Chin. Ann. Math.
29B(2), 2008, 113–120
DOI: 10.1007/s11401-007-0508-1

**Chinese Annals of Mathematics, Series B**


# Smoothness of the Functional Law Generated by a Nonlinear SPDE***

Marta SANZ-SOLÉ* Paul MALLIAVIN**

**Abstract** The authors consider a stochastic heat equation in dimension $d = 1$ driven by an additive space time white noise and having a mild nonlinearity. It is proved that the functional law of its solution is absolutely continuous and possesses a smooth density with respect to the functional law of the corresponding linear SPDE.

**Keywords** Stochastic heat equation, Probability law, Absolute continuity, Divergence operator, Gradient operator
**2000 MR Subject Classification** 60H07, 60H15, 60H60

## 1 Introduction

The smoothness of the law associated to a nonlinear SPDE has been initiated by Ocone in [10] and has been the object of many important subsequent works, for instance in [2, 5–7, 9, 11–13] to cite a sample. This question is generally tackled by using Malliavin Calculus. As Malliavin Calculus is established for the law of $\mathbb{R}^d$-valued random variables, the scope of the present stage of knowledge is essentially limited to finite dimensional evaluation of the underlying functional laws. The problem of extending Malliavin Calculus to infinite dimensional range functionals has been considered in [1] in an abstract setting; this work proceeds from the same motivation raised into a concrete setting. Absolute continuity will result from Girsanov Theorem while the smoothness will result from arguments of an infinite dimensional linear version of Malliavin Calculus.

## 2 Results

Let $M$ be either a connected differentiable manifold of dimension $d$ or a compact domain of $\mathbb{R}^d$ with smooth boundary. We consider on $M$ a measure $\mathrm{d}m$ which is absolutely continuous with respect to the Lebesgue measure and a second order elliptic operator $\Delta$ which has a natural self-adjoint extension on $L^2(\mathrm{d}m)$. The spectrum of $-\Delta$ is a sequence $\{\lambda_n, n \geq 1\}$ of positive numbers such that $\lim\limits_{n\to\infty} \lambda_n = +\infty$; more precisely, the spectral function $N_\Delta(\tau) := \text{cardinal}\{n : \lambda_n < \tau\}$,

Manuscript received December 5, 2007. Published online January 28, 2008.
*Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, E-08007 Barcelona, Spain.
E-mail: marta.sanz@ub.edu
**10, rue Saint Louis-en-l'Ile, F-74004 Paris, France. E-mail: sli@ccr.jussieu.fr
***Project supported by the grant MTM 2006-01351 from the Dirección General de Investigación, Ministerio de Educación y Ciencia, Spain.

$\tau > 0$, satisfies

$$N_\Delta(\tau) \le C\tau^{\frac{d}{2}}.$$

For any $n \ge 1$, we shall denote by $e_n$ the normalized eigenvector corresponding to $\lambda_n$. Define

$$W(t,x) = \sum_{n=1}^{\infty} x_n(t) e_n(x),$$

where $\{x_n(t),\ t \ge 0\}$, $n \ge 1$, is a sequence of standard independent real-valued Brownian motions defined on some probability space, and the series converges in the sense of distributions. The process $\{W(t,x),\ (t,x) \in \mathbb{R}_+ \times M\}$ is termed space-time white noise. It will be identified with the sequence $\{x_n(t),\ t \ge 0\}$, $n \ge 1$. We shall denote by $\mathbf{H}$ its associated Cameron-Martin space and by $(\Omega, \mathbf{H}, P)$ the corresponding abstract Wiener space. In the sequel, we shall consider as reference probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the $\sigma$-field generated by $\{x_n(t),\ t \in \mathbb{R}_+;\ n \ge 1\}$. We will also consider the natural filtration associated with $\{(x_n(s),\ n \ge 1),\ 0 \le s \le t;\ n \ge 1\}$ that we will denote by $(\mathcal{F}_t,\ t \in \mathbb{R}_+)$.

## 2.1 A linear SPDE

Consider the SPDE

$$\mathcal{L}u(t,x) = \dot{W}(t,x), \tag{2.1}$$

$t > 0$, $x \in M$, $\mathcal{L} = \frac{\partial}{\partial t} - \Delta$, with $u(0,\cdot) = 0$ and vanishing boundary conditions in the case of considering the equation in a bounded domain. Let $G(t,x)$ be the Green function associated with the heat operator $\mathcal{L}$. Classical results on SPDE's (see [3, 14]) provide a rigourous formulation of the solution of (2.1) by means of the stochastic convolution as follows:

$$u(t,x) = \int_0^t \int_M G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y). \tag{2.2}$$

Notice that $u(t)$ has a realization in $L^2(M)$ if and only if the integral $\int_0^\infty \mathrm{d}N_\Delta(\tau)$ converges. This property holds true if and only if $d = 1$. In dimensions greater than one, an $L^2(M)$-version of the stochastic convolution can be obtained by colouring the noise.

The next two propositions fix the setting of our main result.

**Proposition 2.1** *The process defined in* (2.2) *admits the representation*

$$u(t) = \sum_{n=1}^{\infty} u_n(t)\, e_n, \tag{2.3}$$

*where* $(u_n, n \ge 1)$ *is the sequence of independent Ornstein-Uhlenbeck processes given by the solution of the stochastic differential equations*

$$\begin{aligned} \mathrm{d}u_n(t) &= \mathrm{d}x_n(t) - \lambda_n\, u_n(t)\mathrm{d}t, \quad t > 0, \\ u_n(0) &= 0. \end{aligned}$$

*That is,*

$$u_n(t) = \int_0^t e^{-\lambda_n (t-s)} x_n(\mathrm{d}s). \tag{2.4}$$

*Assume $d = 1$. Then the mapping $t \in [0, \infty[ \mapsto u(t) \in L^2(M)$ is continuous. Moreover, fix $t > 0$ and denote by $\mu_t$ the probability law on $\mathbb{R}^{\otimes \mathbb{N}}$ of the sequence $\{u_n(t),\ n \geq 1\}$; then $\mu_t$ is absolutely continuous with respect to the measure $\otimes^n \mathrm{N}(0, \frac{1}{2\lambda_n})$ and the density $q_t = \frac{\mathrm{d}\mu_t}{\mathrm{d}\mu_\infty}$ belongs to $\Big(\bigcap_{p\geq 1} L^p(\mu_\infty)\Big) \cap \mathbb{D}^{1,2}(\mu_\infty)$.*

**Proof** A series expansion of the Green function gives

$$u(t,x) = \sum_{n\geq 1} \int_0^t \langle G(t-s, x - \cdot\,), e_n \rangle x_n(\mathrm{d}s),$$

where the notation $\langle \cdot\,, \cdot \rangle$ denotes the scalar product in $L^2(M)$.

Notice that by its very definition,

$$\langle G(t, x - \cdot\,), e_n \rangle = e_n(x) \mathrm{e}^{-\lambda_n t}, \quad t \in \mathbb{R}_+.$$

Consequently,

$$u(t,x) = \sum_{n\geq 1} \Big( \int_0^t \mathrm{e}^{-\lambda_n(t-s)} x_n(\mathrm{d}s) \Big) e_n(x),$$

which proves (2.3).

The isometry property of the stochastic integral yields

$$E(u_n(t))^2 = \frac{1}{2\lambda_n}(1 - \mathrm{e}^{-2\lambda_n t}) \to \frac{1}{2\lambda_n}, \quad \text{as } t \to \infty.$$

Hence the invariant measure of $u_n$ is the Gaussian law $\mathrm{N}(0, \frac{1}{2\lambda_n})$.

For the properties on the smoothness of the density $q_t$, we refer the reader to Theorem 1, Lemma 1 and Theorem 4 in [4] (see also [8]).

### 2.2 A nonlinear SPDE

Let $b : \mathbb{R} \to \mathbb{R}$ be a bounded $\mathcal{C}^1$ function with bounded derivative. Let us consider the quasilinear SPDE

$$\mathcal{L}v(t,x) = b(v(t,x)) + \dot{W}(t,x), \tag{2.5}$$

with the same initial and boundary conditions as in (2.1).

The equation (2.5) can be written in mild form as follows:

$$v(t,x) = \int_0^t \int_M G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y) + \int_0^t \mathrm{d}s \int_M \mathrm{d}y G(t-s, x-y) b(v(s,y)).$$

**Proposition 2.2** *A solution to equation (2.5) can be represented as*

$$v(t) = \sum_{n\geq 1} v_n(t) e_n,$$

*where $\{v_n(t),\ t \in \mathbb{R}_+\}$ satisfy the system of coupled equations*

$$\mathrm{d}v_n(t) = \mathrm{d}x_n(t) - \lambda_n v_n(t) \mathrm{d}t + \langle b(v(t, \cdot\,)), e_n \rangle \mathrm{d}t, \quad n \geq 1. \tag{2.6}$$

*Furthermore, set*

$$G_t = \exp\Big( \sum_{n\geq 1} \int_0^t \langle b(u(s,\,\cdot)), e_n\rangle \mathrm{d}x_n(s) - \frac{1}{2}\int_0^t \|b(u(s,\,\cdot))\|^2_{L^2(M)} \mathrm{d}s \Big)$$

*and define* $\mathrm{d}Q := G_t \mathrm{d}P$ *on* $\mathcal{F}_t$*. Then, we have* $\widetilde{\mu}_t := P\circ(v(t))^{-1} = Q\circ(u(t))^{-1}$*; moreover,* $\widetilde{\mu}_t$ *is absolutely continuous with respect to* $\mu_t$ *and*

$$\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}(z) = E(G_t \mid u(t) = z), \quad z \in \mathbb{R}^{\otimes \mathbb{N}}.$$

**Proof** By the same arguments as in Proposition 2.1, we obtain

$$v_n(t) = \int_0^t \mathrm{e}^{-\lambda_n(t-s)} x_n(\mathrm{d}s) + \int_0^t \mathrm{e}^{-\lambda_n(t-s)} \langle b(v(s,\,\cdot)), e_n\rangle \mathrm{d}s, \quad n\geq 1, \tag{2.7}$$

which is equivalent to the system (2.6).

Notice that

$$\sup_{s\in\mathbb{R}_+} \|b(u(s,\,\cdot))\|_{L^2(M)} \leq \|b\|_\infty m(M)^{\frac{1}{2}}. \tag{2.8}$$

Hence the last statement is a consequence of Girsanov's theorem.

From [4] we know that $\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_\infty}$ defines a random variable in $\mathbb{D}^{1,2}(\mu_\infty)$. Our purpose here is to analyze the properties of the density $\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}$ in terms of the measure $\mu_t$. The next Theorem 2.1 gives a result in this direction.

Before writing the statement, we fix some notations that will be used throughout the proof. For a fixed $t>0$ we denote by $\mathbf{H}_t$ and $\mathcal{H}_t$ the Cameron-Martin space associated with $\{x_n(s),\ 0\leq s\leq t;\ n\geq 1\}$ and the Gaussian variable $u_t = \{u_n(t);\ n\geq 1\}$, respectively. Given an abstract Wiener space $(\mathcal{W}, \mathbf{H}, \nu)$, we shall denote by $\mathbb{D}^{k,p}(\nu)$ the Watanabe-Sobolev spaces and by $\delta_\nu$ the divergence operator. The mathematical expectation with respect to a probability $\nu$ different from $P$ will be written as $E_\nu$.

**Theorem 2.1** *For any* $t>0$*, the density* $\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}$ *satisfies*

$$\log \frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t} \in \mathbb{D}^{1,p}(\mu_t) \tag{2.9}$$

*for any* $p\in[1,\infty[$ .

**Proof** We start by proving that for any $t\in\mathbb{R}_+$,

$$G_t \in \bigcap_{p\in[1,\infty[} \mathbb{D}^{1,p}(P). \tag{2.10}$$

Indeed, the process $\{G_t,\ t\in\mathbb{R}_+\}$ satisfies the linear equation

$$G_t = 1 + \sum_{n\geq 1}\int_0^t G_s \langle b(u(s,\,\cdot)), e_n\rangle \mathrm{d}x_n(s).$$

Fix $p \in [2, \infty[$. Burkholder's inequality together with (2.8) and Hölder's inequality imply

$$E|G_t|^p \leq C_p\Big(1 + E\Big(\int_0^t G_s^2 \sum_{n\geq 1} |\langle b(u(s)), e_n\rangle|^2 \mathrm{d}s\Big)^{\frac{p}{2}}\Big)$$
$$\leq C_p\Big(1 + \|b\|_\infty^p m(M)^{\frac{p}{2}} \int_0^t E|G_s|^p \mathrm{d}s\Big).$$

Thus, by Gronwall's lemma we have

$$E|G_t|^p \leq C_p \exp(C_p \|b\|_\infty^p m(M)^{\frac{p}{2}} t),$$

and therefore

$$G_t \in \bigcap_{p\geq 1} L^p(P). \tag{2.11}$$

Similarly,

$$G_t^{-1} \in \bigcap_{p\geq 1} L^p(P). \tag{2.12}$$

Consider an element $\{h_n,\ n \geq 1\}$ in the Cameron-Martin space $\mathbf{H}$. By using the definition and properties of the gradient operator on $(\Omega, \mathbf{H}, P)$ we obtain $D_{h_j} G_t = G_t \mathcal{K}_t^j$, with

$$\mathcal{K}_t^j = \sum_{n\geq 1} \int_0^t \langle b'(u(s,\cdot)) D_{h_j} u(s,\cdot), e_n\rangle \mathrm{d}x_n(s)$$
$$+ \delta_j^n \int_0^t \langle b(u(s,\cdot)), e_n\rangle \dot{h}_j(s) \mathrm{d}s - \langle b(u), b'(u) D_{h_j} u\rangle_{L^2([0,t]\times M)}, \tag{2.13}$$

where $\delta_j^n$ denotes the Kronecker symbol.

We want to prove that

$$E\Big(\sum_{j\geq 1} |\mathcal{K}_t^j|^2\Big)^{\frac{p}{2}} \leq \infty \tag{2.14}$$

for any $p \in [2, \infty[$.

For this, we first notice that, since the process $\{u(t),\ t > 0\}$ is Gaussian, $Du$ is deterministic, and moreover $\|Du\|_{L^2([0,t]\times M;\mathbf{H})} < \infty$. In fact,

$$D_{h_j} u(t,x) = \int_0^t \langle G(t-s, x-\cdot), e_j\rangle h_j(s) \mathrm{d}s.$$

The first term of the right-hand side of (2.13) is an $\mathbb{R}^{\otimes N}$-valued martingale. By Burkholder's inequality we have

$$E\Big(\sum_{j\geq 1}\Big|\sum_{n\geq 1}\int_0^t \langle b'(u(s,\cdot)) D_{h_j} u(s,\cdot), e_n\rangle \mathrm{d}x_n(s)\Big|^2\Big)^{\frac{p}{2}}$$
$$\leq C_p E\Big(\sum_{n\geq 1}\int_0^t \mathrm{d}s \sum_{j\geq 1} |\langle b'(u(s,\cdot)) D_{h_j} u(s,\cdot), e_n\rangle|^2\Big)^{\frac{p}{2}}$$
$$= C_p E\Big(\sum_{n\geq 1} \|b'(u) D_{h_j} u\|^2_{L^2([0,t]\times M)}\Big)^{\frac{p}{2}}$$
$$\leq C_p \|b'\|_\infty^p \|Du\|^p_{L^2([0,t]\times M;\mathcal{H})}. \tag{2.15}$$

For the second term of the right-hand side of (2.13), by applying two times Schwarz's inequality, we easily obtain

$$E\Big(\sum_{j\geq 1}\Big|\int_0^t \langle b(u(s,\,\cdot\,)), e_j(\,\cdot\,)\rangle h_j(s\mathrm{d}s)\Big|^2\Big)^{\frac{p}{2}} \leq \|h\|^p_{L^2([0,t];\mathbb{R}^{\otimes N})}\|b\|^p_\infty m(M)^{\frac{p}{2}}t^{\frac{p}{2}}. \tag{2.16}$$

As for the third term of the right-hand side of (2.13), we have

$$E\Big(\sum_{j\geq 1}|\langle b(u), b'(u)D_{h_j}u\rangle_{L^2([0,t]\times M)}|^2\Big)^{\frac{p}{2}} \leq \|b\|^p_\infty\|b'\|^p_\infty\|Du\|^p_{L^2([0,t]\times M;\mathcal{H})}\,. \tag{2.17}$$

The property (2.14) clearly follows from (2.15)–(2.17); then (2.11) and (2.14) imply (2.10).

The second step of the proof consists in proving that

$$\log\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t} \in \bigcap_{p\in[1,\infty[} L^p(\mu_t). \tag{2.18}$$

Indeed, the identity $|\log x| = \log^+ x + \log^+\left(\frac{1}{x}\right)$ clearly yields $|\log x| \leq x^\alpha + x^{-\alpha}$, for any $\alpha > 0$. Consequently,

$$\begin{aligned} E_{\mu_t}\Big|\log\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}\Big|^p &= E_{\mu_t}|\log E(G_t|u(t))|^p \\ &\leq E_{\mu_t}|E(G_t|u(t))|^q + E_{\mu_t}|E(G_t^{-1}|u(t))|^q \\ &\leq E|G_t|^q + E|G_t^{-1}|^q < \infty \end{aligned} \tag{2.19}$$

for any $q \in ]p,\infty]$. The last term of (2.19) is finite, by virtue of (2.11) and (2.12).

In the third and last step of the proof, we deal with the differentiability of the random vector $\log\frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}$.

For a fixed $t > 0$, we define a linear mapping $\Psi : \mathbf{H}_t \to \mathbb{R}^{\mathbb{N}}$ by

$$\Psi(h) = \Big(\int_0^t \langle G(t-s, x-\,\cdot\,, e_j)\rangle h_j(s)\mathrm{d}s, \quad j\geq 1\Big).$$

The image of $\Psi$ is the Cameron-Martin space of the Gaussian random variable $u_t$. Clearly, the differential of $\Psi$ is given by

$$\Psi'(h)(\widetilde{h}) = \Big(\int_0^t \langle G(t-s, x-\,\cdot\,, e_j)\rangle h_j(s)\mathrm{d}s, \quad j\geq 1\Big).$$

We have $\mathcal{H}_t = L^2([0,t]\times M)/\mathrm{Ker}\,\Psi'$. Set $\mathcal{D}_t = (\mathrm{Ker}\,\Psi')^\perp$. The restriction of $\Psi'$ to $\mathcal{D}_t$ realizes a linear isomorphism with $\mathcal{H}_t$; denote by $\sigma$ its inverse. Notice that $\sigma$ brings elements of the Cameron-Martin of $u_t$, with $t > 0$ fixed, into elements of $\mathbf{H}_t$. Given $z \in \mathcal{H}_t$, set $Z = \sigma(z)$.

We can now apply Lemma 2.1 below to the following framework: For the fixed value of $t > 0$ that we are considering in this proof, $\Omega_1$ is the canonical space associated with $\{x_n(s),\ 0 \leq s \leq t;\ n \geq 1\}$, $\Omega_2$ is the image of $\Omega_1$ by the action of the linear equation $u$, $\mathbf{H}_1$ is the Cameron-Martin space $\mathbf{H}_t$ and $\mathbf{H}_2 = \mathcal{H}_t$. The restriction of $\Phi$ to $\mathbf{H}_1$ is the mapping that we have denoted by $\Psi$ a few lines before. We will make two choices for the measure $\mu$: $P$ and $Q$. This gives two image measures $\nu$: $\mu_t = P \circ u(t)^{-1}$ and $\widetilde{\mu}_t = Q \circ u(t)^{-1}$, respectively. Thus, for $z \in \mathcal{H}_t$ and $Z = \sigma(z)$ we obtain the following identities for their respective divergences:

$$\delta_{\mu_t}(z) = E^{u(t)}(\delta_P(Z)), \quad \delta_{\widetilde{\mu}_t}(z) = E^{u(t)}(\delta_Q(Z)),$$

where $E^{u(t)}$ stands for the conditional expectation with respect to $u(t)$.

On the other hand, Lemma 2.2 below tells us that

$$\delta_Q(Z) = \delta_P(Z) - D_Z(\log G_t). \tag{2.20}$$

Consequently,

$$D_z\Big(\log \frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}\Big) = \delta_{\mu_t}(z) - \delta_{\widetilde{\mu}_t}(z) = E^{u(t)}(D_Z(\log G_t)).$$

From this identity and the integrability results on $\mathcal{K}$ proved before, we obtain that $\log \frac{\mathrm{d}\widetilde{\mu}_t}{\mathrm{d}\mu_t}$ is differentiable in the directions of $\mathcal{H}_t$ and the derivative belongs to any $L^p(\mu_t)$. This concludes the proof of the theorem.

The last part of the article is devoted to the proof of the technical results that we have invoked throughout the proof of the previous theorem.

**Definition 2.1** *Consider two abstract Wiener spaces* $(\Omega_i, \mathbf{H}_i, P_i)$ *along with a mapping* $\Phi : \Omega_1 \to \Omega_2$ *such that* $\Phi(\mathbf{H}_1) \subset \mathbf{H}_2$. *Assume that the restriction* $\Phi : \mathbf{H}_1 \to \mathbf{H}_2$ *is differentiable. Let* $z \in \mathbf{H}_2$. *An element* $Z \in \mathbf{H}_1$ *is called a covering vector field of* $z$ *if for any* $x \in \mathbf{H}_1$,

$$\Phi'(x)(Z) = z.$$

**Lemma 2.1** *In the framework of Definition* 2.1, *let* $\mu$ *be a measure on* $\Omega_1$ *and let* $\nu = \mu \circ \Phi^{-1}$. *The following formula relating the divergences with respect to the measures* $\mu$ *and* $\nu$, *respectively, holds true:*

$$\delta_\nu(z) = E^\Phi(\delta_\mu(Z)).$$

**Proof** Let $f : \mathcal{Y} \to \mathbb{R}$ be a measurable and bounded function. We have

$$\int_{\mathcal{Y}} f E^\Phi(\delta_\mu(Z))\mathrm{d}\nu = \int_{\mathcal{X}} (f \circ \Phi)(\delta_\mu(Z))\mathrm{d}\mu = \int_{\mathcal{X}} D_Z(f \circ \Phi)\mathrm{d}\mu,$$

where we have applied the duality between the divergence $\delta_\mu$ and the gradient operator $D$.

Since $z = \Phi'(x)Z$, we have $(D_z f)(\Phi(x)) = D_Z(f \circ \Phi)(x)$. Consequently,

$$\int_{\mathcal{X}} D_Z(f \circ \Phi)\mathrm{d}\mu = \int_{\mathcal{X}} (D_z f)(\Phi(x))\mathrm{d}\mu(x) = \int_{\mathcal{Y}} (D_z f)(y)\mathrm{d}\nu(y) = \int_{\mathcal{Y}} f \delta_\nu(z)\mathrm{d}\nu.$$

This ends the proof of the lemma.

The next lemma relates the divergence operators corresponding to the two absolutely continuous measures $P$ and $Q$. It has been used to write (2.20).

**Lemma 2.2** *For any* $h \in \mathbf{H}_t$,

$$\delta_P(h) = D_h(\log G_t) + \delta_Q(h). \tag{2.21}$$

**Proof** Let $\varphi$ be a cylindrical function defined on $\Omega$. Consider the identity

$$D_h(\varphi G_t) = \varphi D_h G_t + G_t D_h \varphi.$$

By applying the duality between gradient and divergence on $(\Omega, \mathbf{H}_t, P)$ and $(\Omega, \mathbf{H}_t, Q)$, respectively, we have

$$E(D_h(\varphi G_t)) = E(\varphi G_t \delta_P(h)) = E_Q(\varphi \delta_P(h)) \tag{2.22}$$

and

$$E(G_t D_h \varphi) = E_Q(D_h \varphi) = E_Q(\varphi \delta_Q(h)). \tag{2.23}$$

Moreover,

$$E(\varphi D_h G_t) = E(\varphi G_t D_h(\log G_t)) = E_Q(\varphi D_h(\log G_t)). \tag{2.24}$$

From (2.22)–(2.24) we obtain

$$E_Q(\varphi \delta_P(h)) = E_Q(\varphi D_h(\log G_t)) + E_Q(\varphi \delta_Q(h)),$$

which clearly implies (2.21).

**Acknowledgement** This paper was completed when the authors were visiting the Institute Mittag-Leffler in Djursholm (Sweden) during a semester devoted to SPDEs. They would like to thank this institution for the very kind hospitality and financial support.

## References

[1] Airault, H., Malliavin, P. and Ren, J., Geometry of foliations on the Wiener space and stochastic calculus of variations, *C. R. Math. Acad. Sci. Paris*, **339**(9), 2004, 637–642.

[2] Carmona, R. and Nualart, D., Random nonlinear wave equations: smoothness of the solutions, *Probab. Theory Related Fields*, **79**(4), 1988, 469–508.

[3] Da Prato, G. and Zabczyck, J., Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 1992.

[4] Gaveau, B. and Mouliner, J.-M., Régularité des mesures et perturbations stochastiques de champs de vecteurs sur des espaces de dimension infinie, Publ. RIMS, Kyoto Univ., **21**, 1985, 593–616.

[5] Márquez-Carreras, D., Mellouk, M. and Sarrà, M., On stochastic partial differential equations with spatially correlated noise: smoothness of the law, *Stoch. Proc. Appl.*, **93**, 2001, 269–284.

[6] Mattingly, J. and Bahtkin, Y., Malliavin calculus for infinite-dimensional systems with additive, *J. Funct. Anal.*, **249**(2), 2007, 307–353.

[7] Millet, A. and Sanz-Solé, M., A stochastic wave equation in two space dimensions: smoothness of the law, *Annals of Probab.*, **27**, 1999, 803–844.

[8] Moulinier, J.-M., Absolute continuité de probabilités de transition par rapport à une mesure gaussienne dans un space de Hilbert, *J. Funct. Anal.*, **64**, 1985, 275–295.

[9] Nualart, D. and Sanz-Solé, M., Malliavin calculus for two-parameter Wiener functionals, *Z. Wahrsch. Verw. Gebiete*, **70**(4), 1985, 573–590.

[10] Ocone, D., Stochastic calculus of variations for stochastic partial differential equations, *J. Funct. Anal.*, **79**(2), 1988, 288–331.

[11] Pardoux, E. and Zhang, T., Absolute continuity of the law of the solution of a parabolic SPDE, *J. Funct. Anal.*, **112**(2), 1993, 447–458.

[12] Quer-Sardanyons, Ll. and Sanz-Solé, M., Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation, *J. Funct. Anal.*, **206**(1), 2004, 1–32.

[13] Quer-Sardanyons, Ll. and Sanz-Solé, M., A stochastic wave equation in dimension 3: smoothness of the law, *Bernoulli*, **10**(1), 2004, 165–186.

[14] Walsh, J. B., An introduction to Stochastic partial differential equations, École d'été de Probabilités de Saint-Flour, XIV, P. L. Hennequin (ed.), Lect. Notes Math., **1180**, Springer-Verlag, Berlin, 1986, 265–439.