4] Bers, L., An extremal problem for quasiconformal mappings and a theorem by Thurston, *Acta Math.*, **141**, 1978, 73–98.
- [5] Birman, J. S., Braids, Links and Mapping Class Groups, Ann. of Math. Studies, No. 82, Princeton University Press, Princeton, 1974.
- [6] Fathi, A., Dehn twists and pseudo-Anosov diffeomorphism, *Invent. Math.*, **87**, 1987, 129–151.
- [7] Fathi, A., Laudenbach, F. and Poenaru, V., Travaux de Thurston sur les surfaces, Seminaire Orsay, Asterisque, Vol. 66–67, Soc. Math. de France, Paris, 1979.
- [8] Ivanov, N. V., Mapping class groups, Manuscript, December 21, 1998.
- [9] Kra, I., On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces, *Acta Math.*, **146**, 1981, 231–270.
- [10] Long, D. and Morton, H., Hyperbolic 3-manifolds and surface automorphisms, *Topology*, **25**, 1986, 575–583.
- [11] Nag, S., Non-geodesic discs embedded in Teichmüller spaces, *Amer. J. Math.*, **104**, 1982, 339–408.
- [12] Penner, R. C., A construction of pseudo-Anosov homeomorphisms, *Proc. Amer. Math. Soc.*, **104**, 1988, 1–19.
- [13] Thurston, W. P., On the geometry and dynamics of diffeomorphisms of surfaces, *Bull. Amer. Math. Soc. (N. S.)*, **19**, 1988, 417–431.
- [14] Zhang, C., Hyperbolic mapping classes and their lifts on Bers fiber space, *Chin. Ann. Math.*, **28B**(1), 2007, 55–66.
- [15] Zhang, C., Projections of hyperbolic mapping classes to simple Dehn twists, *Complex Var. and Elliptic Equ.*, to appear, 2007.
- [16] Zhang, C., A characterization of a pseudo-Anosov mapping class on non-compact Riemann surfaces, preprint, 2006.
- [17] Zhang, C., Singularities of quadratic differentials and extremal Teichmüller mappings of Riemann surfaces, preprint, 2006.