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Embedding Theorems in B-Spaces and Applications

Veli B. SHAKHMUROV*

Abstract This study focuses on the anisotropic Besov-Lions type spaces $B^l_{p,\theta}(\Omega; E_0, E)$ associated with Banach spaces E_0 and E. Under certain conditions, depending on $l = (l_1, l_2, \cdots, l_n)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, the most regular class of interpolation space E_α between E_0 and E are found so that the mixed differential operators D^α are bounded and compact from $B^{l+s}_{p,\theta}(\Omega; E_0, E)$ to $B^s_{p,\theta}(\Omega; E_\alpha)$. These results are applied to concrete vector-valued function spaces and to anisotropic differential-operator equations with parameters to obtain conditions that guarantee the uniform B separability with respect to these parameters. By these results the maximal B-regularity for parabolic Cauchy problem is obtained. These results are also applied to infinite systems of the quasi-elliptic partial differential equations and parabolic Cauchy problems with parameters to obtain sufficient conditions that ensure the same properties.

Keywords Embedding theorems, Banach-valued function spaces, Differential-operator equations, B-Separability, Operator-valued Fourier multipliers, Interpolation of Banach spaces
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1 Introduction

Embedding theorems in function spaces have been elaborated in [8, 27, 38]. A comprehensive introduction to the theory of embedding of function spaces and historical references may be also found in [37]. In abstract function spaces embedding theorems have been studied in [3, 5, 19, 21, 23, 28, 31–37, 41]. Lions-Peetre [20] showed that, if

$$u \in L_2(0, T; H_0), \quad u^{(m)} \in L_2(0, T; H),$$

then

$$u^{(i)} \in L_2(0, T; [H, H_0]_{\frac{i}{m}}), \quad i = 1, 2, \cdots, m-1,$$

where H_0, H are Hilbert spaces, H_0 is continuously and densely embedded in H and $[H_0, H]_\theta$ are interpolation spaces between H_0 and H for $0 \le \theta \le 1$. The similar questions for anisotropic Sobolev spaces $W^l_p(\Omega; H_0, H)$, $\Omega \subset R^n$ and for corresponding weighted spaces have been investigated in [31–34] and [24], respectively. Embedding theorems in Banach-valued Besov spaces have been studied in [3, 5, 35–37]. The solvability and the spectrum of boundary value problems for elliptic differential-operator equations (DOEs) have been refined in [3–7, 11, 31–34, 40–41]. A comprehensive introduction to DOEs and historical references may be found in [14,

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*Department of Mathematics, Okan University, Akfirat Beldesi, Tuzla, 34959, Istanbul, Turkey.
E-mail: veli.sahmurov@okan.edu.tr

16, 41]. In these works Hilbert-valued function spaces essentially have been considered. The maximal L_p regularity and fredholmness of partial elliptic equations in smooth regions have been studied, e.g., in [1, 2, 21]. For DOEs the similar problems have been investigated in [11, 29, 30–34, 40–41].

Let $l = (l_1, l_2, \cdots, l_n)$, $s = (s_1, s_2, \cdots, s_n)$ and $l_i > s_i$. Let A be a positive operator in a Banach spaces E with domain $D(A)$. In the present paper the Banach-valued Besov spaces $B^l_{p,\theta}(\Omega; D(A), E) = B^s_{p,\theta}(\Omega; D(A)) \cap B^l_{p,\theta}(\Omega; E)$ are introduced. The boundedness of embedding operators in this space for $\Omega = R^n$ was studied in [36]. In the present paper the most regular interpolation class E_α between E_0 and E are found so that the appropriate mixed differential operators D^α are bounded from $B^{s+l}_{p,\theta}(\Omega; D(A), E)$ to $B^s_{q,\theta}(\Omega; E(A^{1-\varkappa}))$ and $B^s_{q,\theta}(\Omega; (D(A), E)_{\varkappa,p})$ for domains $\Omega \subset R^n$. More precisely, the Ehrling-Nirenberg-Gagliardo type sharp estimates for parameterized norms are established; in turn which allows us to obtain the compactness of operator D^α from $B^{s+l}_{p,\theta}(\Omega; D(A), E)$ to

$$B^s_{q,\theta}(\Omega; D(A^{1-\varkappa-\mu})), \quad B^s_{q,\theta}(\Omega; (D(A), E)_{\varkappa+\mu,p})$$

for some $\mu > 0$. By applying these results, the B-separability of the anisotropic partial DOE with parameters in principal part are derived. The paper is organized as follows. Section 2 collects notations and definitions. Section 3 presents the embedding theorems in Besov-Lions type space $B^{s+l}_{p,\theta}(\Omega; D(A), E)$. Section 4 contains applications of the abstract embedding to vector-valued function spaces and Section 5 is devoted to uniform B-separability of the anisotropic DOE with parameters. Then by these results the uniform maximal B-regularity of parabolic Cauchy problem with parameters are shown. In Section 6, these DOE are applied to the BVP's and the Cauchy problem for finite and infinite systems of quasi-elliptic and parabolic PDE with parameters, respectively.

2 Notations and Definitions

Let E be a Banach space. Let $L_p(\Omega, E)$ denote the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset R^n$ with the norm

$$\|f\|_{L_p(\Omega;E)} = \Big(\int_\Omega \|f(x)\|^p_E \mathrm{d}x\Big)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|f\|_{L_\infty(\Omega;E)} = \operatorname*{ess\,sup}_{x\in\Omega} [\|f(x)\|_E], \quad x = (x_1, x_2, \cdots, x_n).$$

Let $S = S(R^n; E)$ denote a Schwartz class, i.e., the space of all E-valued rapidly decreasing smooth functions φ on R^n and $S'(R^n; E)$ denotes the space of all E-valued tempered distributions. Let $h \in R$, $m \in N$ and e_i, $i = 1, 2, \cdots, n$ be the standard unit vectors in R^n. Let (see [8, §16])

$$\begin{aligned}\Delta_i(h)f(x) &= f(x + he_i) - f(x), \cdots, \Delta^m_i(h)f(x)\\ &= \Delta_i(h)[\Delta^{m-1}_i(h)f(x)] = \sum_{k=0}^{m}(-1)^{m+k} C^k_m f(x + khe_i).\end{aligned}$$

Let

$$\Delta_i(\Omega,h)=\begin{cases}\Delta_i(h) & \text{for } [x,x+mye_i]\subset\Omega,\\ 0 & \text{for } [x,x+mye_i]\notin\Omega.\end{cases}$$

Let $L_p^*(E)$ denote the space of all E-valued function space such that

$$\|u\|_{L_p^*(E)}=\Big(\int_0^\infty \|u(t)\|_E^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}}<\infty.$$

Let m_i be positive integers, k_i be nonnegative integers, s_i be positive numbers and $m_i > s_i-k_i>0,\ i=1,2,\cdots,n,\ s=(s_1,s_2,\cdots,s_n),\ 1\le p\le\infty,\ 1\le q\le\infty,\ 0<y_0<\infty$. Let F denote the Fourier transform. The Banach-valued Besov space $B_{p,q}^s(\Omega;E)$ are defined as

$$\begin{aligned}B_{p,q}^s(\Omega;E)=\Big\{&f: f\in L_p(\Omega;E),\ \|f\|_{B_{p,q}^s(\Omega;E)}=\|f\|_{B_{p,q}^s}=\|f\|_{L_p(\Omega;E)}\\&+\sum_{i=1}^n\Big(\int_0^{h_0}h^{-[(s_i-k_i)q+1]}\|\Delta_i^{m_i}(h,\Omega)D_i^{k_i}f\|_{L_p(\Omega;E)}^q\mathrm{d}y\Big)^{\frac{1}{q}}<\infty \text{ for } 1\le q<\infty,\\&\text{and } \|f\|_{B_{p,q}^s(\Omega;E)}=\sum_{i=1}^n\sup_{0<h<h_0}\frac{\|\Delta_i^{m_i}(h,\Omega)D_i^{k_i}f\|_{L_p(\Omega;E)}}{h^{s_i-k_i}} \text{ for } q=\infty\Big\}.\end{aligned}$$

For $E=\mathbf{C}$ we obtain the scalar-valued anisotropic Besov space $B_{p,q}^s(\Omega)$ (see [8, §18]).

The Banach space E is said to be a UMD spaces (see [9, 10, 12, 26]) if the Hilbert operator

$$(Hf)(x)=\lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}\mathrm{d}y$$

is bounded in $L_p(R;E),\ p\in(1,\infty)$. The UMD spaces include, e.g. L_p, l_p spaces and the Lorentz spaces L_{pq}, $p,\ q\in(1,\infty)$.

A Banach space E has the property (α) (see e.g. [13]) if there exists a constant α such that

$$\Big\|\sum_{i,j=1}^N\alpha_{ij}\varepsilon_i\varepsilon_j'x_{ij}\Big\|_{L_2(\Omega\times\Omega';E)}\le\alpha\Big\|\sum_{i,j=1}^N\varepsilon_i\varepsilon_j'x_{ij}\Big\|_{L_2(\Omega\times\Omega';E)}$$

for all $N\in\mathbf{N}$, $x_{i,j}\in E$, $\alpha_{ij}\in\{0,1\}$, $i,j=1,2,\cdots,N$, and all choices of independent, symmetric, $\{-1,1\}$-valued random variables $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N,\varepsilon_1',\varepsilon_2',\cdots,\varepsilon_N'$ on probability spaces Ω, Ω'. For example the spaces $L_p(\Omega)$, $1\le p<\infty$ has the property (α).

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi=\{\lambda;\ \lambda\in\mathbf{C},\ |\arg\lambda|\le\varphi\}\cup\{0\},\quad 0\le\varphi<\pi.$$

A linear operator A is said to be a φ-positive in a Banach space E, with bound $M>0$, if $D(A)$ is dense on E and

$$\|(A+\lambda I)^{-1}\|_{L(E)}\le M(1+|\lambda|)^{-1},$$

where $\lambda\in S_\varphi,\varphi\in[0,\pi)$, I is the identity operator in E and $L(E)$ is the space of all bounded linear operators in E. Sometimes $A+\lambda I$ will be written as $A+\lambda$ and denoted by A_λ. It is known that there exists fractional powers A^θ of the positive operator A (see [38, §1.15.1]). Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm

$$\|u\|_{E(A^\theta)}=(\|u\|^p+\|A^\theta u\|^p)^{\frac{1}{p}},\quad 1\le p<\infty,\ -\infty<\theta<\infty.$$

Let E_0 and E be two Banach spaces. By $(E_0,E)_{\sigma,p}$, $0<\sigma<1, 1\leq p\leq\infty$, we will denote the interpolation spaces obtained from $\{E_1,E_2\}$ by the K-method (see [38, §1.3.1]).

Let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, where α_i are integers. An E-valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions of the generalized function $f\in S'(R^n,E)$, if the equality

$$\langle D^\alpha f,\varphi\rangle=(-1)^{|\alpha|}\langle f,D^\alpha\varphi\rangle \tag{2.1}$$

holds for all $\varphi\in S$.

By using (2.1) the following relations

$$F(D_x^\alpha f)=(\mathrm{i}\xi_1)^{\alpha_1}\cdots(\mathrm{i}\xi_n)^{\alpha_n}\widehat{f},\quad D_\xi^\alpha(F(f))=F[(-\mathrm{i}x_n)^{\alpha_1}\cdots(-\mathrm{i}x_n)^{\alpha_n}f] \tag{2.2}$$

are obtained for all $f\in S'(R^n;E)$.

Let $l=(l_1,l_2,\cdots,l_n)$, $s=(s_1,s_2,\cdots,s_n)$, where l_k's are integers and $s_k\in(0,\infty)$; let $W^lB^s_{p,q}(\Omega;E)$ denote an E-valued Sobolev-Besov space of all functions $u\in B^s_{p,q}(\Omega;E)$ such that they have the generalized derivatives $D_k^{l_k}u=\frac{\partial^{l_k}}{\partial x_k^{l_k}}u\in B^s_{p,q}(\Omega;E)$, $k=1,2,\cdots,n$ with the norm

$$\|u\|_{W^lB^s_{p,q}(\Omega;E)}=\|u\|_{B^s_{p,q}(\Omega;E)}+\sum_{k=1}^n\|D_k^{l_k}u\|_{B^s_{p,q}(\Omega;E)}<\infty.$$

Let E_0 be continuously and densely embedded into E. Let $W^lB^s_{p,q}(\Omega;E_0,E)$ denote a space all functions $u\in B^s_{p,q}(\Omega;E_0)\cap W^lB^s_{p,q}(\Omega;E)$ with the norm

$$\|u\|_{W^lB^s_{p,q}}=\|u\|_{W^lB^s_{p,q}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q}(\Omega;E_0)}+\sum_{k=1}^n\|D_k^{l_k}u\|_{B^s_{p,q}(\Omega;E)}<\infty.$$

Let $l_i>s_i$. $B^l_{p,q}(\Omega;E_0,E)$ is a space of all functions $u\in B^s_{p,q}(\Omega;E_0)\cap B^l_{p,q}(\Omega;E)$ with the norm

$$\|u\|_{B^l_{p,q}}=\|u\|_{B^l_{p,q}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q}(\Omega;E_0)}+\|u\|_{B^l_{p,q}(\Omega;E)}.$$

For $E_0=E$, the spaces $W^lB^s_{p,q}(\Omega;E_0,E)$, $B^l_{p,q}(\Omega;E_0,E)$ will be denoted by $W^lB^s_{p,q}(\Omega;E)$, $B^l_{p,q}(\Omega;E)$, respectively. Let $t=(t_1,t_2,\cdots,t_n)$, where $t_j>0$ are parameters. We define in $W^lB^s_{p,q}(\Omega;E_0,E)$, $B^l_{p,q}(\Omega;E_0,E)$ the parameterized norms

$$\|u\|_{W^lB^s_{p,q,t}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q}(\Omega;E_0)}+\sum_{k=1}^n\|t_kD_k^{l_k}u\|_{B^s_{p,q}(\Omega;E)},$$
$$\|u\|_{B^l_{p,q,t}}=\|u\|_{B^l_{p,q,t}(\Omega;E_0,E)}=\|u\|_{B^s_{p,q}(\Omega;E_0)}+\|u\|_{B^l_{p,q,t}(\Omega;E)},$$

respectively, where

$$\|f\|_{B^l_{p,q,t}(\Omega;E)}=\|f\|_{L_p(\Omega;E)}+\sum_{i=1}^n t_i\Big(\int_0^{h_0}h^{-[(l_i-k_i)q+1]}\|\Delta_i^{m_i}(h,\Omega)D_i^{k_i}f\|^q_{L_p(\Omega;E)}\mathrm{d}y\Big)^{\frac{1}{\theta}}<\infty,$$
$$1\leq\theta<\infty,\ 1\leq p<\infty,$$
$$\|f\|_{B^l_{p,q,t}(\Omega;E)}=\sum_{i=1}^n\sup_{0<h<h_0}\frac{t_i\|\Delta_i^{m_i}(h,\Omega)D_i^{k_i}f\|_{L_p(\Omega;E)}}{h^{l_i-k_i}}\quad\text{for } q=\infty.$$

Let m be a positive integer. Let $C(\Omega;E)$ and $C^{(m)}(\Omega;E)$ denote the spaces of all E-valued bounded continuous and m-times continuously differentiable bounded functions on Ω, respectively. Let E_1 and E_2 be two Banach spaces. A function

$$\Psi \in C^{(m)}(R^n; L(E_1, E_2))$$

is called a multiplier from $B^s_{p,\theta}(R^n;E_1)$ to $B^s_{q,\theta}(R^n;E_2)$, if there exists a constant $C > 0$ such that

$$\|F^{-1}\Psi(\xi)Fu\|_{B^s_{q,\theta}(R^n;E_2)} \le C\|u\|_{B^s_{p,\theta}(R^n;E_1)}$$

for all $u \in B^s_{p,\theta}(R^n;E_1)$. The set of all multipliers from $B^s_{p,\theta}(R^n;E_1)$ to $B^s_{q,\theta}(R^n;E_2)$ will be denoted by $M^{q,\theta}_{p,\theta}(s,E_1,E_2)$. For $E_1 = E_2 = E$ it will be denoted by $M^{q,\theta}_{p,\theta}(s,E)$. The scalar-valued and operator-valued multipliers in Banach-valued function spaces have been studied, e.g. in [20], [38, §2.2.2] and [3, 10, 12, 15, 23], respectively.

Example 2.1 We note that if $\delta \in C^\infty(R)$ with $\delta(y) \ge 0$ for all $y \ge 0$, $\delta(y) = 0$ for $|y| \le \frac{1}{2}$, $\delta(y) = 1$ for $y \ge 1$ and $\delta(-y) = -\delta(y)$ for all y, then $\delta \in M^{q,\theta}_{p,\theta}(s,R)$.

Let K be a domain in R^m and $h = (h_1, h_2, \cdots, h_m) \in K$. Let

$$H_k = \{\Psi_h \in M^{q,\theta}_{p,\theta}(s,E_1,E_2),\ h \in K\}$$

be a collection of multipliers in $M^{q,\theta}_{p,\theta}(s,E_1,E_2)$ depending on h. We say that H_k is a uniform collection of multipliers, if there exists a constant $C > 0$, independent of $h \in K$, such that

$$\|F^{-1}\Psi_h Fu\|_{B^s_{p,\theta}(R^n;E_2)} \le C\|u\|_{B^s_{q,\theta}(R^n;E_1)}$$

for all $h \in K$ and $u \in B^s_{p,\theta}(R^n;E_1)$.

Let $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ be multiindexes. We also define

$$V_n = \{\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in R^n,\ \xi_i \ne 0,\ i = 1, 2, \cdots, n\},$$
$$U_n = \{\beta : |\beta| \le n\}, \quad \xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2}, \cdots, \xi_n^{\beta_n}, \quad \nu = \frac{1}{p} - \frac{1}{q}.$$

Definition 2.1 *A Banach space E satisfies a B-multiplier condition with respect to p, q, θ and s (or with respect to p, θ and s for the case of $p = q$) when $\Psi \in C^n(R^n;B(E))$, $1 \le p \le q \le \infty$, $\beta \in U_n$ and $\xi \in V_n$, if the estimate*

$$|\xi_1|^{\beta_1+\nu}|\xi_2|^{\beta_2+\nu}\cdots|\xi_n|^{\beta_n+\nu}\|D^\beta\Psi(\xi)\|_{L(E)} \le C$$

implies $\Psi \in M^{q,\theta}_{p,\theta}(s,E)$.

It is well-known that there are Banach spaces satisfying the B-multiplier condition (for isotropic case), e.g. UMD spaces (see [3, 15]).

The expression $\|u\|_{E_1} \sim \|u\|_{E_2}$ means that there exist positive constants C_1 and C_2 such that

$$C_1\|u\|_{E_1} \le \|u\|_{E_2} \le C_2\|u\|_{E_1}$$

for all $u \in E_1 \cap E_2$.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be nonnegative and $l_1, l_2, \cdots, l_n$ be positive integers and

$$|\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}, \quad \varkappa = \sum_{k=1}^{n} \frac{\alpha_k + \frac{1}{p} - \frac{1}{q}}{l_k},$$
$$D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{k=1}^{n} \alpha_k.$$

Consider the anisotropic differential-operator equation with parameters

$$(L_t + \lambda)u = \sum_{k=1}^{n} a_k t_k D^{l_k} u + A_\lambda u + \sum_{|\alpha:l|<1} \prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} A_\alpha(x) D^\alpha u = f \tag{2.3}$$

in $B^s_{p,\theta}(R^n; E)$, where A, $A_\alpha(x)$ are possible unbounded operators in a Banach space E, a_k's are complex numbers, t_k's are positive and λ is complex parameter. For $l_1 = l_2 = \cdots = l_n$ we obtain the isotropic equations containing the elliptic class of DOE with parameters.

The function belonging to $B^{s+l}_{p,\theta}(R^n; E(A), E)$ and satisfying the equation (2.3) a.e. on R^n is said to be a solution of the equation (2.3) on R^n.

Definition 2.2 *The problem* (2.3) *is said to be uniform B-separable* (*or $B^s_{p,\theta}(R^n; E)$-separable*) *with respect to the parameter $t = (t_1, t_2, \cdots, t_n)$, if the problem* (2.3) *for all $f \in B^s_{p,\theta}(R^n; E)$ has a unique solution $u \in B^{s+l}_{p,\theta}(R^n; E(A), E)$ and there exists a positive constant C independent of f and t such that we have the coercive estimate*

$$\|Au\|_{B^s_{p,\theta}(R^n;E)} + \sum_{|\alpha:l|=1} t_k \|D_k^{l_k} u\|_{B^s_{p,\theta}(R^n;E)} \le C\|f\|_{B^s_{p,\theta}(R^n;E)}.$$

The above estimate implies that if $f \in B^s_{p,\theta}(R^n; E)$ and u is the solution of the BVP's (2.3) then all terms of the equation (2.3) belong to $B^s_{p,\theta}(R^n; E)$ (i.e., all terms are separable in $B^s_{p,\theta}(R^n; E)$).

Consider a parabolic Cauchy problem

$$D_y u(y, x) + (L_t + \lambda)u(y, x) = f(y, x), \quad u(0, x) = 0, \quad y \in R_+, \ x \in R^n, \tag{2.4}$$

where L_t is the realization differential operator in $B^s_{p,\theta}(R^n; E)$ generated by problem (2.3).

We say that the parabolic Cauchy problem (2.4) is maximal B-regular, if for all $f \in B^s_{p,\theta}(R^{n+1}_+; E)$ there exists a unique solution u satisfying (2.4) almost everywhere on R^{n+1}_+ and there exists a positive constant C independent of f, such that we have the estimate

$$\|D_y u(y, x)\|_{B^s_{p,\theta}(R^{n+1}_+;E)} + \|L_t u\|_{B^s_{p,\theta}(R^{n+1}_+;E)} \le C\|f\|_{B^s_{p,\theta}(R^{n+1}_+;E)}.$$

3 Embedding Theorems

In this section, we prove the boundedness of the mixed differential operators D^α in the Banach-valued Besov-Lions spaces. From [36, Lemma 1] we have

Lemma 3.1 *Let A be a positive operator on a Banach space E, b be a nonnegative real number and $r=(r_1,r_2,\cdots,r_n)$, $t=(t_1,t_2,\cdots,t_n)$, $0<t_k\le T<\infty$, $k=1,2,\cdots,n$, $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $l=(l_1,l_2,\cdots,l_n)$, where $\varphi\in[0,\pi)$, $r_k\in\{0,b\}$, l_k are positive and α_k are nonnegative integers such that $\varkappa=|(\alpha+r):l|\le 1$. Let δ be a multiplier of the form described in Example* 2.1. *For $0<h\le h_0<\infty$ and $0\le\mu\le 1-\varkappa$, the operator-function*

$$\Psi_t(\xi)=\Psi_{t,h,\mu}(\xi)=\prod_{k=1}^n\big[t_k^{\frac{\alpha_k+r_k}{l_k}}|\xi_k|^{r_k}\big](\mathrm{i}\xi)^\alpha A^{1-\varkappa-\mu}h^{-\mu}[A+\eta(t,\xi)]^{-1}$$

is a bounded operator in E uniformly with respect to ξ, h and t, i.e., there is a constant C_μ such that

$$\|\Psi_{t,h,\mu}(\xi)\|_{L(E)}\le C_\mu \tag{3.1}$$

for all $\xi\in R^n$, where

$$\eta=\eta(t,\xi)=\sum_{k=1}^n t_k[\delta(\xi_k)\xi_k]^{l_k}+h^{-1}.$$

Lemma 3.2 (see [36]) *Let E be a UMD space with (α) property, $p\in(1,\infty)$, $\theta\in[1,\infty]$ and let for all k, $j\in(1,n)$,*

$$\frac{s_k}{l_k+s_k}+\frac{s_j}{l_j+s_j}\le 1. \tag{3.2}$$

Then the spaces $B^{l+s}_{p,\theta}(R^n;E)$ and $W^lB^s_{p,\theta}(R^n;E)$ coincide.

Theorem 3.1 *Suppose that the following conditions hold:*

(1) *E is a UMD space with the (α) property satisfying the B-multiplier condition with respect to p, $q\in(1,\infty)$, $\theta\in[1,\infty]$ and s;*

(2) *$t=(t_1,t_2,\cdots,t_n)$, $0<t_k\le T<\infty$, $k=1,2,\cdots,n$, $0<h\le h_0<\infty$;*

(3) *$\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$, $l=(l_1,l_2,\cdots,l_n)$, $s=(s_1,s_2,\cdots,s_n)$, where α_k's are nonnegative, l_k's are positive integers and s_k's are positive numbers such that*

$$\varkappa=\Big|\Big(\alpha+\frac{1}{p}-\frac{1}{q}\Big):l\Big|\le 1,\quad \frac{s_k}{l_k+s_k}+\frac{s_j}{l_j+s_j}\le 1,\quad k,j\in(1,n)$$

and $0\le\mu\le 1-\varkappa$;

(4) *A is a φ-positive operator in E, where $\varphi\in[0,\pi)$.*

Then an embedding

$$D^\alpha B^{s+l}_{p,\theta}(R^n;E(A),E)\subset B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))$$

is continuous and there exists a positive constant C_μ, depending only on μ, such that

$$\prod_{k=1}^n t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|D^\alpha u\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))}\le C_\mu[h^\mu\|u\|_{B^{s+l}_{p,\theta,t}(R^n;E(A),E)}+h^{-(1-\mu)}\|u\|_{B^s_{p,\theta}(R^n;E)}] \tag{3.3}$$

for all $u\in B^{s+l}_{p,\theta}(R^n;E(A),E)$.

Proof We have

$$\|D^\alpha u\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))}=\|A^{1-\varkappa-\mu}D^\alpha u\|_{B^s_{q,\theta}(R^n;E)} \tag{3.4}$$

for all u such that

$$\|D^{\alpha}u\|_{B^{s}_{q,\theta}(R^{n};E(A^{1-\varkappa-\mu}))}<\infty.$$

On the other hand, using the relation (2.2), we have

$$\begin{aligned} A^{1-\alpha-\mu}D^{\alpha}u &= F^{-'}FA^{1-\varkappa-\mu}D^{\alpha}u = F^{-'}A^{1-\varkappa-\mu}FD^{\alpha}u \\ &= F^{-'}A^{1-\varkappa-\mu}(\mathrm{i}\xi)^{\alpha}Fu = F^{-'}(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}Fu. \end{aligned} \tag{3.5}$$

Hence, denoting Fu by $\widehat{u}$, we get from the relations (3.4) and (3.5)

$$\|D^{\alpha}u\|_{B^{s}_{q,\theta}(R^{n};E(A^{1-\varkappa-\mu}))}\backsim\|F^{-'}(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\|_{B^{s}_{q,\theta}(R^{n};E)}.$$

Similarly, by virtue of Lemma 3.2 we have

$$\begin{aligned} \|u\|_{B^{s+l}_{p,\theta,t}(R^{n};E(A),E)} &= \|u\|_{W^{l}B^{s}_{p,\theta,t}(R^{n};E(A),E)} \\ &= \|u\|_{B^{s}_{p,\theta}(R^{n};E(A))}+\sum_{k=1}^{n}\|t_kD_k^{l_k}u\|_{B^{s}_{p,\theta}(R^{n};E)} \\ &= \|F^{-'}\widehat{u}\|_{B^{s}_{p,\theta}(R^{n};E(A))}+\sum_{k=1}^{n}\|t_kF^{-'}[(\mathrm{i}\xi_k)^{l_k}\widehat{u}]\|_{B^{s}_{p,\theta}(R^{n};E)} \\ &\sim \|F^{-1}A\widehat{u}\|_{B^{s}_{p,\theta}(R^{n};E)}+\sum_{k=1}^{n}\|t_kF^{-'}[(\mathrm{i}\xi_k)^{l_k}\widehat{u}]\|_{B^{s}_{p,\theta}(R^{n};E)} \end{aligned}$$

for all $u\in B^{s+l}_{p,\theta}(R^{n};E(A),E)$. Thus proving the inequality (3.3) for some constants C_μ is equivalent to proving

$$\begin{aligned} &\prod_{k=1}^{n}t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|F^{-'}(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\|_{B^{s}_{q,\theta}(R^{n};E)} \\ &\le C_\mu\Big[h^{\mu}\Big(\|F^{-'}A\widehat{u}\|_{B^{s}_{p,\theta}(R^{n};E)}+\sum_{k=1}^{n}\|t_kF^{-'}[(\mathrm{i}\xi_k)^{l_k}\widehat{u}]\|_{B^{s}_{p,\theta}(R^{n};E)}\Big)+h^{-(1-\mu)}\|F^{-'}\widehat{u}\|_{B^{s}_{p,\theta}(R^{n};E)}\Big]. \end{aligned} \tag{3.6}$$

Since δ is a multiplier in $B^{s}_{p,\theta}(R^{n};E)$, the inequality (3.6) will follow if we prove the following inequality

$$\prod_{k=1}^{n}t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|F^{-'}[(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}]\|_{B^{s}_{p,\theta}(R^{n};E)}\le C_\mu\|F^{-'}[h^{\mu}(A+\eta)]\widehat{u}\|_{B^{s}_{p,\theta}(R^{n};E)} \tag{3.7}$$

for a suitable $C_\mu>0$ and for all $u\in B^{s+l}_{p,\theta}(R^{n};E(A),E)$, where $\eta=\eta(t,\xi)$ has the same expression as defined in Lemma 3.1. Let us express the left-hand side of (3.7) as follows:

$$\begin{aligned} &\prod_{k=1}^{n}t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|F^{-'}[(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}]\|_{B^{s}_{q,\theta}(R^{n};E)} \\ &=\prod_{k=1}^{n}t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|F^{-'}(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}[h^{\mu}(A+\eta)]^{-1}[h^{\mu}(A+\eta)]\widehat{u}\|_{B^{s}_{q,\theta}(R^{n};E)}. \end{aligned} \tag{3.8}$$

(Since A is a positive operator in E and $-\eta(t,\xi)\in S(\varphi)$, it is possible.) By virtue of Definition 2.1, it is clear that the inequality (3.4) will follow immediately from (3.8) if we can prove that the operator-function

$$\Psi_t=\Psi_{t,h,\mu}=\prod_{k=1}^{n} t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}(\mathrm{i}\xi)^{\alpha}A^{1-\varkappa-\mu}[h^{\mu}(A+\eta)]^{-1}$$

is a multiplier in $M_{p,\theta}^{q,\theta}(s,E)$, which is uniform with respect to $h>0$ and t. Then it suffices to show that there exists a constant $M_\mu>0$ with

$$|\xi_1|^{\beta_1+\nu}|\xi_2|^{\beta_2+\nu},\cdots,|\xi_n|^{\beta_n+\nu}\|D_\xi^{\beta}\Psi_t(\xi)\|_{L(E)}\le M_\mu \tag{3.9}$$

for all $\beta\in U_n$, $\xi\in V_n$, $0<t_k\le T<\infty$ and $0<h\le h_0<\infty$. To see this, we apply Lemma 3.1 and get a constant $M_\mu>0$ depending only on μ such that

$$|\xi|^{\nu}\|\Psi_t(\xi)\|_{L(E)}\le M_\mu \tag{3.10}$$

for all $\xi\in R^n$ and $\nu=\frac{1}{p}-\frac{1}{q}$. This shows that the inequality (3.9) is satisfied for $\beta=(0,\cdots,0)$. We next consider (3.9) for $\beta=(\beta_1,\cdots,\beta_n)$ where $\beta_k=1$ and $\beta_j=0$ for $j\ne k$. By differentiation of the operator-function $\Psi_t(\xi)$, by virtue of the positivity of A and by using (3.10), we have

$$\Big\|\frac{\partial}{\partial\xi_k}\Psi_t(\xi)\Big\|_{L(E)}\le M_\mu|\xi_k|^{-(1+\nu)},\quad k=1,2,\cdots,n.$$

Repeating the above process we obtain the estimate (3.9). Thus the operator-function $\Psi_{t,h,\mu}(\xi)$ is a uniform multiplier with respect to h and t, i.e.,

$$\Psi_{t,h,\mu}\in H_K\subset M_{p,\theta}^{q,\theta}(s,E),\quad K=R_+.$$

This completes the proof of Theorem 3.1.

It is possible to state Theorem 3.1 in a more general setting. For this, we use the conception of extension operator.

Condition 3.1 *Let a region $\Omega\subset R^n$ be such that there exists a bounded linear extension operator from $B_{p,\theta}^{s+l}(\Omega;E(A),E)$ to $B_{p,\theta}^{s+l}(R^n;E(A),E)$ for $p,q\in(1,\infty)$ and $\theta\in[1,\infty]$.*

Remark 3.1 If $\Omega\subset R^n$ is a region satisfying the strong l-horn condition (see [8, §18.5]) $E=R$, $A=I$, then there exists a bounded linear extension operator from $B_{p,\theta}^{s}(\Omega)=B_{p,\theta}^{s}(\Omega;R,R)$ to $B_{p,\theta}^{s}(R^n)=B_{p,\theta}^{s}(R^n;R,R)$.

Theorem 3.2 *Suppose that all conditions of Theorem* 3.1 *and Condition* 3.1 *hold. Then an embedding*

$$D^{\alpha}B_{p,\theta}^{s+l}(\Omega;E(A),E)\subset B_{q,\theta}^{s}(\Omega;E(A^{1-\varkappa-\mu}))$$

is continuous and there exists a constant C_μ depending only on μ such that

$$\prod_{k=1}^{n} t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|D^{\alpha}u\|_{B_{q,\theta}^{s}(\Omega;E(A^{1-\varkappa-\mu}))}\le C_\mu[h^{\mu}\|u\|_{B_{p,\theta,t}^{s+l}(\Omega;E(A),E)}+h^{-(1-\mu)}\|u\|_{B_{p,\theta}^{s}(\Omega;E)}] \tag{3.11}$$

for all $u\in B_{p,\theta}^{s+l}(\Omega;E(A),E)$.

Proof It suffices to prove the estimate (3.11). Let P be a bounded linear extension operator from $B^s_{q,\theta}(\Omega;E)$ to $B^s_{q,\theta}(R^n;E)$ and also from $B^{s+l}_{p,\theta}(\Omega;E(A),E)$ to $B^{s+l}_{p,\theta}(R^n;E(A),E)$. Let P_Ω be a restriction operator from R^n to Ω. Then for any $u\in B^{s+l}_{p,\theta}(\Omega;E(A),E)$, we have

$$\begin{aligned}\|D^\alpha u\|_{B^s_{q,\theta}(\Omega;E(A^{1-\varkappa-\mu}))} &= \|D^\alpha P_\Omega Pu\|_{B^s_{q,\theta}(\Omega;E(A^{1-\varkappa-\mu}))}\\ &\le C\|D^\alpha Pu\|_{B^s_{q,\theta}(R^n;E(A^{1-\varkappa-\mu}))}\\ &\le C_\mu[h^\mu\|Pu\|_{B^{s+l}_{p,\theta}(R^n;E(A),E)}+h^{-(1-\mu)}\|Pu\|_{B^s_{p,\theta}(R^n;E)}]\\ &\le C_\mu[h^\mu\|u\|_{B^{s+l}_{p,\theta}(\Omega;E(A)E)}+h^{-(1-\mu)}\|u\|_{B^s_{p,\theta}(\Omega;E)}].\end{aligned}$$

Result 3.1 *Let all conditions of Theorem 3.2 hold. Then for all $u\in B^{s+l}_{p,\theta}(\Omega;E(A),E)$, we have a multiplicative estimate*

$$\|D^\alpha u\|_{B^s_{q,\theta}(\Omega;E(A^{1-\varkappa-\mu}))}\le C_\mu\|u\|^{1-\mu}_{B^{s+l}_{p,\theta}(\Omega;E(A),E)}\|u\|^{\mu}_{B^s_{p,\theta}(\Omega;E)}. \tag{3.12}$$

Indeed setting $h=\|u\|_{B^s_{p,\theta}(\Omega;E)}\cdot\|u\|^{-1}_{B^{s+l}_{p,\theta}(\Omega;E(A),E)}$ in the estimate (3.11), we obtain (3.12).

Theorem 3.3 *Assume that all conditions of Theorem 3.2 are satisfied; let Ω be a bounded region in R^n and A^{-1} be a compact operator in E. Then for $0<\mu\le 1-\varkappa$, the embedding*

$$D^\alpha B^{s+l}_{p,\theta}(\Omega;E(A),E)\subset B^s_{q,\theta}(\Omega;E(A^{1-\varkappa-\mu}))$$

is compact.

Proof By virtue of [5], the embedding

$$B^{s+l}_{p,\theta}(\Omega;E(A),E)\subset B^s_{q,\theta}(\Omega;E)$$

is compact. Then in view of (3.12), we obtain the assertion of Theorem 3.3.

Theorem 3.4 *Suppose that all conditions of Theorem 3.2 hold and $\varphi\in[0,\pi)$. Then for $0<\mu<1-\varkappa$, the embedding*

$$D^\alpha B^{s+l}_{p,\theta}(\Omega;E(A),E)\subset B^s_{q,\theta}(\Omega;(E(A),E)_{\varkappa+\mu,p})$$

is continuous and there exists a constant C_μ depending only on μ such that

$$\|D^\alpha u\|_{B^s_{q,\theta}(\Omega;(E(A),E)_{\varkappa+\mu,p})}\le C_\mu[h^\mu\|u\|_{B^{s+l}_{p,\theta,t}(\Omega,E(A),E)}+h^{-(1-\mu)}\|u\|_{B^s_{p,\theta}(\Omega,E)}] \tag{3.13}$$

for all $u\in B^{s+l}_{p,\theta}(\Omega;E(A),E)$.

Proof Let us at first show the theorem for the case $\Omega=R^n$. Then it is sufficient to prove the estimate

$$\begin{aligned}&\prod_{k=1}^n t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}}\|D^\alpha u\|_{B^s_{q,\theta}(R^n;(E(A),E)_{\varkappa+\mu,p})}\\ &\le C_\mu[h^\mu\|u\|_{B^{s+l}_{p,\theta,t}(R^n;E(A),E)}+h^{-(1-\mu)}\|u\|_{B^s_{p,\theta}(R^n;E)}]\end{aligned} \tag{3.14}$$

for all $u \in B^{s+l}_{p,\theta}(R^n; E(A), E)$. By the definition of interpolation spaces $(E(A), E)_{\varkappa+\mu,p}$ (see [38, §1.14.5]) the estimate (3.14) is equivalent to the inequality

$$\prod_{k=1}^{n} t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}} \|F^{-1} y^{1-\varkappa-\mu-\frac{1}{p}}[A^{\chi+\mu}(A+y)^{-1}]\xi^\alpha \widehat{u}\|_{B^s_{q,\theta}(R^n;L_p(R_+;E))}$$
$$\le C_\mu \Big\| F^{-\prime}\Big[h^\mu\Big(A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k}\Big)+h^{-(1-\mu)}\Big]\widehat{u}\Big\|_{B^s_{p,\theta}(R^n;E)}. \tag{3.15}$$

The inequality (3.14) will follow immediately from (3.15), if we prove that the operator-function

$$\Psi_{t,h,\mu} = (\mathrm{i}\xi)^\alpha \prod_{k=1}^{n} t_k^{\frac{\alpha_k+\frac{1}{p}-\frac{1}{q}}{l_k}} y^{1-\varkappa-\mu-\frac{1}{p}}[A^{\chi+\mu}(A+y)^{-1}]\Big[h^\mu(A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k})+h^{-(1-\mu)}\Big]^{-1}$$

is a uniform collection of multiplier from $B^s_{p,\theta}(R^n; E)$ to $B^s_{q,\theta}(R^n; L_1(R_+; E))$. This fact is proved in a similar manner as Theorem 3.1. Therefore we get the estimate (3.15) which implies (3.14). Then by using the extension operator we obtain (3.13).

Result 3.2 *Let all conditions of Theorem* 3.2 *hold. Then for all* $u \in B^{s+l}_{p,\theta}(\Omega; E(A), E)$ *we have the multiplicative estimate*

$$\|D^\alpha u\|_{B^s_{q,\theta}(\Omega;(E(A),E)_{\varkappa+\mu,1})} \le C_\mu \|u\|^{1-\mu}_{B^{s+l}_{p,\theta}(\Omega;E(A),E)} \|u\|^\mu_{B^s_{p,\theta}(\Omega;E)}. \tag{3.16}$$

Indeed setting $h = \|u\|_{B^s_{p,\theta}(\Omega;E)} \cdot \|u\|^{-1}_{B^{s+l}_{p,\theta}(\Omega;E(A),E)}$ in (3.13) we obtain (3.16).

Theorem 3.5 *Assume that all conditions of Theorem* 3.4 *are satisfied,* Ω *is a bounded region in* R^n *and* A^{-1} *is a compact operator in* E*. Then for* $0 < \mu < 1 - \varkappa$*, the embedding*

$$D^\alpha B^{s+l}_{p,\theta}(\Omega; E(A), E) \subset B^s_{q,\theta}(\Omega; (E(A), E)_{\varkappa+\mu,p})$$

is compact.

Proof By virtue of [5], the embedding

$$B^{s+l}_{p,\theta}(\Omega; E(A), E) \subset B^s_{q,\theta}(\Omega; E)$$

is compact. Then by the estimate (3.16), we obtain the assertion of Theorem 3.5.

Remark 3.2 It seems from the proof of Theorem 3.1 that the extra condition to space E (E is a UMD space with (α) property) and the second inequality in condition (3) of Theorem 3.1 is due to Lemma 3.2. In fact, the (α) property condition for the space E are required with a view to using Marcinkiewicz-Lizorkin type multiplier theorem (see [13]) in $L_p(R^n; E)$ space. Note that both conditions occur due to anisotropic nature of the spaces $B^s_{p,q}$. For the isotropic case it is trivial.

4 Application to Vector-Valued Function Spaces

By virtue of Theorem 3.2, we have

Result 4.1 *For $A = I$, we obtain the continuity of the embedding $D^{\alpha}B^{s+l}_{p,q}(\Omega;E) \subset B^{s}_{p,q}(\Omega;E)$ and the corresponding estimate* (3.4) *for $0 \leq \mu \leq 1 - \varkappa$ in the Banach-valued Besov space $B^{s+l}_{p,q}(\Omega;E)$.*

Result 4.2 *For $E = R^m$, $A = I$ we obtain the following embedding $D^{\alpha}B^{l+s}_{p,\theta}(\Omega;R^m) \subset B^{s}_{q,\theta}(\Omega;R^m)$ for $0 \leq \mu \leq 1 - \varkappa$ and the corresponding estimate* (3.4)*. For $E = R$, $A = I$ we get the embedding $D^{\alpha}B^{l+s}_{p,\theta}(\Omega) \subset B^{s}_{q,\theta}(\Omega)$ proved in* [8, §18] *for the numerical Besov spaces.*

Result 4.3 *Let $l_1 = l_2 = \cdots = l_n = m$, $s_1 = s_2 = \cdots = s_n = \sigma$ and $p = q$. Then we obtain the continuity of embedding $D^{\alpha}B^{\sigma+m}_{p,\theta}(\Omega;E(A),E) \subset B^{\sigma}_{p,\theta}(\Omega;E(A^{1-\frac{|\alpha|}{m}}))$ and the corresponding estimate* (3.4) *for $|\alpha| \leq m$, in isotropic Besov-Lions spaces $B^{\sigma+m}_{p,\theta}(\Omega;E(A),E)$.*

Result 4.4 *Let σ be a positive number. Consider the following space* (see [37, §1.18.2])

$$l^{\sigma}_{q} = \{u;\ u = \{u_i\},\ i = 1,2,\cdots,\infty,\ u_i \in \mathbf{C}\}$$

with the norm

$$\|u\|_{l^{\sigma}_{q}} = \Big(\sum_{i=1}^{\infty} 2^{iq\sigma}|u_i|^q\Big)^{1/q} < \infty.$$

Note that $l^{0}_{q} = l_q$. Let A be the infinite matrix defined in l_q such that

$$D(A) = l^{\sigma}_{q}, \quad A = [\delta_{ij}2^{si}],$$

where $\delta_{ij} = 0$, when $i \neq j$, $\delta_{ij} = 1$, when $i = j$, $i,j = 1,2,\cdots,\infty$. It is clear to see that this operator A is positive in l_q. Then by Theorem 3.2 *we obtain the embedding*

$$D^{\alpha}B^{l+s}_{p_1,\theta}(\Omega;l^{\sigma}_{q},l_q) \subset B^{s}_{p_2,\theta}(\Omega;l^{\sigma(1-\varkappa-\mu)}_{q}), \quad \varkappa = \sum_{k=1}^{n}\frac{\alpha_k + \frac{1}{p_1} - \frac{1}{p_2}}{l_k},$$

and the corresponding estimate (3.4)*, where $0 \leq \mu \leq 1 - \varkappa$.*

It should be noted that the above embedding has not been obtained by classical methods so far.

5 Maximal *B*-Regular DOE in R^n

Let us consider the differential-operator equations with parameters

$$L_t u = \sum_{k=1}^{n} a_k t_k D^{l_k}_{k} u + A_{\lambda}u + \sum_{|\alpha:l|<1}\prod_{k=1}^{n} t_k^{\frac{\alpha_k}{l_k}} A_{\alpha}(x)D^{\alpha}u = f \tag{5.1}$$

in $B^{s}_{p,\theta}(R^n,E)$, where $A_{\lambda} = A + \lambda I$, $\lambda \in S(\varphi_0)$, A and $A_{\alpha}(x)$ are possible unbounded operators in Banach space E, a_k's are complex numbers, t_k, $k = 1,2,\cdots,n$, are parameters, $l = (l_1,l_2,\cdots,l_n)$, l_i's are positive integers.

Condition 5.1 *Let $-\sum\limits_{k=1}^{n} a_k t_k(\mathrm{i}\xi_k)^{l_k} \in S(\varphi_1)$, $\varphi_0 + \varphi_1 \leq \varphi$ and there is $C > 0$ such that $\Big|\sum\limits_{k=1}^{n} a_k t_k(\mathrm{i}\xi_k)^{l_k}\Big| \geq C\sum\limits_{k=1}^{n} t_k|\xi_k|^{l_k}$ for all $\xi = (\xi_1,\xi_2,\cdots,\xi_n) \in R^n$ and $t_k \in (0,T]$, $T < \infty$.*

Remark 5.1 If $l_k = 2m_k$, $a_k = (-1)^{m_k}$, Condition 5.1 holds for some $\varphi \in [0, \pi)$.

Theorem 5.1 *Suppose that the following conditions hold:*

(1) *Condition* 5.1 *holds and* $s > 0$, $p \in (1, \infty)$, $\theta \in [1, \infty]$, $0 < t_k \le T < \infty$;

(2) *E is a UMD space with (α) property satisfying the B-multiplier condition with respect to $p \in (1, \infty)$, $\theta \in [1, \infty]$ and s; moreover*

$$\frac{s_k}{l_k + s_k} + \frac{s_j}{l_j + s_j} \le 1, \quad k, j \in (1, n);$$

(3) *A is a φ-positive operator in E and*

$$A_\alpha(x) A^{-(1-|\alpha:l|-\mu)} \in L_\infty(R^n; L(E)), \quad 0 < \mu < 1 - |\alpha : l|.$$

Then for all $f \in B^s_{p,\theta}(R^n; E)$, for $\lambda \in S(\varphi_0)$ and for sufficiently large $|\lambda|$, the problem (5.1) *has a unique solution $u(x)$ that belongs to space $B^{s+l}_{p,\theta}(R^n; E(A), E)$ and the coercive uniform estimate for the solution of* (5.1)

$$\sum_{k=1}^{n} t_k \|D_k^{l_k} u\|_{B^s_{p,\theta}(R^n;E)} + \|Au\|_{B^s_{p,\theta}(R^n;E)} \le C\|f\|_{B^s_{p,\theta}(R^n;E)} \tag{5.2}$$

holds with respect to t and λ.

Proof At first, we will consider principal part of the equation (5.1), i.e., the differential-operator equation

$$L_0 u = \sum_{k=1}^{n} a_k t_k D_k^{l_k} u + A_\lambda u = f. \tag{5.3}$$

Then by applying Fourier transform to the equation (5.3) with respect to $x = (x_1, \cdots, x_n)$, we obtain

$$\sum_{k=1}^{n} a_k t_k (\mathrm{i}\xi_k)^{l_k} \widehat{u}(\xi) + A_\lambda \widehat{u}(\xi) = f^\wedge(\xi). \tag{5.4}$$

Since $-\sum\limits_{k=1}^{n} a_k t_k (\mathrm{i}\xi_k)^{l_k} \in S(\varphi)$ for all $\xi = (\xi_1, \cdots, \xi_n) \in R^n$, we have

$$\omega = \omega(t, \lambda, \xi) = -\Big(\lambda + \sum_{k=1}^{n} a_k t_k (\mathrm{i}\xi_k)^{l_k}\Big) \in S(\varphi).$$

That is, the operator $A - \omega I$ is invertible in E. Hence (5.4) implies that the solution of the equation (5.3) can be represented in the form

$$u(x) = F^{-1}(A - \omega I)^{-1} f^\wedge.$$

It is clear to see that the operator-function $\varphi_{\lambda,t}(\xi) = [A - \omega I]^{-1}$ is the multiplier in $B^s_{p,\theta}(R^n; E)$ uniformly with respect to $\lambda \in S(\varphi_0)$. Actually, by virtue of the positivity of operator A and in view of [11, Lemma 2.3], we have

$$\|\varphi_\lambda(\xi)\|_{L(E)} = \|(A - \omega I)^{-1}\| \le M(1 + |\omega|)^{-1} \le M_0.$$

Moreover, it is clear to see that

$$\|\xi_k D_k \varphi_{\lambda,t}\|_{L(E)} \leq l_k t_k |a_k| |\xi_k|^{l_k} \|(A-\omega I)^{-2}\| \leq M. \tag{5.5}$$

Using the estimate (5.5), we obtain the uniform estimate

$$|\xi_1|^{\beta_1} |\xi_2|^{\beta_2} \cdots |\xi_n|^{\beta_n} \|D_\xi^\beta \varphi_{\lambda,t}(\xi)\|_{L(E)} \leq C \tag{5.6}$$

for $\beta = (\beta_1, \cdots, \beta_n) \in U_n$ and $\xi = (\xi_1, \cdots, \xi_n) \in V_n$ with respect to parameters t and λ. In a similar way we prove that for operator-functions $\varphi_{k\lambda,t}(\xi) = \xi_k^{l_k} \varphi_{\lambda,t}$, $k = 1, 2, \cdots, n$ and $\varphi_{0\lambda,t} = A\varphi_{\lambda,t}$ the estimates of type (5.6) are satisfied. So, we conclude that operator-functions $\varphi_{\lambda,t}, \varphi_{k\lambda,t}, \varphi_{0,\lambda,t}$ are uniform multipliers in $B_{p,\theta}^s(R^n; E)$ with respect to t and λ. It is easy to see that

$$\|D_k^{l_k} u\|_{B_{p,\theta}^s} = \|F^{-1}(\mathrm{i}\xi_k)^{l_k} \widehat{u}\|_{B_{p,\theta}^s} = \|F^{-1}(\mathrm{i}\xi_k)^{l_k} (A-\omega I)^{-1} f^\wedge\|_{B_{p,\theta}^s},$$
$$\|Au\|_{B_{p,\theta}^s} = \|F^{-1} A\widehat{u}\|_{B_{p,\theta}^s} = \|F^{-1}[A(A-\omega I)^{-1}] f^\wedge\|_{B_{p,\theta}^s}.$$

We obtain that for all $f \in B_{p,\theta}^s(R^n; E)$ there exists a unique solution of the equation (5.3) in the form

$$u(x) = F^{-1}(A-\omega I)^{-1} f^\wedge,$$

and the estimate

$$\sum_{k=1}^n t_k \|D_k^{l_k} u\|_{B_{p,\theta}^s} + \|Au\|_{B_{p,\theta}^s} \leq C\|f\|_{B_{p,\theta}^s} \tag{5.7}$$

holds. Consider in $B_{p,\theta}^s(R^n; E)$ the differential operator L_{0t} generated by the problem (5.3), that is,

$$D(L_{0t}) = B_{p,\theta}^{s+l}(R^n; E(A), E), \quad L_{0t} u = \sum_{k=1}^n t_k a_k D_k^{l_k} u + A_\lambda u.$$

Let L denote the differential operator in $B_{p,\theta}^s(R^n; E)$ generated by the problem (5.1). Namely,

$$D(L_t) = B_{p,\theta}^{s+l}(R^n; E(A), E), \quad L_t u = L_{0t} u + L_{1t} u,$$

where

$$L_{1t} u = \sum_{|\alpha:l|<1} \prod_{k=1}^n t_k^{\frac{\alpha_k}{l_k}} A_\alpha(x) D^\alpha u.$$

In view of the condition (3) of Theorem 5.1, by virtue of Theorem 3.1 for all $u \in B_{p,\theta}^{s+l}(R^n; E(A), E)$ we have

$$\|L_{1t} u\|_{B_{p,\theta}^s} \leq \sum_{|\alpha:l|<1} \prod_{k=1}^n t_k^{\frac{\alpha_k}{l_k}} \|A_\alpha(x) D^\alpha u\|_{B_{p,\theta}^s} \leq \sum_{|\alpha:l|<1} \prod_{k=1}^n t_k^{\frac{\alpha_k}{l_k}} \|A^{1-|\alpha:l|-\mu} D^\alpha u\|_{B_{p,\theta}^s}$$
$$\leq C\Big[h^\mu \Big(\sum_{k=1}^n t_k \|D_k^{l_k} u\|_{B_{p,\theta}^s} + \|Au\|_{B_{p,\theta}^s}\Big) + h^{-(1-\mu)} \|u\|_{B_{p,\theta}^s}\Big]. \tag{5.8}$$

Then from the estimates (5.7) and (5.8) and for $u \in B_{p,\theta}^{s+l}(R^n; E(A), E)$, we obtain

$$\|L_{1t} u\|_{B_{p,\theta}^s} \leq C[h^\mu \|(L_{0t} + \lambda) u\|_{B_{p,\theta}^s} + h^{-(1-\mu)} \|u\|_{B_{p,\theta}^s}]. \tag{5.9}$$

Since $\|u\|_{B^s_{p,\theta}} = \frac{1}{\lambda}\|(L_{0t}+\lambda)u - L_{0t}u\|_{B^s_{p,\theta}}$ for all $u \in B^{s+l}_{p,\theta}(R^n; E(A), E)$, we get

$$\begin{aligned} \|u\|_{B^s_{p,\theta}} &\le \frac{1}{|\lambda|}[\|(L_{0t}+\lambda)u\|_{B^s_{p,\theta}} + \|L_{0t}u\|_{B^s_{p,\theta}}], \\ \|L_{0t}u\|_{B^s_{p,\theta}} &\le C\Big[\sum_{k=1}^n t_k\|D_k^{l_k}u\|_{B^s_{p,\theta}} + \|Au\|_{B^s_{p,\theta}}\Big]. \end{aligned} \tag{5.10}$$

Then from (5.9)–(5.10) for $u \in B^{s+l}_{p,\theta}(R^n; E(A), E)$, we obtain

$$\|L_{1t}u\| \le Ch^\mu\|(L_{0t}+\lambda)u\|_{B^s_{p,\theta}} + C_1|\lambda|^{-1}h^{-(1-\mu)}\|(L_{0t}+\lambda)u\|_{B^s_{p,\theta}}. \tag{5.11}$$

By virtue of the estimate (5.7) we conclude that the operator $L_{0t}+\lambda$ for $\lambda \in S(\varphi_0)$ is invertible. Then choosing h and λ such that $Ch^\mu < 1,\ C_1|\lambda|^{-1}h^{-(1-\mu)} < 1$ in (5.11), we obtain the uniform estimate

$$\|L_{1t}(L_{0t}+\lambda)^{-1}\|_{L(F)} < 1, \quad F = B^s_{p,\theta}(R^n; E), \tag{5.12}$$

with respect to parameters t and λ. The estimate (5.7) implies that the operator $L_{0t}+\lambda$ for $\lambda \in S(\varphi_0)$ has a bounded inverse from $B^s_{p,\theta}(R^n; E)$ into $B^{s+l}_{p,\theta}(R^n; E(A), E)$. Then by using (5.12) and the perturbation theory of linear operators (see [17]), we obtain that the differential operator $L_t+\lambda$ is invertible from $B^s_{p,\theta}(R^n, E)$ into $B^{s+l}_{p,\theta}(R^n; E(A), E)$ and there is a positive constant C such that the uniform estimate

$$\|(L_t+\lambda)^{-1}\|_{L(F)} \le C$$

holds with respect to t and λ. This implies the estimate (5.2).

Result 5.1 *Theorem* 5.1 *implies that the differential operator L_t has a resolvent operator $(L+\lambda)^{-1}$ for $\lambda \in S(\varphi_0)$ and sufficiently large $|\lambda|$, and the coercive uniform estimate*

$$\sum_{|\alpha:l|\le 1} |\lambda|^{1-|\alpha:l|}\prod_{k=1}^n t_k^{\frac{\alpha_k}{l_k}}\|D^\alpha(L_t+\lambda)^{-1}\|_F + \|A(L_t+\lambda)^{-1}\|_F \le C$$

holds with respect to t and λ.

Theorem 5.2 *Let all conditions of Theorem* 5.1 *hold for $\varphi \in (0, \frac{\pi}{2})$. Then the parabolic Cauchy problem* (2.4) *for $\lambda \in S(\varphi_0)$ and sufficiently large $|\lambda|$ is maximal B-regular.*

Proof Really, the problem (2.4) can be express in space $B^s_{p,\theta}(R_+; F)$ in the following form

$$\frac{\mathrm{d}u(y)}{\mathrm{d}y} + (L_t+\lambda) \quad u(y) = f(t), \quad u(0) = 0, \quad y > 0,$$

where $F = L_p(G; E)$ and L_t is the differential operator in $B^s_{p,\theta}(R^n; E)$ generated by problem (5.1). In view of Result 4.3 the operator L is positive in $B^s_{p,\theta}(R^n; E)$ for $\varphi \in (0, \frac{\pi}{2})$. Then by virtue of [3, Corollary 8.9], we obtain the assertion.

Remark 5.2 There are a lot of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of E and concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of operator A on DOE (5.1), by virtue of Theorem 5.1, we can obtain the maximal regularity of different class of BVP's for partial differential equations or system of equations. Here we give some of its applications.

6 The Applications of Differential-Operator Equations

6.1 Infinite systems of quasielliptic equations

Consider the following infinity systems of boundary value problem with parameters

$$(L_t+\lambda)u_m(x)=\sum_{k=1}^{n}t_ka_kD_k^{l_k}u_m+\sum_{j=1}^{\infty}(d_j+\lambda)u_j(x)$$
$$+\sum_{|\alpha:l|<1}\sum_{j=1}^{\infty}\prod_{k=1}^{n}t_k^{\frac{\alpha_k}{l_k}}d_{\alpha jm}(x)D^{\alpha}u_j(x)=f_m(x),\quad x\in R^n,\ m\in\mathbf{N}. \tag{6.1}$$

Let

$$D=\{d_m\},\quad d_m>0,\quad u=\{u_m\},\quad Du=\{d_mu_m\},\quad m\in\mathbf{N},$$
$$l_q(D)=\Big\{u:u\in l_q,\ \|u\|_{l_q(D)}=\|Du\|_{l_q}=\Big(\sum_{m=1}^{\infty}|d_mu_m|^q\Big)^{\frac{1}{q}}<\infty\Big\},$$
$$\lambda\in S(\varphi_0),\quad x\in G,\quad 1<q<\infty.$$

Let O_t denote a differential operator in $B_{p,\theta}^{s}(R^n;l_q)$ generated by problem (6.1). Let

$$B=L(B_{p,\theta}^{s}(R^n;l_q)).$$

Theorem 6.1 *Let Condition* 5.1 *holds. Let* $\frac{s_k}{l_k+s_k}+\frac{s_j}{l_j+s_j}\leq 1$ *for* $k,j=1,2,\cdots,n,\ p,q\in(1,\infty),\ \theta\in[1,\infty]$ *and* $d_{\alpha km}\in L_\infty(R^n)$ *such that for all* $x\in G$,

$$\sum_{m=1}^{\infty}d_m^{-1}<\infty,\quad \sum_{j=1}^{\infty}\sum_{m=1}^{\infty}d_{\alpha jm}^{q_1}(x)d_m^{-\frac{q_1}{2}}<\infty,\quad \frac{1}{q}+\frac{1}{q_1}=1.$$

Then,

(a) *for all* $f(x)=\{f_m(x)\}_1^\infty\in B_{p,\theta}^{s}(R^n;l_q)$, $\lambda\in S(\varphi_0)$ *and for sufficiently large* $|\lambda|$ *the problem* (6.1) *has a unique solution* $u=\{u_m(x)\}_1^\infty$ *that belongs to* $B_{p,\theta}^{s+l}(R^n,l_q(D),l_q)$ *and coercive uniform estimate for the solution of* (6.1)

$$\sum_{|\alpha:l|\leq 1}\|D^\alpha u\|_{B_{p,\theta}^{s}(R^n;l_q)}+\|Du\|_{B_{p,\theta}^{s}(R^n;l_q)}\leq C\|f\|_{B_{p,\theta}^{s}(R^n;l_q)} \tag{6.2}$$

holds with respect to parameter t;

(b) *for* $\lambda\in S(\varphi_0)$ *and for sufficiently large* $|\lambda|$, *there exists a resolvent* $(O_t+\lambda)^{-1}$ *of operator* O_t *and*

$$\sum_{|\alpha:l|\leq 1}\prod_{k=1}^{n}t_k^{\frac{\alpha_k}{l_k}}(1+|\lambda|)^{1-|\alpha:l|}\|D^\alpha(O_t+\lambda)^{-1}\|_B+\|D(O_t+\lambda)^{-1}\|_B\leq M. \tag{6.3}$$

Proof Really, let $E=l_q$, $A(x)$ and $A_\alpha(x)$ be infinite matrices, such that

$$A=[d_m\delta_{jm}],\quad A_\alpha(x)=[d_{\alpha jm}(x)],\quad k,m\in\mathbf{N}.$$

It is clear to see that the operator A is positive in l_q. Therefore, by virtue of Theorem 5.1, we obtain that the problem (6.1) for all $f\in B_{p,\theta}^{s}(R^n;l_q)$, $\lambda\in S(\varphi_0)$ and sufficiently large $|\lambda|$ has a unique solution u that belongs to space $B_{p,\theta}^{s+l}(R^n;l_q(D),l_q)$ and the estimate (6.2) holds. By virtue of Result 5.1 we obtain (6.3).

6.2 Cauchy problems for infinite systems of parabolic equations

Consider the following infinity systems of parabolic Cauchy problem

$$\frac{\partial u_m(y,x)}{\partial y}+\sum_{k=1}^{n}a_k t_k\frac{\partial^{l_k}u_m(y,x)}{\partial x_k}+\sum_{|\alpha:l|<1}\sum_{j=1}^{\infty}\prod_{k=1}^{n}t_k^{\frac{\alpha_k}{l_k}}d_{\alpha jm}(x)D^{\alpha}u_j(x)$$
$$+\sum_{j=1}^{\infty}(d_j+\lambda)u_j(y,x)=f_m(y,x),\quad u_m(0,x)=0,\quad m\in\mathbf{N},\ y\in R_+,\ x\in R^n. \tag{6.4}$$

Theorem 6.2 *Let all conditions of Theorem* 6.1 *hold. Then the parabolic systems* (6.4) *for* $\lambda\in S(\varphi_0)$ *and for sufficiently large* $|\lambda|$ *is maximal* B*-regular.*

Proof Let $E=l_q$, A and $A_k(x)$ be the infinite matrices, such that

$$A=[d_m\delta_{jm}],\quad A_\alpha(x)=[d_{\alpha jm}(x)],\quad j,m=1,2,\cdots,\infty.$$

Then the problem (6.4) can be expressed as the equation (2.4), where L_t is a differential operator in $B^s_{p,\theta}(R^n;l_q)$ generated by the problem (6.1). Then by virtue of Theorem 5.1 and Theorem 5.2 we obtain the assertion.

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