Chin. Ann. Math.
29B(3), 2008, 247–264
DOI: 10.1007/s11401-007-0104-4

Chinese Annals of Mathematics, Series B

Existence and Concentration of Ground States of Coupled Nonlinear Schrödinger Equations with Bounded Potentials**

Gongming WEI*

Abstract A 2-coupled nonlinear Schrödinger equations with bounded varying potentials and strongly attractive interactions is considered. When the attractive interaction is strong enough, the existence of a ground state for sufficiently small Planck constant is proved. As the Planck constant approaches zero, it is proved that one of the components concentrates at a minimum point of the ground state energy function which is defined in Section 4.

Keywords Concentration, Nehari's manifold, Critical point theory, Concentration-compactness principle
2000 MR Subject Classification 35B25, 35J50, 35Q40

1 Introduction

In this paper, we consider existence and concentration phenomena of ground states of the following coupled nonlinear Schrödinger equations

$$\begin{cases} h^2\Delta u - V_1(x)u + \mu_1 u^3 + \beta uv^2 = 0, \\ h^2\Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

as the small parameter h tends to zero. Precisely, we consider existence of ground states for sufficiently small h and then asymptotical behaviors of these ground states as h tends to zero. The small parameter h refers to Planck constant. In physics, Planck constant is a real constant. From this point of view, we are studying problem of semiclassical limit for Schrödinger type systems and hence the relations between classical and quantum mechanics. In this paper, we assume that $N = 2, 3$, $\mu_i, i = 1, 2$ and β are positive constants, and the potential functions $V_i(x), i = 1, 2$ are bounded. We remark that the ground state here coincides with the definition in [29, 30].

The above systems model many physical problems, especially in nonlinear optics and double Bose-Einstein condensate. In fact, these equations are satisfied by solitary waves of some time-dependent nonlinear Schrödinger systems appearing in nonlinear optics when the potentials are constants. μ_i is positive when the i-th component of the beam is self-focusing. β is the interactions between the first and the second component of the beam. As $\beta > 0$, the interaction is attractive, and the interaction is repulsive if $\beta < 0$. The present paper concerns the case

Manuscript received March 15, 2007. Published online April 23, 2008.
*College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China.
E-mail: gmweixy@163.com
**Project supported by the Research Project of Shanghai Municipal Education Commission (No. 07zz83).

that $\beta > 0$ and sufficiently large. The interaction terms in these equations make difficulty not only in studying existence but also in the analysis of asymptotic behaviors as h tends to zero. Although these systems have variational structures, their associated energy functionals have indefinite sign and (PS) condition is not satisfied. This is a feature of the problem.

The stationary Gierer-Mainhardt system

$$\begin{cases} d\Delta u - u + \dfrac{u^p}{v^q} = 0, \\ D\Delta v - v + \dfrac{u^r}{v^s} = 0, \end{cases} \quad \text{in } \Omega, \tag{GM}$$

$(0 < d \ll 1,\ D \gg 1)$ with Neumann condition and the partial differential equation in the shadow system (the s.s. system is indeed an ODE coupled with a PDE)

$$\varepsilon^2 u - u + u^p = 0, \quad \text{in } \Omega, \tag{SS}$$

$(\varepsilon \ll 1)$, which models different diffusion that can lead to nonhomogeneous distribution of reactants, have been extensively studied by many authors since the work of Lin, Ni and Takagi [28, 33–35, 42]. Interested readers can find a good review in [32] and many recent references in [47]. About the techniques of approximate-solution manifold and Liapunov-Schmidt reduction for Gierer-Mainhadrt type problems, one can find interesting development in [4, 25] and some of the ideas of the present paper follows from these developments. It is not our ambition to give a review of this fast developing field and what we want to emphasize here is that most of the interesting results on (SS) are based on the understanding of the following equation

$$\Delta w - w + w^p = 0, \quad \text{in } \mathbb{R}^N. \tag{Eq.1}$$

This is one of our motives to first consider problems on whole space. Recently, Ramos and Yang [39] studied spiked-layered solutions for a singularly perturbed elliptic system (without interactions) on bounded domain and their Hamiltonian functional is different from here.

Since the milestone work of Floer-Weinstein [22], there have been many contributions to the singularly perturbed Schrödinger equation with potential

$$h^2\Delta u - V(x)u + u^p = 0, \quad \text{in } \mathbb{R}^N. \tag{Eq.2}$$

Under a mathematical restriction on V, using minimax arguments combined with Ekeland's variational principle, Rabinowitz [38] proved the existence of positive ground state. Then Wang [44] studied the behavior of Rabinowitz's ground state as h tends to zero and proved that it concentrates at a global minimum point of V. Wang and Zeng [45] offered a new viewpoint to study nonlinear Schrödinger equations, especially for those with bounded potentials, both for existence and concentration of ground states. Ambrosetti, Malchiodi and Secchi [6] gave multiplicity results of semiclassical solutions (solutions when $h \ll 1$) through studying stationary points of V. Badiale and D'Aprile [7] and Ambrosetti, Malchiodi and Ni [5] firstly proved the existence of concentrating sphere of radially symmetric solutions, especially [5] determines the limit radii as stationary points of an auxiliary potential function. On singularly perturbed Neumann problem with potentials on bounded domain

$$\varepsilon^2 \mathrm{div}(J(x)\nabla u) - V(x)u + u^p = 0, \quad \text{in } \Omega, \tag{Eq.3}$$

using techniques similar to (SS), Pomponio [37] studied the existence of single-peaked solutions and determined the concentrating points through coefficient functions. Other important contributions to this problem include Del Pino-Felmer [18–20], Dance-Wei [21], Grossi [23], Jeanjean-Tanaka [24] and so on. For more general potential cases one can refer to [3, 8, 14], and to [12, 13, 15] for multiplicity results.

Compared with so fruitful results for singularly perturbed single semilinear elliptic equations and reaction-diffusion systems with quite different diffusion rates, there are few results on singularly perturbed systems. This is another motivation of this paper.

Recently, spike-layer solutions of singularly perturbed 2-coupled nonlinear Schrödinger equations

$$\begin{cases} h^2\Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0, \\ h^2\Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \quad \text{in } \Omega, \tag{1.2}$$

(which arises in the Hartree-Fock theory for a double condensate) and ground states of

$$\begin{cases} \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0, \\ \Delta v - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

(which are standing waves of time-dependent 2-coupled nonlinear Schrödinger equations in nonlinear optics) are studied mathematically by Tai-Chia Lin and Juncheng Wei [29, 30]. Similar ideas as for (SS), the study on spikes of (1.2) depends on our knowledge for ground states of (1.3). One of the main tools is the employ of Nehari's solution manifold, which has been used by Conti, Terracini and Verzini [16] to study a class of competing species systems. Their main conclusions for (1.3) are: (a) there exist ground states when $0 < \beta < \beta_0 < \sqrt{\mu_1 \mu_2}$; (b) there do not exist ground states when $\beta < 0$. As β is large, A. Ambrosetti and E. Colorado [1, 2] proved the existence of ground states. A. Ambrosetti and E. Colorado's proof consists of two parts. The first step is to prove the existence of nontrivial least energy solutions using mountain pass theorem; the second step is to prove that the nontrivial least energy solutions are in fact ground states using Morse index. In Section 3, we will give a simplified proof for the second step in A. Ambrosetti and E. Colorado's proof and give a sketch of the whole proof for completeness and later use. For other recent developments of such systems, one can refer to [11, 17].

When this work was finished, we learned that several authors have considered such Schrödinger type system with trapping potentials. To the best knowledge of the author, T. Lin and J. Wei [31] gave the first contribution to spikes of such systems for negative or small β. The present paper will continue to study the existence and concentration of ground states of Schrödinger systems like (1.1) for large β, under the restriction that the potentials are bounded from below and above. The paper is organized as follows.

In Section 2, we give some preliminaries and recall some well-known results. In Section 3, we give a sketch of the proof of the existence of ground state of problem (1.3) for sufficiently large β . In Section 4, under the assumption that the global minimum of $V_i(x)$ is less than its limit at infinity, we prove the existence of ground state of problem (1.1) for sufficiently small h. Asymptotic behaviors of these ground states as h tends to zero are studied in Section 5. It is proved that, along a sequence h_k tending to zero, the ground state sequence converges to a ground state of equations of type (1.3) and at least one of their components concentrates at a finite point whose location is determined by an energy function.

For convenience, we call a solution of system strictly nontrivial if each component of the solution is not identically zero. By nontrivial solution of system, we mean that at least one component is not identically zero. In the paper, ground state is in fact strictly nontrivial least energy solution. Sometimes we call a nontrivial least energy solution a nontrivial ground state.

2 Preliminaries

The energy functional for (1.3) is

$$I(u,v) := \int \frac{1}{2}|\nabla u|^2 + \frac{\lambda_1}{2}u^2 - \frac{\mu_1}{4}u^4 + \frac{1}{2}|\nabla v|^2 + \frac{\lambda_2}{2}v^2 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2v^2,$$

where $(u,v) \in E$ and $E = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Let

$$c = \inf_{(u,v)\in M} I(u,v),$$

where

$$M = \left\{(u,v) \in T \,\Big|\, \int |\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2 = \int \mu_1 u^4 + 2\beta u^2v^2 + \mu_2 v^4\right\}$$

and

$$T = H^1(\mathbb{R}^N) \setminus \{0\} \times H^1(\mathbb{R}^N) \setminus \{0\}.$$

Let

$$\widetilde{c} = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],E) \,|\, \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

For any $(u,v) \in T$, set

$$f(t) = I(\sqrt{t}\,u, \sqrt{t}\,v),$$
$$\phi(u,v) = \max_{t>0} f(t)$$

and let

$$d = \inf_{(u,v)\in T} \phi(u,v).$$

Lemma 2.1 *Suppose $\beta > 0$. Then*

(1) *$\forall\,(u,v) \in T$, there exists unique $t = t(u,v) > 0$ such that $(\sqrt{t}\,u, \sqrt{t}\,v) \in M$ and $\phi(u,v) = f(t)$.*

(2) *$\widetilde{c} = c = d$.*

Proof From the definitions of I and f, we have

$$f(t) = \int \frac{t}{2}|\nabla u|^2 + \frac{t}{2}\lambda_1 u^2 - \frac{t^2\mu_1}{4}u^4 + \frac{t}{2}|\nabla v|^2 + \frac{t}{2}\lambda_2 v^2 - \frac{t^2\mu_2}{4}v^4 - \frac{t^2\beta}{2}u^2v^2$$

for $t \geq 0$. By a direct computation, we get

$$f_t(t_0) = 0 \Leftrightarrow \exists\,\text{unique } t_0 = t_0(u,v) > 0 \text{ s.t. } (\sqrt{t_0}\,u, \sqrt{t_0}\,v) \in M \text{ and } \phi(u,v) = f(t_0).$$

For any $(u,v) \in M$, by the uniqueness of t_0,

$$I(u,v) = \max_{t>0} I(\sqrt{t}\,u, \sqrt{t}\,v) = \phi(u,v) \geq \inf_{(u,v)\in T} \phi(u,v) = d.$$

By the definition of c, $c \geq d$. On the other hand, for any $(u,v) \in T$,

$$d = \inf_{(u,v)\in T} \phi(u,v) = \inf_{(u,v)\in T} I(\sqrt{t(u,v)}\,u, \sqrt{t(u,v)}\,v) \geq c$$

because of $(\sqrt{t(u,v)}\,u, \sqrt{t(u,v)}\,v) \in M$. Thus $c = d$.

For any $(u,v) \in M$, choose $\gamma_0(t) = (tAu, tAv)$ for sufficiently large A so that $I(\gamma_0(1)) < 0$. Then $\gamma_0 \in \Gamma$ and

$$\widetilde{c} \le \max_{t>0} I(\gamma_0(t)) = \max_{t>0} I(tu, tv) = I(u,v).$$

Hence $\widetilde{c} \le c$.

To show $\widetilde{c} \ge c$, we only need to prove that for any $\gamma \in \Gamma$, $\gamma([0,1]) \cap M \neq \emptyset$. When $0 < \|(u,v)\| \ll 1$,

$$\int |\nabla u|^2 + \lambda_1 u^2 + |\nabla v|^2 + \lambda_2 v^2 > \int \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2.$$

Since $I(\gamma(0)) = 0$, γ is continuous in $t \in [0,1]$, $I(\gamma(t))$ is continuous in t and $I(\gamma(t)) > 0$ for $0 < t \ll 1$. If we set $\gamma(t) = (u_t, v_t)$, then

$$\int |\nabla u_t|^2 + \lambda_1 u_t^2 + |\nabla v_t|^2 + \lambda_2 v_t^2 > \int \mu_1 u_t^4 + \mu_2 v_t^4 + 2\beta u_t^2 v_t^2$$

holds for sufficiently small t. By the continuity of $I(\gamma(t))$, if $\gamma \in \Gamma$, $\gamma([0,1]) \cap M = \emptyset$, then

$$\int |\nabla u_1|^2 + \lambda_1 u_1^2 + |\nabla v_1|^2 + \lambda_2 v_1^2 > \int \mu_1 u_1^4 + \mu_2 v_1^4 + 2\beta u_1^2 v_1^2.$$

This is a contradiction with $I(\gamma(1)) < 0$.

The following two lemmas are well-known (see e.g. [9, 29, 30, 41]).

Lemma 2.2 *There exists unique positive ground state w of*

$$\begin{cases} \Delta w - w + w^3 = 0, & x \in \mathbb{R}^N,\ N \le 3, \\ \max\limits_{x \in \mathbb{R}^N} w(x) = w(0), & \\ w(x) \to 0, & \text{as } |x| \to \infty \end{cases}$$

which is radially symmetric about the origin and exponentially decays at infinity.

Set $w_i(x) = \sqrt{\frac{\lambda_i}{\mu_i}}\, w(\sqrt{\lambda_i}\, x)$, where $\lambda_i > 0, \mu_i > 0$ for $i = 1,2$. Then w_i is a positive ground state of

$$\Delta w_i - \lambda_i w + \mu_i w^3 = 0, \quad x \in \mathbb{R}^N$$

and the corresponding least energies are

$$\mathrm{I}_i = \int \frac{1}{2}(|\nabla w_i|^2 + \lambda_i w_i^2) - \frac{\mu_i}{4} w_i^4 = \lambda_i^{\frac{4-N}{2}} \mu_i^{-1} \mathrm{I}_0,$$

where

$$\mathrm{I}_0 = \int \frac{1}{2}(|\nabla w|^2 + w^2) - \frac{1}{4} w^4 = \frac{1}{4} \int |\nabla w|^2 + w^2.$$

Lemma 2.3 *$H_r^1(\mathbb{R}^N)$ is compactly imbedded in $L^p(\mathbb{R}^N)$ for $2 \le p < 2^*$ and $N \ge 2$, where $2^* = \frac{2N}{(N-2)_+}$, $H_r^1(\mathbb{R}^N)$ denote the set of functions in $H^1(\mathbb{R}^N)$ which are radially symmetric.*

This lemma is due to Strauss.

3 Constant Potentials

The results in this section have been essentially obtained by A. Ambrosetti and E. Colorado in [1, 2]. For completeness and later use, we give a sketch of the proof and a simplified proof of the second step in A. Ambrosetti and E. Colorado's proof.

Firstly, we consider the existence of ground states of

$$\begin{cases} \Delta u - \lambda u + \mu_1 u^3 + \beta u v^2 = 0, \\ \Delta v - \lambda v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \quad \text{in } \mathbb{R}^N, \ N = 2, 3. \tag{3.1}$$

Theorem 3.1 *Assume $N = 2, 3$ and $\beta > \max\{\mu_1, \mu_2\}$. Then problem (3.1) has a ground state.*

Proof First we consider this problem in the space of radial functions $E_r = H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. The associated energy functional is

$$I_r(u, v) = \int \frac{1}{2}|\nabla u|^2 + \frac{\lambda}{2}u^2 - \frac{\mu_1}{4}u^4 + \frac{1}{2}|\nabla v|^2 + \frac{\lambda}{2}v^2 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2 v^2,$$

where $(u, v) \in E_r$.

Define

$$c_r = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I_r(\gamma(\theta)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], E_r) \mid \gamma(0) = 0, \ I_r(\gamma(1)) < 0\}.$$

Recall that, by Lemma 2.3, $H_r^1(\mathbb{R}^N)$ is compactly imbedded into $L^p(\mathbb{R}^N)$ for $2 \le p < 2^*$ and $N \ge 2$. So it follows that I_r satisfies (PS) condition on E_r. By mountain pass theorem (see e.g. [46]), c_r is a critical value of I_r in E_r and c_r is the least energy of radial solutions. A similar proof as for Lemma 2.1 implies that

$$c_r = c_r^* := \inf_{(u,v) \in M_r} I_r(u, v),$$

where

$$M_r = \Big\{(u, v) \in T_r \,\Big|\, \int |\nabla u|^2 + \lambda u^2 + |\nabla v|^2 + \lambda v^2 = \int \mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2\Big\},$$
$$T_r = \{(u, v) \in E_r \mid u \neq 0 \text{ or } v \neq 0 \ \}.$$

Suppose that (u_r, v_r) is a minimizer and a critical point of $I_r|_{M_r}$. Since

$$I_r(u, v) = \frac{1}{2}\|(u, v)\|^2 + o(\|(u, v)\|^2),$$

where $c_r \ge \overline{c}$ for some positive constant $\overline{c}$, it follows that (u_r, v_r) is a nontrivial solution of (3.1). Next we will show that (u_r, v_r) is strictly nontrivial.

Let $w, \mathrm{I}_0, w_i, \mathrm{I}_i, i = 1, 2$ be as in Lemma 2.2 with $\lambda = \lambda_i$.

Let $u = aw_1, \ v = bw_2$, where $a^2 = \frac{1-\mu_2^{-1}\beta}{1-\beta^2\mu_1^{-1}\mu_2^{-1}}, \ b^2 = \frac{1-\mu_1^{-1}\beta}{1-\beta^2\mu_1^{-1}\mu_2^{-1}}$. Then $(u, v) \in M_r$ and

$$\begin{aligned} I_r(u, v) &= \frac{1}{4}\int |\nabla u|^2 + \lambda u^2 + |\nabla v|^2 + \lambda v^2 \\ &= \frac{1}{4}\int a^2(|\nabla w_1|^2 + \lambda w_1^2) + b^2(|\nabla w_2|^2 + \lambda w_2^2) \\ &= a^2 \mathrm{I}_1 + b^2 \mathrm{I}_2. \end{aligned}$$

In fact, (u, v) satisfies

$$\int |\nabla u|^2 + \lambda u^2 = \int \mu_1 u^4 + \beta u^2 v^2, \quad \int |\nabla v|^2 + \lambda v^2 = \int \mu_2 v^4 + \beta u^2 v^2.$$

Assume $\mu_1 \geq \mu_2$. By direct computation, noting that $\mathrm{I}_i = \lambda^{\frac{4-N}{2}} \mu_i^{-1} \mathrm{I}_0, i = 1, 2$, we have

$$a^2 \mathrm{I}_1 + b^2 \mathrm{I}_2 < \min\{\mathrm{I}_1, \mathrm{I}_2\} \Leftrightarrow \frac{\mu_1^2 - 2\mu_1 \beta + \beta^2}{\mu_1(\mu_1\mu_2 - \beta^2)} < 0.$$

Therefore, when $\beta > \max\{\mu_1, \mu_2\}$,

$$c_r = c_r^* \leq I(aw_1, bw_2) = a^2 \mathrm{I}_1 + b^2 \mathrm{I}_2 < \min\{\mathrm{I}_1, \mathrm{I}_2\}.$$

This implies that u_r and v_r are not trivial. By elliptic regularity and maximum principle, (u_r, v_r) is a positive classical solution of (3.1). From [10, 40], all the positive solutions vanishing at infinity are radially symmetric at the origin. Hence the radial ground state (u_r, v_r) is also a ground state. This finishes the proof.

Remark 3.1 Since M_r includes the Nehari manifold with two equations defined in [30], the ground state we get coincides with the definition in [30].

As for $\lambda_1 \neq \lambda_2$, we extend Theorem 3.1 to the following corollary.

Corollary 3.1 *There exists* $\Lambda_0 = \Lambda_0(\lambda_1, \lambda_2, \mu_1, \mu_2, w)$ *such that, for* $\beta \geq \Lambda_0$*, problem* (1.3) *has a ground state.*

Proof Let $w, I_0, w_i, I_i, i = 1, 2$ be as in Lemma 2.2. The same arguments as in the proof of Theorem 3.1 imply the existence of a nontrivial solution (u_0, v_0). What we need to show is that this nontrivial solution is a ground state.

Let $u = aw_1,\ v = bw_2$, where

$$a^2 = \frac{1 - \beta\lambda_1\lambda_2^{-1}\mu_1^{-1}\delta_1}{1 - \beta^2\mu_1^{-1}\mu_2^{-1}\delta_1\delta_2}, \quad b^2 = \frac{1 - \beta\lambda_1^{-1}\lambda_2\mu_2^{-1}\delta_2}{1 - \beta^2\mu_1^{-1}\mu_2^{-1}\delta_1\delta_2},$$

and δ_1, δ_2 are given by

$$\int w^2(x) w^2\Big(\sqrt{\frac{\lambda_2}{\lambda_1}}\, x\Big) = \delta_1 \int w^4, \quad \int w^2(x) w^2\Big(\sqrt{\frac{\lambda_1}{\lambda_2}}\, x\Big) = \delta_2 \int w^4.$$

For $\beta \gg 1$, we have $0 < a^2 \ll 1, 0 < b^2 \ll 1$. By direct computation, $(u, v) \in M$. In fact,

$$\int |\nabla u|^2 + \lambda_1 u^2 = \int \mu_1 u^4 + \beta u^2 v^2, \quad \int |\nabla v|^2 + \lambda_2 v^2 = \int \mu_2 v^4 + \beta u^2 v^2.$$

Hence

$$\begin{aligned} I(u, v) &= \int \frac{1}{2}|\nabla u|^2 + \frac{\lambda_1}{2}u^2 - \frac{\mu_1}{4}u^4 + \frac{1}{2}|\nabla v|^2 + \frac{\lambda_2}{2}v^2 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2v^2 \\ &= \frac{1}{4}(a^2\lambda_1^{\frac{4-N}{2}}\mu_1^{-1} + a^2\lambda_1^{\frac{4-N}{2}}\mu_1^{-1}) \int |\nabla w|^2 + w^2 \\ &= a^2 \mathrm{I}_1 + b^2 \mathrm{I}_2 \\ &< \min\{\mathrm{I}_1, \mathrm{I}_2\} \end{aligned}$$

when β is sufficiently large. This implies that u_0 and v_0 are both non zero functions and (u_0, v_0) must be a ground state.

4 Existence of Ground States for Varying Potentials

In this and the next section, following the idea in [45], we study the existence and concentration of ground states of

$$\begin{cases} h^2\Delta u - V_1(x)u + \mu_1 u^3 + \beta uv^2 = 0, \\ h^2\Delta v - V_2(x)v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

under the assumption that $N = 2, 3$ and $V_i(x), i = 1, 2$, satisfy

$$V_i \in C^1(\mathbb{R}^N), \quad \lambda_i \le V_i(x) \le \Lambda_i, \tag{V}$$

where λ_i, Λ_i are positive constants. Note that the λ_i in this section is not the λ_i in section 3. We always assume that this condition holds in the following sections.

Let

$$u^h(x) = u_h(hx), \quad v^h(x) = v_h(hx). \tag{4.1}$$

If (u^h, v^h) is a ground state of (1.1), then (u_h, v_h) is a ground state of

$$\begin{cases} \Delta u - V_1(hx)u + \mu_1 u^3 + \beta uv^2 = 0, \\ \Delta v - V_2(hx)v + \mu_2 v^3 + \beta u^2 v = 0. \end{cases} \tag{4.2}$$

Define

$$c_h = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_h(\gamma(t)),$$
$$c_h^* = \inf_{(u,v)\in M_h} I_h(u, v),$$

where

$$I_h(u, v) = \int \frac{1}{2}(|\nabla u|^2 + V_1(hx)u^2 + |\nabla v|^2 + V_2(hx)v^2) - \frac{\mu_1}{4}u^4 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2v^2,$$
$$\Gamma_h = \{\gamma \in C([0, 1], E_h) \mid \gamma(0) = 0,\ I_h(\gamma(1)) < 0\},$$
$$E_h = \Big\{(u, v) \,\Big|\, u, v \in H^1(\mathbb{R}^N), \int V_1(hx)u^2 < \infty,\ \int V_2(hx)v^2 < \infty\Big\},$$
$$M_h = \Big\{(u, v) \in T_h \,\Big|\, \int |\nabla u|^2 + V_1(hx)u^2 + |\nabla v|^2 + V_2(hx)v^2 = \int \mu_1 u^4 + 2\beta u^2v^2 + \mu_2 v^4\Big\},$$
$$T_h = \{(u, v) \in E_h \mid u \ne 0 \text{ or } v \ne 0\}.$$

Just as in the proof of Lemma 2.1, it can be easily shown that $c_h = c_h^*$.

For fixed $s \in \mathbb{R}^N$, define

$$E(s) = \inf_{(u,v)\in M_s} I_s(u, v),$$

where

$$I_s(u, v) = \int \frac{1}{2}(|\nabla u|^2 + V_1(s)u^2 + |\nabla v|^2 + V_2(s)v^2) - \frac{\mu_1}{4}u^4 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2v^2,$$
$$M_s = \Big\{(u, v) \in T \,\Big|\, \int |\nabla u|^2 + V_1(s)u^2 + |\nabla v|^2 + V_2(s)v^2 = \int \mu_1 u^4 + 2\beta u^2v^2 + \mu_2 v^4\Big\}.$$

From Corollary 3.1, for sufficiently large β, there exists a ground state (u_s, v_s) of

$$\begin{cases} \Delta u - V_1(s)u + \mu_1 u^3 + \beta uv^2 = 0, \\ \Delta v - V_2(s)v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \tag{4.3}$$

i.e. $E(s) = I_s(u_s, v_s)$, $(u_s, v_s) \in M_s$ and (u_s, v_s) is strictly nontrivial.

Proposition 4.1 *There exists $\Lambda_0 = \Lambda_0(\lambda_i, \Lambda_i, \mu_i, w, V_i) > 0$ such that, for $\beta > \Lambda_0$, any nontrivial least energy solution of* (4.2) *is strictly nontrivial.*

Proof Let w, I_0 be as in Lemma 2.2 and $w_i(x) = \sqrt{\frac{\Lambda_i}{\mu_i}}\, w(\sqrt{\Lambda_i}\, x)$, $\widetilde{w}_i(x) = \sqrt{\frac{\lambda_i}{\mu_i}}\, w(\sqrt{\lambda_i}\, x)$ for $i = 1, 2$. Then w_i and $\widetilde{w}_i$ are ground states of

$$\Delta w_i - \Lambda_i w_i + \mu_i w_i^3 = 0$$

and

$$\Delta \widetilde{w}_i - \lambda_i \widetilde{w}_i + \mu_i \widetilde{w}_i^3 = 0,$$

with corresponding energies $\mathrm{I}_i = \Lambda_i^{\frac{4-N}{2}} \mu_i^{-1} \mathrm{I}_0$ and $\widetilde{\mathrm{I}}_i = \lambda_i^{\frac{4-N}{2}} \mu_i^{-1} \mathrm{I}_0$.

Let $u = aw_1, v = bw_2$, where

$$a^2 = \frac{\alpha_1 - \beta \Lambda_1^{-1} \Lambda_2 \mu_2^{-1} \delta_1 \alpha_2}{1 - \beta^2 \mu_1^{-1} \mu_2^{-1} \delta_1 \delta_2}, \quad b^2 = \frac{\alpha_2 - \beta \Lambda_1 \Lambda_2^{-1} \mu_1^{-1} \delta_2 \alpha_1}{1 - \beta^2 \mu_1^{-1} \mu_2^{-1} \delta_1 \delta_2}$$

and

$$\int |\nabla w|^2 + \Lambda_i^{-1} V_i\big(\sqrt{\Lambda_i^{-1}}\, x\big) w^2 = \alpha_i \int |\nabla w|^2 + w^2, \quad i = 1, 2,$$
$$\int w_1^2(x) w_2^2\Big(\sqrt{\frac{\Lambda_2}{\Lambda_1}}\, x\Big) = \delta_1 \int w^4, \quad w_1^2\Big(\sqrt{\frac{\Lambda_1}{\Lambda_2}}\, x\Big) w_2^2(x) = \delta_2 \int w^4.$$

Then $(u, v) \in M_h$ (similar computation as in Section 3) and $0 < a^2 \ll 1, 0 < b^2 \ll 1$ when $\beta \gg 1$. Thus

$$\begin{aligned} I_h(u, v) &= \frac{1}{4} \int |\nabla u|^2 + V_1(hx) u^2 + |\nabla v|^2 + V_2(hx) v^2 \\ &\le \frac{1}{4} \int |\nabla u|^2 + \Lambda_1 u^2 + |\nabla v|^2 + \Lambda_2 v^2 \\ &= a^2 \mathrm{I}_1 + b^2 \mathrm{I}_2 \\ &< \min\{\widetilde{\mathrm{I}}_1, \widetilde{\mathrm{I}}_2\} \end{aligned}$$

for sufficiently large β. It is routine to prove that the least energy for solutions to

$$\Delta w_i - V_i(x) w_i + \mu_i w_i^3 = 0$$

is no less than $\widetilde{\mathrm{I}}_i$ for $i = 1, 2$. Indeed, the corresponding least energy of the above equation satisfies

$$\begin{aligned} I_{V_i} &= \inf_{w \neq 0} \max_{t > 0} \int \frac{t}{2}(|\nabla w|^2 + V_i(x) w^2) - \frac{t^2 \mu_i}{4} w^4 \\ &\ge \inf_{w \neq 0} \max_{t > 0} \int \frac{t}{2}(|\nabla w|^2 + \lambda_i(x) w^2) - \frac{t^2 \mu_i}{4} w^4 \\ &= \widetilde{\mathrm{I}}_i. \end{aligned}$$

This finishes the proof.

The next lemma will be applied in the next section and is also of its own interests.

Lemma 4.1 *$E(s)$ is locally Lipschitz continuous in $s \in \mathbb{R}^N$.*

Proof Let $w_t=(u_t,v_t)$ be a positive ground state of (4.3) with $t=s$. Then there exists a unique $\theta=\theta(s,t)>0$ such that $\theta w_t\in M_s$, i.e.

$$\int |\nabla u_t|^2+V_1(s)u_t^2+|\nabla v_t|^2+V_2(s)v_t^2=\theta^2\int \mu_1u_t^4+\mu_2v_t^4+2\beta u_t^2v_t^2.$$

By the bounds (V) of V_i, we have $E_{\lambda_1,\lambda_2}\le E(s)\le E_{\Lambda_1,\Lambda_2}$, where E_{λ_1,λ_2} and E_{Λ_1,Λ_2} are the corresponding least energies with $V_i=\lambda_i$ and $V_i=\Lambda_i$, respectively. This means that $E(s)$ is bounded from below and above. Since

$$\begin{aligned}E(t)&=\frac{1}{4}\int |\nabla u_t|^2+V_1(t)u_t^2+|\nabla v_t|^2+V_2(t)v_t^2\\&=\frac{1}{4}\int \mu_1u_t^4+\mu_2v_t^4+2\beta u_t^2v_t^2,\end{aligned}$$

it follows that θ^2 and $|\nabla_s\theta^2|$ are locally bounded from below and above.

Since $\theta w_t\in M_s$, we have

$$\begin{aligned}I_s(\theta w_t)&=\frac{\theta^2}{4}\int |\nabla u_t|^2+V_1(s)u_t^2+|\nabla v_t|^2+V_2(s)v_t^2\\&=\frac{\theta^4}{4}\int \mu_1u_t^4+\mu_2v_t^4+2\beta u_t^2v_t^2\end{aligned}$$

and hence $\nabla_sI_s(\theta w_t)$ is locally bounded from below and above.

Thus, for s_1,s_2 in some finite domain,

$$\begin{aligned}E(s_1)-E(s_2)&\le I_{s_1}(\theta w_{s_2})-I_{s_2}(w_{s_2})\\&=(s_1-s_2)\nabla_sI_s(\theta w_{s_2})|_{s\in[s_1,s_2]}\\&\le M|s_1-s_2|.\end{aligned}$$

Similarly, we can show $E(s_1)-E(s_2)\ge M|s_1-s_2|$. Thus $|E(s_1)-E(s_2)|\le M|s_1-s_2|$ for s_1,s_2 in a finite domain.

Lemma 4.2 $\limsup\limits_{h\to0}c_h\le\inf\limits_{s\in R^N}E(s)$.

Proof Let $w_{s_0}=(u_{s_0},v_{s_0})$ be a ground state of (4.3) with $s=s_0$ and ϕ_R be a cut-off function: $\phi_R\in C_c^\infty(\mathbb{R}^N),\ 0\le\phi_R\le1,\ \phi(x)=1$ for $|x|\le R$, $\phi_R=0$ for $|x|\ge R+1$. Define $w_R=w_{s_0}\phi_R=(u_{s_0}\phi_R,v_{s_0}\phi_R)=(u_R,v_R)$ and $w(x)=w_R(x-\frac{s_0}{h})$. Then (by the same proof of Lemma 2.1) there exists $\theta>0$ such that $\theta w\in M_h$, i.e.

$$\int |\nabla u_R|^2+V_1(hx+s_0)u_R^2+|\nabla v_R|^2+V_2(hx+s_0)v_R^2=\theta^2\int \mu_1u_R^4+\mu_2v_R^4+2\beta u_R^2v_R^2.$$

Since $u_R\to u_{s_0},v_R\to v_{s_0}$ and (u_{s_0},v_{s_0}) is a solution of (4.3), we have $\lim\limits_{R\to\infty}\lim\limits_{h\to0}\theta^2=1$. Thus

$$c_h=\inf_{(u,v)\in M_h}I(u,v)\le I(\theta w)=\frac{\theta^4}{4}\int \mu_1u_R^4+\mu_2v_R^4+2\beta u_R^2v_R^2=:f(R,h)$$

and $\lim\limits_{R\to\infty}\lim\limits_{h\to0}f(R,h)=E(s_0)$. Since s_0 is arbitrary, this proves the lemma.

Let $V_i^\infty = \liminf_{x\to\infty} V_i(x),\ i = 1, 2$ and define

$$E_\infty = \inf_{(u,v)\in M_\infty} I_\infty(u, v),$$

where

$$I_\infty(u, v) = \int \frac{1}{2}(|\nabla u|^2 + V_1^\infty u^2 + |\nabla v|^2 + V_2^\infty v^2) - \frac{\mu_1}{4}u^4 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2v^2,$$

$$M_\infty = \Big\{(u, v) \in T \Big| \int |\nabla u|^2 + V_1^\infty u^2 + |\nabla v|^2 + V_2^\infty v^2 = \int \mu_1 u^4 + 2\beta u^2v^2 + \mu_2 v^4\Big\},$$

whose associated equations are

$$\begin{cases}\Delta u - V_1^\infty u + \mu_1 u^3 + \beta uv^2 = 0,\\ \Delta v - V_2^\infty v + \mu_2 v^3 + \beta u^2 v = 0.\end{cases}$$

Theorem 4.1 *Assume*

$$E_\infty > \inf_{s\in\mathbb{R}^N} E(s).$$

Then for small h, problem (4.2) *has a nontrivial ground state.*

Proof From the definition of I_h,

$$I_h(u, v) = \frac{1}{2}\|(u, v)\|_{E_h}^2 + o(\|(u, v)\|_{E_h}^2).$$

Hence there exists a constant $\overline{c} > 0$ such that $c_h \geq \overline{c}$ for any h.

By general minimax principle (see e.g. [46] or [38]) and since $c_h = c_h^*$, there exists $w_m = (u_m, v_m) \in E_h$ such that

$$I_h(w_m) \to c_h, \quad I_h'(w_m) \to 0, \quad \text{as } m \to \infty$$

and we can choose w_m so that $I_h(w_m) = \sup_{\theta>0} I_h(\theta w_m)$. This implies that $w_m \in M_h$ and $\|w_m\|_{E_h}$ is bounded. Hence there exists a subsequence, still denoted by w_m, such that w_m converges to some $w_0 = (u_0, v_0)$ weakly in E_h, strongly in $L^p_{\text{loc}}(\mathbb{R}^N)$ $(2 \leq p < 2^*)$, almost everywhere in $\mathbb{R}^N$, classically in $C^2_{\text{loc}}(\mathbb{R}^N)$, and (u_0, v_0) is a solution of (4.2).

By direct computation, we get

$$I_h'(w_m), w_m) = \int |\nabla u_m|^2 + V_1(hx)u_m^2 + |\nabla v_m|^2 + V_2(hx)v_m^2 + \mu_1 u_m^4 + \mu_2 v_m^4 + 2\beta u_m^2 v_m^2.$$

If (u_0, v_0) is nontrivial, then

$$\begin{aligned} c_h &\leq I_h(u_0, v_0)\\ &= \frac{1}{4}\int |\nabla u_0|^2 + V_1(hx)u_0^2 + |\nabla v_0|^2 + V_2(hx)v_0^2\\ &\leq \liminf_{m\to\infty} \frac{1}{4}\int |\nabla u_m|^2 + V_1(hx)u_m^2 + |\nabla v_m|^2 + V_2(hx)v_m^2\\ &= \lim_{m\to\infty}\Big(I_h(w_m) - \frac{1}{4}(I_h'(w_m), w_m)\Big)\\ &\to c_h. \end{aligned}$$

This implies that (u_0, v_0) is a nontrivial ground state of (4.2) (nontrivial solution with the least energy among all nontrivial solutions). In the following, what we need to show is that (u_0, v_0) is nontrivial.

Otherwise, $w_m \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ $(2 \le p < 2^*)$.

Claim 1 $\|w_m\|_{L^4}$ is bounded from below.

Assuming the contrary, we would have $\|u_m\|_{L^4} \to 0,\ \|v_m\|_{L^4} \to 0$, and then

$$\begin{aligned} c_h &= \lim_{m\to\infty} \Big[I_h(w_m) - \frac{1}{2}(I'_h(w_m), w_m) \Big] \\ &= \lim_{m\to\infty} \frac{1}{4} \int \mu_1 u_m^4 + \mu_2 v_m^4 + 2\beta u_m^2 v_m^2 \\ &\to 0. \end{aligned}$$

This is in contradiction with $c_h \ge \overline{c} > 0$.

By the definition of V_i^∞, $\forall \varepsilon > 0,\ \exists \rho > 0$ such that $V_i(x) > V_i^\infty - \varepsilon,\ i = 1, 2$ for all $x : |x| \ge \rho$.

Consider the system

$$\begin{cases} \Delta u - (V_1^\infty - \varepsilon)u + \mu_1 u^3 + \beta u v^2 = 0, \\ \Delta v - (V_2^\infty - \varepsilon)v + \mu_2 v^3 + \beta u^2 v = 0, \end{cases} \tag{4.4}$$

and let $I^\varepsilon, M^\varepsilon$ be defined as above, namely

$$M^\varepsilon = \Big\{ (u, v) \in T \,\Big|\, \int |\nabla u|^2 + (V_1^\infty - \varepsilon)u^2 + |\nabla v|^2 + (V_2^\infty - \varepsilon)v^2 = \int \mu_1 u^4 + 2\beta u^2 v^2 + \mu_2 v^4 \Big\},$$

$$I^\varepsilon(u, v) = \int \frac{1}{2}(|\nabla u|^2 + (V_1^\infty - \varepsilon)u^2 + |\nabla v|^2 + (V_2^\infty - \varepsilon)v^2) - \frac{\mu_1}{4}u^4 - \frac{\mu_2}{4}v^4 - \frac{\beta}{2}u^2 v^2.$$

Then there exists $\theta_m > 0$ such that $\theta_m w_m \in M^\varepsilon$.

Define

$$c^\varepsilon = \inf_{(u,v)\in M^\varepsilon} I^\varepsilon(u, v).$$

Claim 2 θ_m is bounded.

Indeed, by condition (V),

$$\begin{aligned} C_1\|w_m\|^2_{E_h} &\ge \int |\nabla u_m|^2 + (V_1^\infty - \varepsilon)u_m^2 + |\nabla v_m|^2 + (V_2^\infty - \varepsilon)v_m^2 \\ &= \theta_m^2 \int \mu_1 u_m^4 + 2\beta u_m^2 v_m^2 + \mu_2 v_m^4 \\ &\ge C_2 \theta_m^2 \|w_m\|^4_{L^4}. \end{aligned}$$

From Claim 1, θ_m is bounded.

Therefore,

$$\begin{aligned} c_h &= \lim_{m\to\infty} \max_{\theta>0} I_h(\theta w_m) \ge \limsup_{m\to\infty} I_h(\theta_m w_m) \\ &= \limsup_{m\to\infty} \int \frac{1}{2}(|\nabla(\theta_m u_m)|^2 + V_1(hx)(\theta_m u_m)^2 + |\nabla(\theta_m v_m)|^2 + V_2(hx)(\theta_m v_m)^2) \\ &\quad - \frac{\mu_1}{4}(\theta_m u_m)^4 - \frac{\mu_2}{4}(\theta_m v_m)^4 - \frac{\beta}{2}(\theta_m u_m)^2(\theta_m v_m)^2 \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{m\to\infty} \int \frac{1}{2}(|\nabla(\theta_m w_m)|^2 + (V_1^\infty - \varepsilon)(\theta_m w_m)^2 + |\nabla(\theta_m v_m)|^2 + (V_2^\infty - \varepsilon)(\theta_m v_m)^2) \\
&\quad - \frac{\mu_1}{4}(\theta_m u_m)^4 - \frac{\mu_2}{4}(\theta_m v_m)^4 - \frac{\beta}{2}(\theta_m u_m)^2(\theta_m v_m)^2 \\
&\quad + \frac{1}{2}\int (V_1(hx) - (V_1^\infty - \varepsilon))(\theta_m u_m)^2 + (V_2(hx) - (V_2^\infty - \varepsilon))(\theta_m v_m)^2 \\
&\geq: c^\varepsilon + \frac{1}{2}R,
\end{aligned}
$$

where $R = \int_{|x|<\frac{\rho}{h}} (V_1(hx) - (V_1^\infty - \varepsilon))(\theta_m u_m)^2 + (V_2(hx) - (V_2^\infty - \varepsilon))(\theta_m v_m)^2$.

Since $\|\theta_m w_m\|_{L^2_{\rm loc}(\mathbb{R}^N)} \to 0$ as $m \to \infty$, we have $c_h \geq c^\varepsilon > \inf\limits_{s\in\mathbb{R}^N} E(s)$. This is impossible for small h. Therefore (u_0, v_0) is nontrivial and we are done.

Theorem 4.2 *Under the assumption of Theorem* 4.1, *for sufficiently large β and small h, problem* (4.2) *has a ground state.*

Proof It is a consequence of Proposition 4.1 and Theorem 4.1.

5 Concentration

Suppose that (u_h, v_h) is a ground state of (4.2). Now we study the concentration of these ground states as $h \to 0$.

Theorem 5.1 *Assume that $E_\infty > \inf\limits_{s\in\mathbb{R}^N} E(s)$ and β is sufficiently large. Then there exists a sequence $\{h_k\} \to 0$ such that u^{h_k} or v^{h_k} concentrates at the global minimum point x_0 of $E(s)$.*

Define

$$
\mu_h(\Omega) = \frac{1}{4}\int_\Omega |\nabla u_h|^2 + V_1(hx)u_h^2 + |\nabla v_h|^2 + V_2(hx)v_h^2.
$$

Then along a sequence if necessary, as $h \to 0$, $\mu_h = c_h \to \widetilde{c} \leq \inf\limits_{s\in\mathbb{R}^N} E(s)$. By the concentration-compactness principle of P. L. Lions in [27, Part 1], there are three possibilities:

(i) (Compactness) There exists a sequence $\{y_{h_k}\}$ that satisfies: $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ such that

$$
\int_{B_\rho(y_{h_k})} \mathrm{d}\mu_{h_k} \geq \widetilde{c} - \varepsilon.
$$

(ii) (Vanishing) There exists a sequence $\{h_k\} \to 0$ such that for all $\rho > 0$,

$$
\lim_{h_k\to 0} \sup_{y\in\mathbb{R}^N} \int_{B_\rho(y)} \mathrm{d}\mu_k = 0.
$$

(iii) (Dichotomy) There exists a constant $\widetilde{c}'$ with $0 < \widetilde{c}' < \widetilde{c}$, sequences $\{\rho_{h_k}\} \to \infty$ and $\{y_{h_k}\} \subset \mathbb{R}^N$, and two nonnegative measures $\mu_{h_k}^1$ and $\mu_{h_k}^2$ such that

$$
\begin{aligned}
&0 \leq \mu_{h_k}^1 + \mu_{h_k}^2 \leq \mu_{h_k}, \\
&\mathrm{supp}(\mu_{h_k}^1) \subset B_{\rho_{h_k}}(y_{h_k}), \quad \mathrm{supp}(\mu_{h_k}^2) \subset B_{2\rho_{h_k}}^c(y_{h_k}), \\
&\mu_{h_k}^1(\mathbb{R}^N) \to \widetilde{c}', \quad \mu_{h_k}^2(\mathbb{R}^N) \to \widetilde{c} - \widetilde{c}'.
\end{aligned}
$$

For convenience, sometimes we write $u_{h_k} = u_k$, $v_{h_k} = v_k$, $y_{h_k} = y_k$, $\rho_{h_k} = \rho_k$ and $\mu_{h_k} = \mu_k$.

Lemma 5.1 *Only compactness occurs.*

Proof Claim 1 Vanishing does not occur.

Otherwise, from [27, Part 2], $u_k \to 0,\ v_k \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Hence

$$0 = \lim_{h_k \to 0} \frac{1}{4} \int \mu_1 u_k^4 + \mu_2 v_k^4 + 2\beta u_k^2 v_k^2 = \lim_{h_k \to 0} \mu_k(\mathbb{R}^N) = \widetilde{c} > 0,$$

a contradiction.

Claim 2 Dichotomy does not occur.

Otherwise, take $\phi_h \in C_0^1(\mathbb{R}^N)$ such that $\phi_h = 1$ in $B_{\rho_h}(y_h)$, $\phi_h = 0$ in $B_{2\rho_h}^c(y_h)$ and $0 \le \phi_h \le 1$, $|\nabla \phi_h| \le \frac{2}{\rho_h}$. Write

$$u_h = \phi_h u_h + (1 - \phi_h) u_h := u_{1h} + u_{2h}, \quad v_h = \phi_h v_h + (1 - \phi_h) v_h := v_{1h} + v_{2h}.$$

Then as $h_k \to 0$,

$$I_{h_k}(u_{1h_k}, v_{1h_k}) \ge \mu_k(B_{\rho_k}(y_k)) \ge \mu_k^1(B_{\rho_k}(y_k)) = \mu_k^1(\mathbb{R}^N) \to \widetilde{c}\,'.$$

Similarly,

$$I_{h_k}(u_{2h_k}, v_{2h_k}) \ge \mu_k(B_{2\rho_k}^c(y_k)) \ge \mu_k^2(B_{2\rho_k}^c(y_k)) = \mu_k^2(\mathbb{R}^N) \to \widetilde{c} - \widetilde{c}\,'.$$

Let $\Omega_h = B_{2\rho_h}(y_h) \setminus B_{\rho_h}(y_h)$. Then

$$\begin{aligned}
&\frac{1}{4} \int_{\Omega_{h_k}} |\nabla u_k|^2 + V_1(h_k x) u_k^2 + |\nabla v_k|^2 + V_2(h_k x) v_k^2 \\
&= \mu_k(\Omega_{h_k}) = \mu_k(\mathbb{R}^N) - \mu_k(B_{\rho_k}(y_k)) - \mu_k(B_{2\rho_k}^c(y_k)) \\
&\le \mu_k(\mathbb{R}^N) - \mu_k^1(\mathbb{R}^N) - \mu_k^2(\mathbb{R}^N) \\
&\to 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\widetilde{c} &= \lim_{h_k \to 0} I_{h_k}(u_k, v_k) \\
&= \lim_{h_k \to 0} \frac{1}{4} \int |\nabla u_k|^2 + V_1(h_k x) u_k^2 + |\nabla v_k|^2 + V_2(h_k x) v_k^2 \\
&= \lim_{h_k \to 0} I_{h_k}(u_{1h_k}, v_{1h_k}) + I_{h_k}(u_{2h_k}, v_{2h_k}) + J_{h_k} \\
&\ge \widetilde{c},
\end{aligned}$$

where $J_{h_k} = o(1)$ as $h_k \to 0$. Hence

$$\lim_{h_k \to 0} I_{h_k}(u_{1h_k}, v_{1h_k}) = \widetilde{c}\,', \quad \lim_{h_k \to 0} I_{h_k}(u_{2h_k}, v_{2h_k}) = \widetilde{c} - \widetilde{c}\,'.$$

Set

$$J_{h_k}^1 = \int |\nabla u_{1h_k}|^2 + V_1(h_k x) u_{1h_k}^2 + |\nabla v_{1h_k}|^2 + V_2(h_k x) v_{1h_k}^2 - \mu_1 u_{1h_k}^4 - \mu_2 v_{1h_k}^4 - 2\beta u_{1h_k}^2 v_{1h_k}^2,$$

$$J_{h_k}^2 = \int |\nabla u_{2h_k}|^2 + V_1(h_k x) u_{2h_k}^2 + |\nabla v_{2h_k}|^2 + V_2(h_k x) v_{2h_k}^2 - \mu_1 u_{2h_k}^4 - \mu_2 v_{2h_k}^4 - 2\beta u_{2h_k}^2 v_{2h_k}^2.$$

Since $(u_k, v_k) \in M_{h_k}$, $J^1_{h_k} + J^2_{h_k} = o(1)$.

Case 1 $J^1_{h_k} \le 0$.

Since (u_{1h_k}, v_{1h_k}) is nontrivial, there exists $\theta > 0$ such that $(\theta u_{1h_k}, \theta v_{1h_k}) \in M_{h_k}$, i.e.

$$\int |\nabla u_{1h_k}|^2 + V_1(h_k x) u^2_{1h_k} + |\nabla v_{1h_k}|^2 + V_2(h_k x) v^2_{1h_k} = \theta^2 \int \mu_1 u^4_{1h_k} + \mu_2 v^4_{1h_k} + 2\beta u^2_{1h_k} v^2_{1h_k}.$$

Hence $\theta \le 1$ and

$$c_{h_k} \le I_{h_k}(\theta u_{1h_k}, \theta v_{1h_k}) \le I_{h_k}(u_{1h_k}, v_{1h_k}) \to \widetilde{c}\,' < \widetilde{c}.$$

This is a contradiction.

Case 2 $J^2_{h_k} \le 0$.

Similar to Case 1, this also leads to a contradiction.

Case 3 $J^1_{h_k} > 0$ and $J^2_{h_k} > 0$.

This implies $J^1_{h_k} = o(1)$ and $J^2_{h_k} = o(1)$. If $\theta \le 1 + o(1)$, we are done by arguments similar to those in the proof of case 1. Assume $\lim\limits_{h_k \to 0} \theta = \theta_0 > 1$. Then

$$0 = \lim_{h_k \to 0} J^1_{h_k} = \lim_{h_k \to 0} \frac{2(\theta^2 - 1)}{2\theta^2 - 1} I_{h_k}(u_{1h_k}, v_{h_k}) = \frac{2(\theta_0^2 - 1)}{2\theta_0^2 - 1} \widetilde{c}\,' > 0,$$

a contradiction.

Let

$$w_k(x) = u^{h_k}(h_k x + h_k y_k) = u_k(x + y_k), \quad z_k(x) = v^{h_k}(h_k x + h_k y_k) = v_k(x + y_k).$$

Then (w_k, z_k) is a ground state of

$$\begin{cases} \Delta w_k - V_1(h_k x + y_k h_k) w_k + \mu_1 w_k^3 + \beta w_k z_k^2 = 0, \\ \Delta z_k - V_2(h_k x + y_k h_k) z_k + \mu_2 z_k^3 + \beta w_k^2 z_k = 0. \end{cases} \tag{5.1}$$

Lemma 5.2 *$\{h_k y_k\}$ is bounded and $(w_k, z_k) \to (w_0, z_0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $1 < p < 2^*$.*

Proof Assume $h_k y_k \to \infty$. Since c_{h_k} is bounded, there exists a subsequence of (w_k, z_k), still denoted by (w_k, z_k), such that $(w_k, z_k) \rightharpoonup (w_0, z_0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. By the compactness of μ_{h_k}, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ such that

$$\frac{1}{4} \int_{B^c_\rho} |\nabla w_k|^2 + \lambda_1 w_k^2 + |\nabla z_k|^2 + \lambda_2 z_k^2 \le \mu_{h_k}(B^c_\rho(y_k)) < \varepsilon.$$

This implies that $(w_k, z_k) \to (w_0, z_0)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $1 < p < 2^*$. Observe that

$$\frac{1}{4} \int \mu_1 w_0^4 + \mu_2 z_0^4 + 2\beta w_0^2 z_0^2 = \lim_{h_k \to 0} \frac{1}{4} \int \mu_1 w_k^4 + \mu_2 z_k^4 + 2\beta w_k^2 z_k^2 \ge \limsup_{h_k \to 0} c_{h_k} \ge \overline{c} > 0.$$

Thus (w_0, z_0) is nontrivial.

Since $h_k y_k \to \infty$, (w_0, z_0) satisfies the following equations

$$\begin{cases} \Delta w_0 - \Big(V_1^\infty - \dfrac{\varepsilon}{2}\Big) w_0 + \mu_1 w_0^3 + \beta w_0 z_0^2 \ge 0, \\ \Delta z_0 - \Big(V_2^\infty - \dfrac{\varepsilon}{2}\Big) z_0 + \mu_2 z_0^3 + \beta w_0^2 z_0 \ge 0. \end{cases} \tag{5.2}$$

In particular,

$$\int |\nabla w_0|^2 + (V_1^\infty - \varepsilon)w_0^2 + |\nabla z_0|^2 + (V_2^\infty - \varepsilon)z_0^2 < \int \mu_1 w_0^4 + \mu_2 z_0^4 + 2\beta w_0^2 z_0^2.$$

Take $\theta > 0$ such that $(\theta w_0, \theta z_0) \in M^\varepsilon$. Then $\theta < 1$ and

$$\begin{aligned} c^\varepsilon &\leq \frac{\theta^2}{4} \int |\nabla w_0|^2 + (V_1^\infty - \varepsilon)w_0^2 + |\nabla z_0|^2 + (V_2^\infty - \varepsilon)z_0^2 \\ &\leq \liminf_{h_k \to 0} \frac{1}{4} \int |\nabla w_k|^2 + V_1(h_k x + h_k y_k)w_k^2 + |\nabla z_k|^2 + V_2(h_k x + h_k y_k)z_k^2 \\ &= \liminf_{h_k \to 0} c_{h_k} \\ &\leq \inf_{s \in \mathbb{R}^N} E(s). \end{aligned}$$

By Lemma 4.1, $c^\varepsilon \to E_\infty$ as $\varepsilon \to 0$. This is in contradiction with $E_\infty > \inf_{s\in\mathbb{R}^N} E(s)$.

Assume $\overline{x}_k = h_k y_k \to x_0$. Then (w_0, z_0) satisfies

$$\begin{cases} \Delta w_0 - V_1(x_0)w_0 + \mu_1 w_0^3 + \beta w_0 z_0^2 = 0, \\ \Delta z_0 - V_2(x_0)z_0 + \mu_2 z_0^3 + \beta w_0^2 z_0 = 0. \end{cases} \tag{5.3}$$

Lemma 5.3 *$E(x_0) = \inf_{s\in\mathbb{R}^N} E(s)$ and $(w_k, z_k) \to (w_0, z_0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.*

Proof

$$\begin{aligned} \inf_{s\in\mathbb{R}^N} E(s) \leq E(x_0) &\leq \frac{1}{4} \int |\nabla w_0|^2 + V_1(x_0)w_0^2 + |\nabla z_0|^2 + V_2(x_0)z_0^2 \\ &= \frac{1}{4} \int \mu_1 w_0^4 + \mu_2 z_0^4 + 2\beta w_0^2 z_0^2 \\ &= \lim_{k\to\infty} \frac{1}{4} \int \mu_1 w_k^4 + \mu_2 z_k^4 + 2\beta w_k^2 z_k^2 \\ &= \frac{1}{4} \lim_{k\to\infty} \int |\nabla w_k|^2 + V_1(h_k x + \overline{x}_k)w_k^2 + |\nabla z_k|^2 + V_2(h_k x + \overline{x}_k)z_k^2 \\ &= \liminf_{h_k\to 0} c_{h_k} \leq \inf_{s\in\mathbb{R}^N} E(s). \end{aligned}$$

Thus $E(x_0) = \inf_{s\in\mathbb{R}^N} E(s)$ and there exists a subsequence (w_k, z_k) such that

$$\begin{aligned} &\int |\nabla w_k|^2 + V_1(h_k x + \overline{x}_k)w_k^2 + |\nabla z_k|^2 + V_2(h_k x + \overline{x}_k)z_k^2 \\ &\to \int |\nabla w_0|^2 + V_1(x_0)w_0^2 + |\nabla z_0|^2 + V_2(x_0)z_0^2. \end{aligned}$$

By integration on the complement of large balls and using Fatou's lemma, we have

$$\int |\nabla w_k|^2 + |\nabla z_k|^2 \to \int |\nabla w_0|^2 + |\nabla z_0|^2.$$

Hence $(w_k, z_k) \to (w_0, z_0)$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Proof of Theorem 5.1 Let x_{1k} be a local maximum point of w_k and x_{2k} a local maximum point of z_k. Then $\Delta w_k(x_{1k}) \leq 0$ and $\Delta z_k(x_{2k}) \leq 0$. From (5.1),

$$\mu_1 w_k^2(x_{1k}) + \beta z_k^2(x_{1k}) \geq V_1(h_k x_{1k} + \overline{x}_k) \geq \lambda_1$$

and

$$\beta w_k^2(x_{2k}) + \mu_2 z_k^2(x_{2k}) \geq V_2(h_k x_{2k} + \overline{x}_k) \geq \lambda_2.$$

Hence, along a subsequence if necessary, $w_k(x_{1k}) > c_1$ or $z_k(x_{2k}) > c_2$ for some positive constant c_1 and c_2 .

Suppose $w_k(x_{1k}) > c_1$. From Lemma 5.3, $w_k(x) \to 0$ as $x \to \infty$ uniformly with respect to k. Since $u^{h_k}(x) = w_k(\frac{x-\overline{x}_k}{h_k})$ and $\overline{x}_k \to x_0$, we conclude that u^{h_k} concentrates at x_0.

Acknowledgements This work was done while the author was in Fudan University as a postdoctor fellow. He is very grateful to professor Hong Jiaxing for his helpful suggestions and advices. The author would like to thank the referees for their careful reading of the manuscript and useful comments, and especially for causing the author's attention of the references [1, 2] and the physical meaning of Planck constant in BEC and nonlinear optics.

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