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# Sharp Growth Theorems and Coefficient Bounds for Starlike Mappings in Several Complex Variables***

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**Abstract** Let $B$ be the unit ball in a complex Banach space. Let $S^*_{k+1}(B)$ be the family of normalized starlike mappings $f$ on $B$ such that $z=0$ is a zero of order $k+1$ of $f(z)-z$. The authors obtain sharp growth and covering theorems, as well as sharp coefficient bounds for various subsets of $S^*_{k+1}(B)$.

**Keywords** Sharp coefficient bound, Sharp covering theorem, Sharp growth theorem, Starlike mapping, Zero of order $k$
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## 1 Introduction

In the case of one complex variable, the following growth theorem and de Branges theorem are well-known (cf. [28]).

**Theorem 1.1** *Let* $f(z)=z+\sum\limits_{m=2}^{\infty}a_m z^m$ *be a normalized univalent holomorphic function on the unit disc* $U$ *in* $\mathbb{C}$. *Then*

(i)
$$\frac{|z|}{(1+|z|)^2}\le|f(z)|\le\frac{|z|}{(1-|z|)^2}.\tag{1.1}$$

(ii)
$$a_m|\le m.$$

However, in the case of several complex variables, Cartan [3] pointed out that the above theorem does not hold.

Barnard, FitzGerald and Gong [2], Chuaqui [4] extended the growth theorem (1.1) to normalized starlike mappings on the Euclidean unit ball in $\mathbb{C}^n$. Kohr [20] obtained a sharp growth theorem for normalized starlike mappings of order $\alpha$ on the Euclidean unit ball in $\mathbb{C}^n$. Liu and Liu [24] generalized these results and obtained a sharp growth theorem for a normalized starlike mapping $f$ or a normalized starlike mapping $f$ of order $\alpha$ on the unit ball $B$ in a complex Banach space such that $f(z)-z$ has a zero of order $k+1$ at $z=0$. On the other hand, Graham, Hamada and Kohr [8] obtained a growth theorem for the set $S^0_g(B)$ of mappings which have

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$g$-parametric representation on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm. Hamada, Honda and Kohr [12] generalized this result to the set $S^0_{g,k+1}(B)$ of mappings $f$ for which there exists a $g$-Loewner chain $f(z,t)$ such that $\{\mathrm{e}^{-t}f(z,t)\}_{t\geq 0}$ is a normal family on the unit ball $B$ in $\mathbb{C}^n$ with respect to an arbitrary norm, $f = f(\cdot, 0)$ and $z = 0$ is a zero of order $k+1$ of $\mathrm{e}^{-t}f(z,t) - z$ for each $t \geq 0$. For the growth theorem of convex mappings, see [6, 15, 18, 34].

Concerning the bounds for coefficients of subclasses of normalized biholomorphic mappings, Kohr [21] obtained a sharp bound for the second coefficient of starlike mappings or starlike mappings of order $\alpha$ on the Euclidean unit ball in $\mathbb{C}^n$. Gong [7] obtained bounds for the second and third coefficients of starlike mappings on the unit polydisc in $\mathbb{C}^n$. Liu and Liu [25] obtained bounds for the $m$-th coefficient ($m = k+1, \cdots, 2k$) of starlike mappings or starlike mappings of order $\alpha$ on the unit ball in a complex Banach space. Especially, the bound is sharp when $m = k+1$. On the other hand, Graham, Hamada and Kohr [8] obtained a bound for the second coefficient of mappings which have $g$-parametric representation on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm. Hamada, Honda and Kohr [12] obtained a bound for the $(k+1)$-th coefficient of mappings in the set $S^0_{g,k+1}(B)$, where $B$ is the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm.

We do not know whether the growth theorems and coefficient bounds in [8] and [12] are sharp or not. In this paper, we shall obtain sharp growth and covering theorems, as well as sharp coefficient bounds for various subsets of $S^*_{k+1}(B)$, where $B$ is the unit ball in a complex Banach space and $S^*_{k+1}(B)$ is the family of normalized starlike mappings $f$ on $B$ such that $z = 0$ is a zero of order $k+1$ of $f(z) - z$. These results are generalizations of the above sharp results.

## 2 Preliminaries

Let $X$ be a complex Banach space with respect to a norm $\|\cdot\|$. Let $B_r = \{z \in X : \|z\| < r\}$ and $B = B_1$. When $X = \mathbb{C}$, $B_r$ is denoted by $U_r$ and $B_1$ by $U$. For a domain $G \subset X$, let $H(G)$ be the set of holomorphic mappings from $G$ into $X$. When $f \in H(B)$, we say that $f$ is biholomorphic on $B$ if $f(B)$ is a domain and the inverse exists and is holomorphic on $f(B)$. Let $L(X,Y)$ denote the set of continuous linear operators from $X$ into a complex Banach space $Y$. Let $I$ be the identity in $L(X,X)$. For each $z \in X \setminus \{0\}$, let

$$T(z) = \{l_z \in L(X,\mathbb{C}) : \ l_z(z) = \|z\|, \ \|l_z\| = 1\}.$$

This set is nonempty by the Hahn-Banach theorem.

When $f \in H(B)$, we say that $f$ is normalized if $f(0) = 0$ and $Df(0) = I$. When $f \in H(B)$, we say that $f$ is starlike if $f$ is biholomorphic on $B$ and $f(B)$ is starlike with respect to the origin. Let $S^*(B)$ be the set of normalized starlike mappings on $B$. When $B = U$, the set $S^*(U)$ is denoted by $S^*$.

**Assumption 2.1** *Let $g : U \to \mathbb{C}$ be a univalent holomorphic function such that $g(0) = 1$, $g(\overline{\zeta}) = \overline{g(\zeta)}$ for $\zeta \in U$ (so, $g$ has real coefficients in its power series expansion), $g'(0) < 0$ and*

$\Re g(\zeta) > 0$ *on* $U$. *We assume that* $g$ *satisfies the conditions*

$$\begin{cases} \min\limits_{|\zeta|=r} \Re g(\zeta) = g(r), \\ \max\limits_{|\zeta|=r} \Re g(\zeta) = g(-r) \end{cases}$$

*for* $r \in (0,1)$.

We mention that there are many functions which satisfy the above assumption (cf. [8]).

The following set $\mathcal{M}$ of normalized mappings of "positive real part" on $B$ plays a fundamental role in the study of the Loewner differential equations. Let

$$\mathcal{M} = \{p \in H(B) : p(0) = 0, Dp(0) = I, \Re l_z(p(z)) > 0, z \in B \setminus \{0\}, l_z \in T(z)\}.$$

As in [8, 12, 22], we shall introduce various subsets of $\mathcal{M}$. Let

$$\mathcal{M}_g = \Big\{p \in H(B) : p(0) = 0, Dp(0) = I, \frac{1}{\|z\|} l_z(p(z)) \in g(U), z \in B \setminus \{0\}, \ l_z \in T(z)\Big\}.$$

If $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, then we obtain $\mathcal{M}_g = \mathcal{M}$. However, there are other choices of $g$ which provide interesting properties of the set $\mathcal{M}_g$.

Let $f \in H(B)$ be a normalized locally biholomorphic mapping. Then $f$ is starlike if and only if $[Df(z)]^{-1}f(z) \in \mathcal{M}$ (cf. [10, 16, 33]). Let $S_g^*(B)$ denote the subset of $S^*(B)$ consisting of those normalized locally biholomorphic mappings $f$ such that $[Df(z)]^{-1}f(z) \in \mathcal{M}_g$.

Now, we will give particular subsets of $S^*(B)$.

**Definition 2.1** *Let* $0 \le p < 1$. *A normalized locally biholomorphic mapping* $f \in H(B)$ *is said to be starlike of order* $p$ *if*

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

*where* $g(\zeta) = \frac{1-\zeta}{1-(2p-1)\zeta}$.

We denote by $S_p^*(B)$ the set of all starlike mappings of order $p$ on $B$.

**Definition 2.2** *Let* $0 < \alpha \le 1$. *A normalized locally biholomorphic mapping* $f \in H(B)$ *is said to be strongly starlike of order* $\alpha$ *if*

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

*where* $g(\zeta) = \frac{(1-\zeta)^\alpha}{(1+\zeta)^\alpha}$ *and the branches of the power functions are chosen such that* $(1-\zeta)^\alpha|_{\zeta=0} = (1+\zeta)^\alpha|_{\zeta=0} = 1$.

We denote by $SS_\alpha^*(B)$ the set of all strongly starlike mappings of order $\alpha$ on $B$.

**Definition 2.3** *Let* $0 \le \alpha < 1$. *A normalized locally biholomorphic mapping* $f \in H(B)$ *is said to be almost starlike of order* $\alpha$ *if*

$$[Df(z)]^{-1}f(z) \in \mathcal{M}_g,$$

*where* $g(\zeta) = (1-\alpha)\frac{1-\zeta}{1+\zeta} + \alpha$.

We denote by $AS_\alpha^*(B)$ the set of all almost starlike mappings of order $\alpha$ on $B$.

**Definition 2.4** *Let*

$$q_\rho(\zeta) = 1 + \frac{4(1-\rho)}{\pi^2}\Big(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\Big)^2,$$

*where we choose the branch of the square root such that* $\sqrt{\zeta}\,|_{\zeta=1} = 1$ *and the branch of logarithm function such that* $\log 1 = 0$, $0 \le \rho < 1$. *A normalized locally biholomorphic mapping* $f \in H(B)$ *is said to be parabolic starlike of order* $\rho$ *if*

$$[Df(z)]^{-1} f(z) \in \mathcal{M}_g,$$

*where* $g = \frac{1}{q_\rho}$.

We denote by $PS^*_\rho(B)$ the set of all parabolic starlike mappings of order $\rho$ on $B$.

When $B = U$, $S^*(U)$ (respectively $S^*_g(U)$, $S^*_p(U)$, $SS^*_\alpha(U)$, $AS^*_\alpha(U)$, $PS^*_\rho(U)$) is denoted by $S^*$ (respectively $S^*_g$, $S^*_p$, $SS^*_\alpha$, $AS^*_\alpha$, $PS^*_\rho$).

Let $f \in H(B)$ and let $k$ be a positive integer. We say that $z = 0$ is a zero of order $k$ of $f(z)$ if $f(0) = 0, \cdots, D^{k-1}f(0) = 0$ and $D^k f(0) \ne 0$ (compare with [26]).

Also, we denote by $S^*_{k+1}(B)$ (respectively $S^*_{g,k+1}(B)$, $S^*_{p,k+1}(B)$, $SS^*_{\alpha,k+1}(B)$, $AS^*_{\alpha,k+1}(B)$, $PS^*_{\rho,k+1}(B)$) the subset of $S^*(B)$ (respectively $S^*_g(B)$, $S^*_p(B)$, $SS^*_\alpha(B)$, $AS^*_\alpha(B)$, $PS^*_\rho(B)$) of mappings $f$ such that $z = 0$ is a zero of order $k+1$ of $f(z) - z$. When $B = U$, $S^*_{k+1}(U)$ (respectively $S^*_{g,k+1}(U)$, $S^*_{p,k+1}(U)$, $SS^*_{\alpha,k+1}(U)$, $AS^*_{\alpha,k+1}(U)$, $PS^*_{\rho,k+1}(U)$) is denoted by $S^*_{k+1}$ (respectively $S^*_{g,k+1}$, $S^*_{p,k+1}$, $SS^*_{\alpha,k+1}$, $AS^*_{\alpha,k+1}$, $PS^*_{\rho,k+1}$).

## 3 Sharpness of Growth Theorems

For the set $S^*_{g,k+1}(B)$, we obtain the following growth result by an argument similar to that in the proof of [12, Theorem 10] (cf. [8, Theorem 2.2] and [22, Theorem 2.3]).

**Theorem 3.1** *Let* $g : U \to \mathbb{C}$ *satisfy the conditions of Assumption* 2.1 *and* $f \in S^*_{g,k+1}(B)$. *Then*

$$\|z\| \exp\int_0^{\|z\|}\Big[\frac{1}{g(-x^k)} - 1\Big]\frac{\mathrm{d}x}{x} \le \|f(z)\| \le \|z\| \exp\int_0^{\|z\|}\Big[\frac{1}{g(x^k)} - 1\Big]\frac{\mathrm{d}x}{x}, \quad z \in B. \qquad (3.1)$$

Later, we will show that the above estimations (3.1) are sharp. First, we give a lemma.

Let $b \in S^*_g$ be defined by $b(0) = b'(0) - 1 = 0$ and

$$\frac{\zeta b'(\zeta)}{b(\zeta)} = \frac{1}{g(\zeta)}, \quad \zeta \in U.$$

For a positive integer $k$, let

$$b_k(\zeta) = \zeta[\varphi(\zeta^k)]^{\frac{1}{k}},$$

where

$$\varphi(\zeta) = \frac{b(\zeta)}{\zeta}.$$

The branches of the power functions are chosen so that

$$(\varphi(\zeta^k))^{\frac{1}{k}}|_{\zeta=0} = 1.$$

Also, for $u \in \partial B$, let

$$f_u(z) = \frac{b_k(l_u(z))}{l_u(z)} z, \quad z \in B. \tag{3.2}$$

Then, we obtain the following lemma by direct computations.

**Lemma 3.1** *Let $b_k$ and $f_u$ be as above. Then*
(i) $b_k(\zeta) = \zeta - \frac{1}{k} g'(0)\zeta^{k+1} + \cdots$, $\zeta \in U$ *and*

$$\frac{\zeta b_k'(\zeta)}{b_k(\zeta)} = \frac{1}{g(\zeta^k)}, \quad \zeta \in U.$$

*Thus,* $b_k \in S^*_{g,k+1}$ *and* $b_k(0) = b_k'(0) - 1 = 0$.
(ii) $f_u \in S^*_{g,k+1}(B)$ *and*

$$f_u(\zeta u) = b_k(\zeta) u = \Big(\zeta - \frac{1}{k} g'(0)\zeta^{k+1} + \cdots\Big) u, \quad \zeta \in U.$$

Now, we obtain the following equivalent formulation of Theorem 3.1 (cf. [1, Theorem 2.5], [13] and [27]).

**Theorem 3.2** *Let $g : U \to \mathbb{C}$ satisfy the conditions of Assumption* 2.1. *If $f \in S^*_{g,k+1}(B)$, then*

$$\mathrm{e}^{-\frac{\pi \mathrm{i}}{k}} b_k(\mathrm{e}^{\frac{\pi \mathrm{i}}{k}} \|z\|) \le \|f(z)\| \le b_k(\|z\|), \quad z \in B. \tag{3.3}$$

*These estimations are sharp.*

**Proof** From (3.1) and Lemma 3.1(i), we obtain

$$\exp \int_0^{\|z\|} \Big[\frac{x \widetilde{b}_k'(x)}{\widetilde{b}_k(x)} - 1\Big] \frac{\mathrm{d}x}{x} \le \frac{\|f(z)\|}{\|z\|} \le \exp \int_0^{\|z\|} \Big[\frac{x b_k'(x)}{b_k(x)} - 1\Big] \frac{\mathrm{d}x}{x}$$

for $z \in B$, where $\widetilde{b}_k(\zeta) = \mathrm{e}^{-\frac{\pi \mathrm{i}}{k}} b_k(\mathrm{e}^{\frac{\pi \mathrm{i}}{k}} \zeta)$. Then, we obtain

$$\exp \Big[\log \frac{\widetilde{b}_k(\|z\|)}{\|z\|} - \log \widetilde{b}_k'(0)\Big] \le \frac{\|f(z)\|}{\|z\|} \le \exp \Big[\log \frac{b_k(\|z\|)}{\|z\|} - \log b_k'(0)\Big]$$

for $z \in B$, since $\widetilde{b}_k(x), b_k(x) > 0$ for $x > 0$. This implies (3.3).

Next, we will show that the estimations (3.3) are sharp. Let $f_u \in S^*_{g,k+1}(B)$ be as in (3.2). Since $\|f_u(ru)\| = b_k(r)$ and $\|f_u(\mathrm{e}^{\frac{\pi \mathrm{i}}{k}} ru)\| = |b_k(\mathrm{e}^{\frac{\pi \mathrm{i}}{k}} r)|$, the equalities of the estimations (3.3) hold. This completes the proof.

**Remark 3.1** The equivalence of (3.1) and (3.3) implies that the estimations (3.1) are sharp.

Now, we obtain the following corollaries from Theorems 3.1 or 3.2.

For $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, we have the following sharp growth and covering results for the set $S^*_{k+1}(B)$ due to Liu and Liu [24, Theorem 1] (cf. [2, 4, 11, 35]).

**Corollary 3.1** *If $f \in S^*_{k+1}(B)$, then*

$$\frac{\|z\|}{(1 + \|z\|^k)^{\frac{2}{k}}} \le \|f(z)\| \le \frac{\|z\|}{(1 - \|z\|^k)^{\frac{2}{k}}}, \quad z \in B.$$

*Consequently,* $f(B) \supset B_{2^{-\frac{2}{k}}}$. *These estimations are sharp.*

For $g(\zeta)=\frac{1-\zeta}{1-(2p-1)\zeta}$, where $0\le p<1$, we have the following sharp growth and covering results for the set $S^*_{p,k+1}(B)$ due to Liu and Liu [24, Theorem 2] (cf. [17, 20]).

**Corollary 3.2** *If $f\in S^*_{p,k+1}(B)$, then*

$$\frac{\|z\|}{(1+\|z\|^k)^{\frac{2(1-p)}{k}}}\le\|f(z)\|\le\frac{\|z\|}{(1-\|z\|^k)^{\frac{2(1-p)}{k}}},\quad z\in B.$$

*Consequently, $f(B)\supset B_{2^{-\frac{2(1-p)}{k}}}$. These estimations are sharp.*

For $g(\zeta)=\frac{(1-\zeta)^\alpha}{(1+\zeta)^\alpha}$, where $0<\alpha\le 1$, we have the following sharp growth and covering results for the set $SS^*_{\alpha,k+1}(B)$ (cf. [23]).

**Corollary 3.3** *Let $f\in SS^*_{\alpha,k+1}(B)$. Then*

$$\|z\|\exp\int_0^{\|z\|}\Big[\Big(\frac{1-x^k}{1+x^k}\Big)^\alpha-1\Big]\frac{\mathrm{d}x}{x}\le\|f(z)\|\le\|z\|\exp\int_0^{\|z\|}\Big[\Big(\frac{1+x^k}{1-x^k}\Big)^\alpha-1\Big]\frac{\mathrm{d}x}{x},\quad z\in B.$$

*Consequently, $f(B)\supset B_r$, where*

$$r=\exp\int_0^1\Big[\Big(\frac{1-x^k}{1+x^k}\Big)^\alpha-1\Big]\frac{\mathrm{d}x}{x}.$$

*These estimations are sharp.*

For $g(\zeta)=(1-\alpha)\frac{1-\zeta}{1+\zeta}+\alpha$, where $0\le\alpha<1$, we have the following sharp growth and covering results for the set $AS^*_{\alpha,k+1}(B)$ (cf. [5, 19]).

**Corollary 3.4** *Let $f\in AS^*_{\alpha,k+1}(B)$.*

(i) *If $\alpha\in[0,1)$ and $\alpha\ne\frac{1}{2}$, then*

$$\frac{\|z\|}{(1+(1-2\alpha)\|z\|^k)^{\frac{2(1-\alpha)}{k(1-2\alpha)}}}\le\|f(z)\|\le\frac{\|z\|}{(1-(1-2\alpha)\|z\|^k)^{\frac{2(1-\alpha)}{k(1-2\alpha)}}}$$

*for $z\in B$. Consequently, $f(B)\supset B_r$, where*

$$r=\frac{1}{(2-2\alpha)^{\frac{2(1-\alpha)}{k(1-2\alpha)}}}.$$

*These estimations are sharp.*

(ii) *If $\alpha=\frac{1}{2}$, then*

$$\|z\|\exp\Big(-\frac{1}{k}\|z\|^k\Big)\le\|f(z)\|\le\|z\|\exp\Big(\frac{1}{k}\|z\|^k\Big),\quad z\in B.$$

*Consequently, $f(B)\supset B_r$, where*

$$r=\exp\Big(-\frac{1}{k}\Big).$$

*These estimations are sharp.*

For $g=\frac{1}{q_\rho}$, where

$$q_\rho(\zeta)=1+\frac{4(1-\rho)}{\pi^2}\Big(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\Big)^2$$

and $0\le\rho<1$, we have the following sharp growth and covering results for the set $PS^*_{\rho,k+1}(B)$ (cf. [1, Theorem 2.5], [13], [27] and [31, Theorem 2.3]).

**Corollary 3.5** *If $f \in PS^*_{\alpha,k+1}(B)$, then*

$$-\frac{16(1-\rho)}{\pi^2}\int_0^{\|z\|}\frac{1}{x}(\tan^{-1}\sqrt{x^k})^2\mathrm{d}x \le \log\frac{\|f(z)\|}{\|z\|} \le \frac{4(1-\rho)}{\pi^2}\int_0^{\|z\|}\frac{1}{x}\Big(\log\frac{1+\sqrt{x^k}}{1-\sqrt{x^k}}\Big)^2\mathrm{d}x.$$

*Consequently, $f(B) \supset B_r$, where*

$$r = \exp\Big(-\frac{16(1-\rho)}{\pi^2}\int_0^1 \frac{1}{x}(\tan^{-1}\sqrt{x^k}\,)^2\mathrm{d}x\Big).$$

*These estimations are sharp.*

## 4 Coefficient Bounds

We now obtain an estimate for the $(k+1)$-th order coefficients of mappings in the set $S^*_{g,k+1}(B)$ as in [12, Theorem 24] (cf. [8, Theorem 2.14], [22, Theorem 2.4] and [29, Theorem 3]). Moreover, we will show that this estimation is sharp.

**Theorem 4.1** *Let $g : U \to \mathbb{C}$ satisfy the conditions of Assumption* 2.1 *and $f \in S^*_{g,k+1}(B)$. Then*

$$\Big|\frac{1}{(k+1)!}l_w(D^{k+1}f(0)(w^{k+1}))\Big| \le \frac{1}{k}|g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w). \tag{4.1}$$

*This estimation is sharp.*

**Proof** Since $f \in S^*_{g,k+1}(B)$, $f$ has the Taylor expansion

$$f(z) = z + \frac{1}{(k+1)!}D^{k+1}f(0)(z^{k+1}) + \cdots, \quad z \in B.$$

Let $h(z) = [Df(z)]^{-1}f(z)$. Then $h$ has the Taylor expansion

$$h(z) = z - \frac{k}{(k+1)!}D^{k+1}f(0)(z^{k+1}) + \cdots, \quad z \in B.$$

Fix $w \in \partial B$, $l_w \in T(w)$. Let

$$p(\zeta) = \begin{cases} \dfrac{1}{\zeta}l_w(h(\zeta w)), & \zeta \in U \setminus \{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Then $p$ is a holomorphic function on $U$ and has the Taylor expansion

$$p(\zeta) = 1 - \frac{k}{(k+1)!}l_w(D^{k+1}f(0)(w^{k+1}))\zeta^k + \cdots, \quad \zeta \in U.$$

Therefore, we obtain

$$p^{(k)}(0) = -\frac{k}{k+1}l_w(D^{k+1}f(0)(w^{k+1})).$$

Also, there exists a holomorphic function $\widetilde{p}$ on $U$ such that $p(\zeta) = 1 + \zeta^k\widetilde{p}(\zeta)$ for $\zeta \in U$. Since $h \in \mathcal{M}_g$, we deduce that $p(\zeta) \in g(U)$ for $\zeta \in U$. Therefore, $g^{-1} \circ p(\,\cdot\,) : U \to U$ and $g^{-1} \circ p(0) = 0$. Since $g^{-1}(1) = 0$, there exists a holomorphic function $G(w)$ on a neighborhood of 1 such that $g^{-1}(w) = (w-1)G(w)$. Therefore, we obtain $g^{-1}\circ p(\zeta) = \zeta^k \widetilde{p}(\zeta) G(p(\zeta))$ on a neighborhood of 0. Then, by the Schwarz lemma, we obtain $|g^{-1}\circ p(\zeta)| \le |\zeta|^k$ for $\zeta \in U$. Thus, there exists a holomorphic function $H(\zeta)$ on $U$ such that $g^{-1}\circ p(\zeta) = \zeta^k H(\zeta)$ on $U$ and $|H(\zeta)| \le 1$ on $U$. Then, $p(\zeta) = g(\zeta^k H(\zeta))$. Hence we obtain $|p^{(k)}(0)| \le k!|g'(0)|$. Thus, we obtain

$$\Big|\frac{1}{(k+1)!} l_w(D^{k+1}f(0)(w^{k+1}))\Big| = \frac{1}{k}\frac{1}{k!}|p^{(k)}(0)| \le \frac{1}{k}|g'(0)|.$$

We will show that the estimation (4.1) is sharp. Let $f_u$ be as in (3.2). Then

$$f_w(\zeta w) = \zeta w - \frac{1}{k} g'(0)\zeta^{k+1} w + \cdots$$

by Lemma 3.1(ii). Therefore,

$$\Big|\frac{1}{(k+1)!} l_w(D^{k+1}f_w(0)(w^{k+1}))\Big| = \frac{1}{k}|g'(0)|.$$

This completes the proof.

Moreover, if $g$ is convex, then we obtain the following theorem.

**Theorem 4.2** *Let $g: U \to \mathbb{C}$ be a convex function which satisfies the conditions of Assumption* 2.1 *and $f \in S^*_{g,k+1}(B)$. Then*

$$\Big|\frac{1}{m!} l_w(D^m f(0)(w^m))\Big| \le \frac{1}{m-1}|g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w)$$

*for $m = k+1, \cdots, 2k$.*

**Proof** Let $h(z) = [Df(z)]^{-1} f(z)$ and

$$p(\zeta) = \begin{cases} \dfrac{1}{\zeta} l_w(h(\zeta w)), & \zeta \in U \setminus \{0\}, \\ 1, & \zeta = 0. \end{cases}$$

Since $g$ is convex and $p$ is subordinate to $g$, we obtain

$$\Big|\frac{p^{(m)}}{m!}(0)\Big| \le |g'(0)|, \quad m \ge 1 \tag{4.2}$$

by Rogosinski's Theorem (cf. [30]). Since

$$p(\zeta) = \sum_{m=1}^{\infty} l_w\Big(\frac{1}{m!} D^m h(0)(w^m)\Big)\zeta^{m-1},$$

we obtain

$$\Big| l_w\Big(\frac{1}{m!} D^m h(0)(w^m)\Big)\Big| \le |g'(0)| \tag{4.3}$$

by (4.2). Since $f(z) - z$ has a zero of order $k+1$ at $z = 0$,

$$\frac{1}{m!} D^m f(0) = \frac{-1}{m-1}\frac{1}{m!} D^m h(0), \quad m = k+1, \cdots, 2k \tag{4.4}$$

by [25, (2.3)]. From (4.3) and (4.4), we obtain

$$\Big|\frac{1}{m!} l_w(D^m f(0)(w^m))\Big| \le \frac{1}{m-1}|g'(0)|, \quad \|w\| = 1, \quad l_w \in T(w)$$

for $m = k+1, \cdots, 2k$. This completes the proof.

Now, we obtain the following corollaries from Theorem 4.2. For the norm of the $m$-th order Fréchet derivative of a mapping in $S^*_{g,k+1}(B)$, where $m = k+1, \cdots, 2k$, we have the following estimate by an argument similar to that in the proof of [12, Corollary 25] (cf. [8, Corollary 2.15]).

**Corollary 4.1** *Let $g : U \to \mathbb{C}$ be a convex function which satisfies the conditions of Assumption* 2.1 *and $f \in S^*_{g,k+1}(B)$. Then*

$$\left\| \frac{1}{m!} D^m f(0) \right\| \le c_m |g'(0)|$$

*for $m = k+1, \cdots, 2k$, where $c_m = \frac{m^{\frac{m}{m-1}}}{m-1}$.*

For the mappings in $S^*_{k+1}(B)$ (respectively $S^*_{p,k+1}(B)$, $SS^*_{\alpha,k+1}(B)$, $AS^*_{\alpha,k+1}(B)$, $PS^*_{\rho,k+1}(B)$), we obtain the following estimates (cf. [12, Corollaries 26 and 27], [13], [21] and [25, Theorems 4 and 5]).

**Corollary 4.2** *For $m = k+1, \cdots, 2k$, $\|w\| = 1$ and $l_w \in T(w)$, we have the following estimates:*

( i ) *If $f \in S^*_{k+1}(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \le \frac{2}{m-1}.$$

( ii ) *If $f \in S^*_{p,k+1}(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \le \frac{2(1-p)}{m-1}.$$

(iii) *If $f \in SS^*_{\alpha,k+1}(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \le \frac{2\alpha}{m-1}.$$

(iv) *If $f \in AS^*_{\alpha,k+1}(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \le \frac{2(1-\alpha)}{m-1}.$$

( v ) *If $f \in PS^*_{\rho,k+1}(B)$, then*

$$\left| \frac{1}{m!} l_w(D^m f(0)(w^m)) \right| \le \frac{16(1-\rho)}{\pi^2(m-1)}.$$

*These estimations are sharp for $m = k+1$.*

**Corollary 4.3** *Let $c_m$ be as in Corollary* 4.1*. For $m = k+1, \cdots, 2k$, we have the following estimates:*

( i ) *If $f \in S^*_{k+1}(B)$, then*

$$\left\| \frac{1}{m!} D^m f(0) \right\| \le 2c_m.$$

(ii) *If* $f \in S^*_{p,k+1}(B)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le 2(1-p)c_m.$$

(iii) *If* $f \in SS^*_{\alpha,k+1}(B)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le 2\alpha c_m.$$

(iv) *If* $f \in AS^*_{\alpha,k+1}(B)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le 2(1-\alpha)c_m.$$

(v) *If* $f \in PS^*_{\rho,k+1}(B)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le \frac{16(1-\rho)c_m}{\pi^2}.$$

Let $B$ be the unit polydisc $U^n$ in $\mathbb{C}^n$. Since

$$\begin{aligned}\Big\|\frac{1}{m!}D^m f(0)\Big\| &= \sup_{\|w\|=1}\Big\|\frac{1}{m!}D^m f(0)(w^m)\Big\| \\ &= \sup_{|w_1|=|w_2|=\cdots=|w_m|=1}\Big\|\frac{1}{m!}D^m f(0)(w^m)\Big\| \\ &= \sup_{|w_1|=|w_2|=\cdots=|w_m|=1}\max_{1\le i\le n}\Big|l_i\Big(\frac{1}{m!}D^m f(0)(w^m)\Big)\Big|,\end{aligned}$$

where $l_i \in T(w)$ with $l_i(y) = \frac{y_i}{w_i}$, we obtain the following corollaries (cf. [25, Theorems 1 and 2]).

**Corollary 4.4** *Let* $g : U \to \mathbb{C}$ *be a convex function which satisfies the conditions of Assumption* 2.1 *and* $f \in S^*_{g,k+1}(U^n)$. *Then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le \frac{1}{m-1}|g'(0)|, \quad \|w\| = 1$$

*for* $m = k+1, \cdots, 2k$. *This estimation is sharp for* $m = k+1$.

**Corollary 4.5** *For* $m = k+1, \cdots, 2k$, *we have the following estimates:*

(i) *If* $f \in S^*_{k+1}(U^n)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le \frac{2}{m-1}.$$

(ii) *If* $f \in S^*_{p,k+1}(U^n)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le \frac{2(1-p)}{m-1}.$$

(iii) *If* $f \in SS^*_{\alpha,k+1}(U^n)$, *then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\| \le \frac{2\alpha}{m-1}.$$

(iv) *If* $f \in AS^*_{\alpha,k+1}(U^n)$, *then*

$$\Big\| \frac{1}{m!} D^m f(0) \Big\| \le \frac{2(1-\alpha)}{m-1}.$$

( v ) *If* $f \in PS^*_{\rho,k+1}(U^n)$, *then*

$$\Big\| \frac{1}{m!} D^m f(0) \Big\| \le \frac{16(1-\rho)}{\pi^2(m-1)}.$$

*These estimations are sharp for* $m = k+1$.

Let $g$ be a function which satisfies the conditions of Assumption 2.1. Moreover, we assume that $\frac{1}{g}$ is convex. In this case, we will obtain the estimation of $\|\frac{1}{2}D^2 f(0)\|$ and $\|\frac{1}{3!}D^3 f(0)\|$ for $S^*_g(U^n)$ (cf. [1, Theorem 3.4], [7, Theorem 5.3.1], [13] and [32, Theorem 5]). First, we give a lemma.

**Lemma 4.1** *Let* $g$ *be a function which satisfies the conditions of Assumption* 2.1. *Moreover, we assume that* $\frac{1}{g}$ *is convex. Let* $f \in S^*_g(U^n)$, $h(z) = [Df(z)]^{-1} f(z)$ *and let the Taylor expansion of* $\frac{z_i}{h_i(z)}$ *at* $z = 0$ *be*

$$\frac{z_i}{h_i(z)} = 1 + \sum_{m=1}^{\infty} Q_i^{(m)}(z),$$

*where* $Q_i^{(m)}$ *is a homogeneous polynomial of degree* $m$ *in* $z$. *Then*

$$|Q_i^{(m)}(z)| \le |g'(0)| \|z\|^m \tag{4.5}$$

*holds for all* $m = 1, 2, 3, \cdots$ *and* $z \in U^n$.

**Proof** Let $z \in U^n \setminus \{0\}$ be fixed. We may assume that $|z_i| = \|z\|$ for all $i$ by the maximum principle. Let

$$\psi(\zeta) = \frac{\zeta \frac{z_i}{\|z\|}}{h_i(\zeta \frac{z}{\|z\|})}, \quad \zeta \in U.$$

Since

$$l_a(y) = \|a\| \frac{y_i}{a_i} \in T(a)$$

for all $a \in U^n \setminus \{0\}$ with $|a_i| = \|a\|$, we have

$$\frac{1}{\|a\|} l_a(h(a)) = \frac{h_i(a)}{a_i} \in g(U)$$

for all $a \in U^n \setminus \{0\}$ with $|a_i| = \|a\|$. This implies that $\psi(\zeta) \in \frac{1}{g}(U)$ for all $\zeta \in U$. Since $\psi(0) = \frac{1}{g}(0) = 1$, we have $\psi \prec \frac{1}{g}$. Since $\frac{1}{g}$ is convex and

$$\psi(\zeta) = 1 + \sum_{m=1}^{\infty} Q_i^{(m)}\Big(\frac{z}{\|z\|}\Big) \zeta^m,$$

we obtain

$$\Big| Q_i^{(m)}\Big(\frac{z}{\|z\|}\Big) \Big| \le |g'(0)|$$

by Rogosinski's theorem (cf. [30]). This implies the inequality (4.5). This completes the proof.

**Theorem 4.3** *Let $g$ be a function which satisfies the conditions of Assumption* 2.1. *Moreover, we assume that $\frac{1}{g}$ is convex. If $f \in S_g^*(U^n)$, then*

$$\left\|\frac{1}{2!}D^2 f(0)\right\| \leq |g'(0)|, \quad \left\|\frac{1}{3!}D^3 f(0)\right\| \leq \frac{|g'(0)|}{2}(1+|g'(0)|).$$

**Proof** Let

$$\varphi^{(2)}(z) = \frac{1}{2!}D^2 f(0)(z^2), \quad \varphi^{(3)}(z) = \frac{1}{3!}D^3 f(0)(z^3).$$

By [7, (5.3.8)],

$$\varphi^{(2)}(z) = (Q_1^{(1)}(z)z_1, \cdots, Q_n^{(1)}(z)z_n).$$

Then, by Lemma 4.1, we obtain

$$\|\varphi^{(2)}(z)\| \leq |g'(0)|\|z\|^2.$$

By the proof of [7, Theorem 5.3.1],

$$2\varphi^{(3)}(z) = \begin{pmatrix} Q_1^{(2)}(z)z_1 \\ \vdots \\ Q_n^{(2)}(z)z_n \end{pmatrix} + \begin{pmatrix} a_{11}z_1\varphi_1^{(2)}(z) + \cdots + a_{1n}z_1\varphi_n^{(2)}(z) \\ \vdots \\ a_{n1}z_n\varphi_1^{(2)}(z) + \cdots + a_{nn}z_n\varphi_n^{(2)}(z) \end{pmatrix},$$

where

$$Q_i^{(1)}(z) = a_{i1}z_1 + \cdots + a_{in}z_n.$$

By Lemma 4.1, we obtain

$$2\|\varphi^{(3)}(z)\| \leq |g'(0)|\|z\|^3 + |g'(0)|^2\|z\|^3.$$

This completes the proof.

**Corollary 4.6** ( i ) *If $f \in S^*(U^n)$, then*

$$\left\|\frac{1}{3!}D^3 f(0)\right\| \leq 3. \tag{4.6}$$

*This estimation is sharp.*

( ii ) *If $f \in S_p^*(U^n)$, then*

$$\left\|\frac{1}{3!}D^3 f(0)\right\| \leq (1-p)(3-2p). \tag{4.7}$$

*This estimation is sharp.*

(iii) *If $f \in SS_\alpha^*(U^n)$, then*

$$\left\|\frac{1}{3!}D^3 f(0)\right\| \leq \alpha(1+2\alpha).$$

(iv) *If $f \in AS_\alpha^*(U^n)$, then*

$$\left\|\frac{1}{3!}D^3 f(0)\right\| \leq (1-\alpha)(3-2\alpha).$$

( v ) *If $f \in PS_\rho^*(U^n)$, then*

$$\left\|\frac{1}{3!}D^3 f(0)\right\| \leq \frac{8(1-\rho)}{\pi^2}\Big(1+\frac{16(1-\rho)}{\pi^2}\Big).$$

**Proof** It suffices to show that the estimations (4.6) and (4.7) are sharp. We can verify that the mapping

$$f(z)=\Big(\frac{z_1}{(1-z_1)^{2(1-p)}},z_2,\cdots,z_n\Big),\quad z=(z_1,\cdots,z_n)\in U^n$$

attains the equalities in (4.6) and (4.7).

**Remark 4.1** The upper bounds in (iii), (iv) and (v) of Corollary 4.6 may not be sharp. The reason is that the sharp upper bounds for the third coefficients of functions in $SS^*_\alpha$, $AS^*_\alpha$ and $PS^*_\rho$ are less than the bounds in Corollary 4.6 (cf. [27, Theorem 3]).

For $S^*_{g,k+1}(U^n)$, we obtain the following estimation.

**Theorem 4.4** *Let $g$ be a function which satisfies the conditions of Assumption* 2.1. *Moreover, we assume that $\frac{1}{g}$ is convex. If $f\in S^*_{g,k+1}(U^n)$, then*

$$\Big\|\frac{1}{m!}D^m f(0)\Big\|\le\frac{1}{m-1}|g'(0)|$$

*for $m=k+1,\cdots,2k$. This estimation is sharp for $m=k+1$.*

**Proof** Let

$$\varphi^{(m)}(z)=\frac{1}{m!}D^m f(0)(z^m).$$

Since $f(z)-z$ has a zero of order $k+1$ at $z=0$, we have $\varphi^{(m)}=0$ for $m=2,\cdots,k$. Then we obtain

$$\begin{aligned}\sum_{m=k+1}^{\infty}(m-1)\varphi^{(m)}(z)=&\,[(I+J_{\varphi^{(k+1)}}(z)+\cdots)\\ &\times\begin{pmatrix}\sum\limits_{m=1}^{\infty}Q_1^{(m)}(z) & & 0\\ & \ddots & \\ 0 & & \sum\limits_{m=1}^{\infty}Q_n^{(m)}(z)\end{pmatrix}\\ &\times(I-J_{\varphi^{(k+1)}}(z)+\cdots)]\cdot(z+\varphi^{(k+1)}(z)+\cdots)\end{aligned}\tag{4.8}$$

from [7, p. 173]. Comparing $m$-th degree terms on both sides of (4.8), where $m=2,\cdots,k$, we obtain

$$Q_i^{(m)}(z)=0\quad\text{for } i=1,\cdots,n \text{ and } m=1,\cdots,k-1.$$

Therefore, comparing $m$-th degree terms on both sides of (4.8), where $m=k+1,\cdots,2k$, we obtain

$$(m-1)\varphi^{(m)}(z)=(Q_1^{(m-1)}(z)z_1,\cdots,Q_n^{(m-1)}(z)z_n)$$

for $m=k+1,\cdots,2k$. Then by Lemma 4.1, we obtain

$$\|\varphi^{(m)}(z)\|\le\frac{1}{m-1}|g'(0)|\|z\|^m$$

for $m=k+1,\cdots,2k$. This completes the proof.

## 5 Examples

Let $B^n(p)$ denote the unit ball in $\mathbb{C}^n$ with respect to a $p$-norm $\|\cdot\|$, $1 \le p \le \infty$, where

$$\|z\| = \begin{cases} \Big[\sum_{j=1}^{n} |z_j|^p\Big]^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{1\le j\le n} |z_j|, & p = \infty. \end{cases}$$

In Lemma 3.1, we give an example of a mapping for $S^*_{g,k+1}(B)$. In this section, we will give other examples in the case $B = B^n(p)$ and $g(U)$ is a starlike domain with respect to 1.

Let $\alpha \in [0,1]$. Hamada, Honda and Kohr [13] showed that if $f$ is a parabolic starlike mapping of order $\rho$ on $U$, then $\Psi_{n,\alpha}(f)$ is a parabolic starlike mapping of order $\rho$ on $B^n(p)$, where

$$\Psi_{n,\alpha}(f)(z) = \Big(f(z_1), \widetilde{z}\Big(\frac{f(z_1)}{z_1}\Big)^{\alpha}\Big)$$

for $z = (z_1, \widetilde{z}) \in B^n(p)$. The branch of the power function is chosen so that

$$\Big(\frac{f(z_1)}{z_1}\Big)^{\alpha}\Big|_{z_1=0} = 1.$$

We will generalize the above result to $f \in S^*_{g,k+1}$ in the case where $g(U)$ is a starlike domain with respect to 1. This result gives many examples of mappings in $S^*_{g,k+1}(B^n(p))$.

**Theorem 5.1** *Assume that $g$ satisfies Assumption* 2.1 *and $g(U)$ is a starlike domain with respect to* 1. *Let $\alpha \in [0,1]$. Let $\Psi_{n,\alpha}(f)$ be as above. If $f \in S^*_{g,k+1}$, then $F_\alpha = \Psi_{n,\alpha}(f) \in S^*_{g,k+1}(B^n(p))$, where $1 \le p \le \infty$.*

**Proof** When $1 < p < \infty$, $T(z)$ $(z \ne 0)$ consists of those functionals given by

$$l_z(w) = \sum_{j=1}^{n} \frac{1}{\|z\|^{p-1}} |z_j|^p \frac{w_j}{z_j}$$

(cf. [33]). Since $f \in S^*_{g,k+1}$, we obtain

$$h(z_1) = \frac{f(z_1)}{z_1 f'(z_1)} \in g(U). \tag{5.1}$$

By direct computations, we have

$$[DF_\alpha(z)]^{-1} F_\alpha(z) = (z_1 h(z_1), (1 - \alpha + \alpha h(z_1))\widetilde{z})$$

for $z = (z_1, \widetilde{z}) \in B^n(p)$. Then

$$\frac{1}{\|z\|} l_z([DF_\alpha(z)]^{-1} F_\alpha(z)) = \frac{|z_1|^p + \alpha\|\widetilde{z}\|^p}{\|z\|^p} h(z_1) + \frac{(1-\alpha)\|\widetilde{z}\|^p}{\|z\|^p} \in g(U)$$

by (5.1). Therefore, $F_\alpha \in S^*_{g,k+1}(B^n(p))$. The proof for the case $p = 1$ or $p = \infty$ is similar. This completes the proof.

**Theorem 5.2** *Assume that $g$ satisfies Assumption* 2.1 *and $g(U)$ is starlike with respect to* 1. *Then $f_1 \in S^*_{g,k+1}$ if and only if $f(z) = (f_1(z_1), z_2, \cdots, z_n) \in S^*_{g,k+1}(B^n(p))$, where $1 \le p \le \infty$.*

**Proof** When $p=\infty$, $T(z)$ $(z\neq 0)$ consists of those functionals $l_z$ given by

$$l_z(w)=\sum_{|z_k|=\|z\|} t_k\|z\|\frac{w_k}{z_k},\quad t_k\geq 0,\quad \sum_{|z_k|=\|z\|} t_k=1$$

(cf. [33]). Then

$$\frac{1}{\|z\|}l_z([Df(z)]^{-1}f(z))=\begin{cases}1, & |z_1|\neq\|z\|,\\ t_1\dfrac{f_1}{z_1f_1'}+(1-t_1)1, & |z_1|=\|z\|.\end{cases}$$

Since $g(U)$ is a starlike domain with respect to 1, $f_1\in S^*_{g,k+1}$ if and only if $f\in S^*_{g,k+1}(B^n(p))$. The proof in the case $p<\infty$ is similar.

Furthermore, if $g(U)$ is convex, then we obtain the following theorem.

**Theorem 5.3** *Assume that $g$ satisfies Assumption* 2.1 *and $g(U)$ is convex. Then $f_1,f_2,\cdots,f_n\in S^*_{g,k+1}$ if and only if $f(z)=(f_1(z_1),f_2(z_2),\cdots,f_n(z_n))\in S^*_{g,k+1}(B^n(p))$, where $1\leq p\leq\infty$.*

**Proof** When $p=1$, $T(z)$ $(z\neq 0)$ consists of those functionals given by

$$l_z(w)=\sum_{z_j\neq 0}\frac{|z_j|}{z_j}w_j+\sum_{z_j=0}\alpha_jw_j,\quad |\alpha_j|\leq 1.$$

(cf. [33]). Then

$$\frac{1}{\|z\|}l_z([Df(z)]^{-1}f(z))=\sum_{z_j\neq 0}\frac{|z_j|}{\|z\|}\frac{f_j(z_j)}{z_jf_j'(z_j)}.$$

Since $g(U)$ is convex, $f_1,f_2\cdots,f_n\in S^*_{g,k+1}$ if and only if $f\in S^*_{g,k+1}(B^n(p))$. The proof in the case $p>1$ is similar.

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