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The Semiclassical Limit in the Quantum Drift-Diffusion Equations with Isentropic Pressure***

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Abstract The semiclassical limit in the transient quantum drift-diffusion equations with isentropic pressure in one space dimension is rigorously proved. The equations are supplemented with homogeneous Neumann boundary conditions. It is shown that the semiclassical limit of this solution solves the classical drift-diffusion model. In the meanwhile, the global existence of weak solutions is proved.

Keywords Quantum drift-diffusion, Weak solution, Semiclassical limit, Isentropic
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1 Introduction

This is one of the series on the mathematical analysis of the quantum drift-diffusion (QDD) model which is one kind of the quantum macroscopic models for miniaturized semiconductor devices. For semiconductor physics and modelling, we refer to the references [3, 8, 12, 14]. One could find the full picture on quantum models in the review papers [7, 13]. The quantum macroscopic models could be derived from the mixed state Schrödinger system, or equivalently the Wigner equation by Wigner transformation. Compared with those microscopic models like Schrödinger-Poisson or Wigner-Poisson system, these macroscopic quantum models can save more time and energy in numerical simulations.

The scaled QDD model takes the form

$$\begin{aligned} n_t &= \operatorname{div}\Big[-\varepsilon^2 n\nabla\Big(\frac{\triangle\sqrt{n}}{\sqrt{n}}\Big)+\nabla(P_n(n))-n\nabla V\Big],\\ p_t &= \operatorname{div}\Big[-\xi\varepsilon^2 p\nabla\Big(\frac{\triangle\sqrt{p}}{\sqrt{p}}\Big)+\nabla(P_p(p))+p\nabla V\Big],\\ \lambda^2\triangle V &= n-p-C(x), \end{aligned} \tag{1.1}$$

where $n=n(x,t)$ is the electron density, $p=p(x,t)$ is the hole density and $V=V(x,t)$ is the electrostatic potential. The pressure functions P_n and P_p are usually of the forms $P_n(n)=\theta_n n^\alpha$ and $P_p(p)=\theta_p p^\beta$ with $\theta_n>0,\ \theta_p>0,\ \alpha\geq 1$ and $\beta\geq 1$. $\varepsilon>0$ is the scaled Plank constant,

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$\lambda > 0$ is the ratio of the Debye length to the characteristic length (e.g. the device diameter), and $\xi > 0$ is the ratio of the effective masses of electrons and holes. The doping profile $C(x)$ represents the distribution of charged background ions. The system will be considered in a bounded domain $\Omega \in \mathbb{R}^N$ and fixed time interval $(0, T]$ with reasonable boundary and initial conditions. Without loss of generality, θ_n and θ_p are normalized to be 1 throughout this paper.

The QDD equations equal up to a quantum correction of classical drift-diffusion (DD) model. The quantum correction terms are $-\varepsilon^2 n\nabla(\frac{\triangle\sqrt{n}}{\sqrt{n}})$ and $-\xi\varepsilon^2 p\nabla(\frac{\triangle\sqrt{p}}{\sqrt{p}})$, respectively, in $(1.1)_1$ and $(1.1)_2$. The semiclassical limit $\varepsilon \to 0$ describes the relation between QDD model and DD model. Formally, letting $\varepsilon \to 0$, one has the classical DD model

$$\begin{aligned} n_t &= \operatorname{div}(\nabla n^\alpha - n\nabla V), \\ p_t &= \operatorname{div}(\nabla p^\beta + p\nabla V), \\ \lambda^2 \triangle V &= n - p - C(x). \end{aligned}$$

The main purpose of the present paper is to analyze rigorously the semiclassical limit.

The previous mathematical analysis on (1.1) is mainly about isothermal model (i.e. $\alpha = \beta = 1$). For the stationary equations, Abdallah and Unterreiter [2] proved the existence of solutions and studied the semiclassical limit. For the transient unipolar equation (i.e. only one carrier), Jüngel and Pinnau [9, 10] established a positivity preserving scheme and got the existence of the solution in a weak sense for fixed ε and λ in one dimension. Recently, Chen and Ju [4, 6] got the first semiclassical limit for transient unipolar and bipolar equations in one dimension with homogeneous Neumann boundary conditions. Indeed, the treatment of transient QDD in multi-dimension is currently not known.

As far as the isentropic model (i.e. $\alpha > 1$ and $\beta > 1$) is concerned, X. Q. Chen [5] studied the unipolar model with homogeneous Neumann boundary conditions in 1-dimension, and obtained the semiclassical limit under the assumption that the exponent of pressure is less than or equal to $\frac{3}{2}$. Clearly, this is not satisfactory from physical consideration. It also seems impossible to get rid of (or weaken) such assumption by the approaches used in [5]. In this paper, we would employ different ideas to prove the semiclassical limit to the solution of (1.1) with $\alpha, \beta \in (1, 3]$.

We will focus on 1-dimensional case with $\Omega = (0, 1)$. Since all the results discussed here are obtained for fixed $\lambda, \xi > 0$, for convenience, we let $\lambda = \xi = 1$ through the paper. To search for physically reasonable solutions, namely, nonnegative solutions for densities n and p, with the help of the so-called quantum quasi-Fermi levels F and G, it is equivalent to consider the system, with $n = \rho^2$ and $p = \eta^2$, namely

$$\begin{aligned} &(\rho^2)_t = (\rho^2 F_x)_x, \\ &(\eta^2)_t = (\eta^2 G_x)_x, \\ &-\varepsilon^2 \frac{\rho_{xx}}{\rho} + \frac{\alpha}{\alpha - 1}\rho^{2(\alpha-1)} - V = F, \\ &-\varepsilon^2 \frac{\eta_{xx}}{\eta} + \frac{\beta}{\beta - 1}\eta^{2(\beta-1)} + V = G, \\ &V_{xx} = \rho^2 - \eta^2 - C(x). \end{aligned} \tag{1.2}$$

We will consider an insulated model, i.e. a model with the following homogenous Neumann boundary condition,

$$\rho_x = \eta_x = F_x = G_x = V_x = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

and the initial condition

$$\rho|_{t=0} = \rho_0(x) \geq 0, \quad \eta|_{t=0} = \eta_0(x) \geq 0. \tag{1.4}$$

Also, we assume the compatible condition

$$\int_\Omega (\rho_0^2 - \eta_0^2 - C(x))\mathrm{d}x = 0. \tag{1.5}$$

We will use the following notations in this paper.

(1) The Sobolev spaces, $W^{m,p}$ ($W^{m,2} = H^m$).

(2) The Orlicz $L_\Psi(\Omega)$ with the Young function $\Psi(s) = s(\ln s - 1) + 1$, here the definition and basic properties of Orlicz space can be found in [1].

(3) For any Banach space B,

$$L^p(0,T;B) = \{f : \| \|f\|_B\|_{L^p(0,T)} \leq C\}.$$

Let $E = V_x$, $0 < T < \infty$ be given and $Q_T = \Omega \times (0,T]$.

The main results in this paper are the following two theorems.

Theorem 1.1 (Existence of Solutions) *Let $\alpha, \beta > 1$. Assume $C(x) \in L^\infty(\Omega)$ and*

$$\rho_0^2 \in L_\Psi(\Omega), \quad \eta_0^2 \in L_\Psi(\Omega), \tag{1.6}$$

$$\rho_0^2 - \ln\rho_0^2 \in L^1(\Omega), \quad \eta_0^2 - \ln\eta_0^2 \in L^1(\Omega), \tag{1.7}$$

$$(\rho_0)_x \in L^2(\Omega), \quad (\eta_0)_x \in L^2(\Omega). \tag{1.8}$$

Then, there exist functions $(\rho_\varepsilon, \eta_\varepsilon, J_\varepsilon, K_\epsilon, E_\varepsilon)$ satisfying

$$\rho_\varepsilon,\ \eta_\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{1.9}$$

$$(\rho_\varepsilon^2)_t,\ (\eta_\varepsilon^2)_t \in L^2(0,T;(H^1(\Omega))'), \tag{1.10}$$

$$J_\varepsilon,\ K_\varepsilon \in L^2(Q_T), \quad E_\varepsilon \in L^\infty(Q_T), \tag{1.11}$$

such that for all $\phi \in C_0^\infty(Q_T)$, it holds that

$$\int_0^T \langle(\rho_\varepsilon^2)_t, \phi\rangle_{\langle(H^1)',H^1\rangle}\mathrm{d}t + \int_{Q_T} J_\varepsilon\phi_x\mathrm{d}x\mathrm{d}t = 0, \tag{1.12}$$

$$\int_0^T \langle(\eta_\varepsilon^2)_t, \phi\rangle_{\langle(H^1)',H^1\rangle}\mathrm{d}t + \int_{Q_T} K_\varepsilon\phi_x\mathrm{d}x\mathrm{d}t = 0, \tag{1.13}$$

$$\begin{aligned}\int_{Q_T} J_\varepsilon\phi\mathrm{d}x\mathrm{d}t = &\int_{Q_T} \varepsilon^2(\rho_\varepsilon)_{xx}(2(\rho_\varepsilon)_x\phi + \rho_\varepsilon\phi_x)\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} [\alpha(\rho_\varepsilon)^{2\alpha-2}(\rho_\varepsilon^2)_x\phi - E_\varepsilon\rho_\varepsilon^2\phi]\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{1.14}$$

$$\begin{aligned}\int_{Q_T} K_\varepsilon\phi\mathrm{d}x\mathrm{d}t = &\int_{Q_T} \varepsilon^2(\eta_\varepsilon)_{xx}(2(\eta_\varepsilon)_x\phi + \eta_\varepsilon\phi_x)\mathrm{d}x\mathrm{d}t \\ &+ \int_{Q_T} [\beta(\eta_\varepsilon)^{2\beta-2}(\eta_\varepsilon^2)_x\phi + E_\varepsilon\eta_\varepsilon^2\phi]\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{1.15}$$

$$-\int_{Q_T} E_\varepsilon\phi_x\mathrm{d}x\mathrm{d}t = \int_{Q_T} ((\rho_\varepsilon)^2 - (\eta_\varepsilon)^2 - C(x))\phi\mathrm{d}x\mathrm{d}t. \tag{1.16}$$

Remark 1.1 Theorem 1.1 shows that for fixed ε, $(\rho_\varepsilon, \eta_\varepsilon, J_\varepsilon, K_\epsilon, E_\varepsilon)$ is a weak solution to the problem (1.2)–(1.5) in the sense of (1.12)–(1.16).

Remark 1.2 One could show that the solutions ρ_ε and η_ε have the regularity up to the third order by the same method as in [4, 5].

Theorem 1.2 (Semiclassical Limit) *Let $\alpha, \beta \in (1, 3]$. Assume that $(\rho_\varepsilon, \eta_\varepsilon, J_\varepsilon, K_\epsilon, E_\varepsilon)$ is the solution obtained in Theorem 1.1. Then as $\varepsilon \to 0^+$, for a.e. $t \in (0, T)$, it holds that*

$$\rho_\varepsilon^2 \to n \quad \text{strongly in } L^2(0,T; C^{0,\gamma_1}(\Omega)) \text{ for some } 0 < \gamma_1 < 1, \tag{1.17}$$

$$\eta_\varepsilon^2 \to p \quad \text{strongly in } L^2(0,T; C^{0,\gamma_2}(\Omega)) \text{ for some } 0 < \gamma_2 < 1, \tag{1.18}$$

and

$$J_\varepsilon \rightharpoonup J \qquad \text{weakly in } L^2(0,T; L^{\frac{2\alpha}{1+\alpha}}(\Omega)), \tag{1.19}$$

$$K_\varepsilon \rightharpoonup K \qquad \text{weakly in } L^2(0,T; L^{\frac{2\beta}{1+\beta}}(\Omega)), \tag{1.20}$$

$$(\rho_\varepsilon^2)_t \rightharpoonup n_t \qquad \text{weakly in } L^2(0,T; (W^{1,\frac{2\alpha}{1+\alpha}}(\Omega))'), \tag{1.21}$$

$$(\eta_\varepsilon^2)_t \rightharpoonup p_t \qquad \text{weakly in } L^2(0,T; (W^{1,\frac{2\beta}{1+\beta}}(\Omega))'), \tag{1.22}$$

$$E_\varepsilon \rightharpoonup E \qquad \text{weakly-}* \text{ in } L^\infty(Q_T), \tag{1.23}$$

where (n, p, J, K, E) satisfies

$$\int_0^T \langle (n)_t, \phi \rangle_{\langle (W^{1,\frac{2\alpha}{1+\alpha}})', W^{1,\frac{2\alpha}{1+\alpha}} \rangle} \mathrm{d}t + \int_{Q_T} J\phi_x \mathrm{d}x\mathrm{d}t = 0, \tag{1.24}$$

$$\int_0^T \langle (p)_t, \phi \rangle_{\langle (W^{1,\frac{2\beta}{1+\beta}})', W^{1,\frac{2\beta}{1+\beta}} \rangle} \mathrm{d}t + \int_{Q_T} K\phi_x \mathrm{d}x\mathrm{d}t = 0, \tag{1.25}$$

$$J = \alpha n^{\alpha-1} n_x - En, \ \text{a.e.}, \tag{1.26}$$

$$K = \beta p^{\beta-1} p_x + Ep, \ \text{a.e.}, \tag{1.27}$$

$$-\int_{Q_T} E\phi_x \mathrm{d}x\mathrm{d}t = \int_{Q_T} (n - p - C(x))\phi \mathrm{d}x\mathrm{d}t \tag{1.28}$$

for all $\phi \in C_0^\infty(Q_T)$.

Remark 1.3 Theorem 1.2 shows that (n, p, E) is a weak solution to the classical drift-diffusion model

$$n_t = \mathrm{div}(\nabla n^\alpha - nE),$$
$$p_t = \mathrm{div}(\nabla p^\beta + pE),$$
$$\mathrm{div} E = n - p - C(x)$$

with homogeneous Neumann boundary condition in the sense of (1.24)–(1.28).

Remark 1.4 The above results are also true for periodic boundary conditions.

Let us explain the basic ideas involved in this paper briefly. The proofs of Theorems 1.1 and 1.2 depend on the entropy estimates and compactness arguments. The approximation solutions are constructed by semi-discretization in time. Thus the key points are to get uniform estimates on approximation solutions with respect to time step for the existence of weak solutions and to ε for the semiclassical limit, respectively. The difficulties come from the nonlinearity, singularity and the higher order derivatives. In fact, it is much harder to get the uniform estimates on solution with respect to ε. When $1 \le \alpha \le \frac{3}{2}$, [4, 5] get the key estimates on $\|\rho_x\|_{L^\infty(0,T;L^2(\Omega))}$ and quantum terms by using $-\frac{\rho_{xx}}{\rho}$ as test function for the unipolar case. But for $\alpha > \frac{3}{2}$, such arguments fail to work. Roughly speaking, the most difficult point is to deal with $\varepsilon^2\rho_{xx}\rho_x$ from the quantum terms as $\varepsilon \to 0^+$. When $1 < \alpha \le 2$, our idea is to combine the entropy estimates from $1 - \frac{1}{\rho^2}$ and $\ln \rho^2$ in an efficient way to get the uniform estimates on $\|\rho_x\|_{L^2(Q_T)}$ and $\varepsilon\|\rho_{xx}\|_{L^2(Q_T)}$. Then, with the help of the entropy estimates from the quantum-Fermi level F, the semiclassical limit could be obtained. However, when $2 < \alpha \le 3$, we are not able to obtain the uniform estimates on $\|\rho_x\|_{L^2(Q_T)}$. Actually, we find that the estimate on $\|\rho_x\|_{L^2(Q_T)}$ is not necessary in the discussion of semiclassical limit. To get the strong convergence, we use the estimate on $\|(\rho^2)_x\|_{L^2(Q_T)}$ instead. With the help of $\|(\sqrt{\rho_\tau})_x\|_{L^4(Q_T)} \le C\varepsilon^{-\frac{1}{2}}$, which comes from the entropy estimates, we can control $\varepsilon^2\rho_{xx}\rho_x$ by $O(\varepsilon^{\frac{1}{2}})$ through the expression $2\varepsilon^2\rho_{xx}(\sqrt{\rho})_x\sqrt{\rho}$. In the above discussion, we also overcome the difficulties coming from the bipolar coupled terms.

Another interesting limit in (1.1) is the quasineutral limit $\lambda \to 0$. This limit was studied first in [15] for the corresponding thermal equilibrium problem in multi-dimension. In [11], the limit has been shown rigorously for the transient equations in one dimension with Dirichlet-Neumann boundary conditions.

The paper is organized as follows. In Section 2, we will construct the approximation problem, which is a series of elliptic problems. In Section 3, we generate the a priori estimates which are used not only in the proof for existence but also in guaranteeing the semiclassical limit. Theorems 1.1 and 1.2 will be proved in Sections 4 and 5, respectively.

2 Approximation

In this section, we will describe the approximation to (1.2)–(1.5).

We divide the time interval $(0, T]$ into several subintervals $(0, T] = \bigcup_{k=1}^{K} (t_{k-1}, t_k]$ such that $t_k - t_{k-1} = \tau$, $k = 1, \cdots, K$. Given ρ_{k-1} and η_{k-1}, we will solve the following problem

$$
\begin{aligned}
&\frac{\rho_k^2 - \rho_{k-1}^2}{\tau} = (\rho_k^2 (F_k)_x)_x, \\
&\frac{\eta_k^2 - \eta_{k-1}^2}{\tau} = (\eta_k^2 (G_k)_x)_x, \\
&-\varepsilon^2 \frac{(\rho_k)_{xx}}{\rho_k} + \frac{\alpha}{\alpha - 1}\rho_k^{2(\alpha-1)} - V_k = F_k, \\
&-\varepsilon^2 \frac{(\eta_k)_{xx}}{\eta_k} + \frac{\beta}{\beta - 1}\eta_k^{2(\beta-1)} + V_k = G_k, \\
&(V_k)_{xx} = \rho_k^2 - \eta_k^2 - C(x), \\
&(\rho_k)_x = (F_k)_x = 0, \quad \text{on } \partial\Omega, \\
&(\eta_k)_x = (G_k)_x = (V_k)_x = 0, \quad \text{on } \partial\Omega.
\end{aligned}
\tag{2.1}
$$

Also, (2.1) is supplemented with the following compatible condition

$$\int_\Omega (\rho_k^2 - \eta_k^2 - C(x))\mathrm{d}x = 0. \tag{2.2}$$

For the above problem, we have the following existence results.

Theorem 2.1 *Let $\alpha > 1$, $\beta > 1$. Assume $\rho_{k-1}, \eta_{k-1} \in C^{0,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$, $\min\limits_\Omega \rho_{k-1} > 0$ and $\min\limits_\Omega \eta_{k-1} > 0$, $C(x) \in L^\infty(\Omega)$. Then problem (2.1)–(2.2) has a solution $(\rho_k, \eta_k, G_k, F_k, V_k)$ such that $\rho_k, \eta_k \in W^{4,p}(\Omega)$, $F_k, G_k \in W^{2,p}(\Omega)$, $V_k \in W^{2,p}(\Omega)$ $(\forall\, p > 1)$ with $\rho_k \geq c_k > 0$ and $\eta_k \geq c_k > 0$ for some positive constant c_k.*

One can modify slightly the proof in [5] and [4] to get Theorem 2.1, where the key idea is to use exponential transformation. The details are omitted here.

Remark 2.1 It is worth noting that we cannot get the uniform positive lower bound on the approximation solutions ρ_k and η_k, and therefore the limit weakens to be nonnegative.

3 Uniform Estimates — Entropy Inequalities

The approximate solutions for our problem are constructed in the following way. Introduce the functions $\rho_\tau(x,t) = \rho_k(x)$, $\eta_\tau(x,t) = \eta_k(x)$, $V_\tau(x,t) = V_k(x)$ if $x \in \Omega$ and $t \in ((k-1)\tau, k\tau]$. Then $F_\tau(x,t) = F_k(x)$ and $G_\tau(x,t) = G_k(x)$ for $x \in \Omega$ and $t \in ((k-1)\tau, k\tau]$. Let $Q_t = \Omega \times (0,t]$ for $t \in (0,T]$. Next, we will focus on the uniform estimates for the approximate solutions.

Lemma 3.1 *Assume $\alpha > 1$, $\beta > 1$. Let ρ_k, $\eta_k \in W^{4,p}(\Omega)$, F_k, $G_k \in W^{2,p}(\Omega)$ and $V_k \in W^{2,p}(\Omega)$ be the solutions obtained in Theorem 2.1. Then the following inequality holds:*

$$\begin{aligned}
&\epsilon^2 \int_\Omega (|(\rho_k)_x|^2 + |(\eta_k)_x|^2)\mathrm{d}x + \int_\Omega \Big(\frac{1}{\alpha-1}\rho_k^{2\alpha} + \frac{1}{\beta-1}\eta_k^{2\beta}\Big)\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega |(V_k)_x|^2 \mathrm{d}x + \tau \int_\Omega ((\rho_k(F_k)_x)^2 + (\eta_k(G_k)_x)^2)\mathrm{d}x \\
&\leq \epsilon^2 \int_\Omega (|(\rho_{k-1})_x|^2 + |(\eta_{k-1})_x|^2)\mathrm{d}x + \int_\Omega \Big(\frac{1}{\alpha-1}\rho_{k-1}^{2\alpha} + \frac{1}{\beta-1}\eta_{k-1}^{2\beta}\Big)\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega |(V_{k-1})_x|^2 \mathrm{d}x.
\end{aligned} \tag{3.1}$$

Proof Multiplying $(2.1)_1$ and $(2.1)_2$ by F_k and G_k, respectively, we obtain

$$\begin{aligned}
\int_\Omega (\rho_k^2 - \rho_{k-1}^2) F_k \mathrm{d}x &= \tau \int_\Omega (\rho_k^2 (F_k)_x)_x F_k \mathrm{d}x, \\
\int_\Omega (\eta_k^2 - \eta_{k-1}^2) G_k \mathrm{d}x &= \tau \int_\Omega (\eta_k^2 (G_k)_x)_x G_k \mathrm{d}x.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\int_\Omega (\rho_k^2 (F_k)_x)_x F_k \mathrm{d}x &= -\tau \int_\Omega (\rho_k (F_k)_x)^2 \mathrm{d}x, \\
\int_\Omega (\eta_k^2 (G_k)_x)_x G_k \mathrm{d}x &= -\tau \int_\Omega (\eta_k (G_k)_x)^2 \mathrm{d}x.
\end{aligned}$$

It is easy to find that

$$\begin{aligned}
&\int_\Omega(\rho_k^2-\rho_{k-1}^2)F_k\mathrm{d}x\\
=&-\varepsilon^2\int_\Omega(\rho_k^2-\rho_{k-1}^2)\frac{(\rho_k)_{xx}}{\rho_k}\mathrm{d}x+\frac{\alpha}{\alpha-1}\int_\Omega(\rho_k^2-\rho_{k-1}^2)\rho_k^{2(\alpha-1)}\mathrm{d}x-\int_\Omega(\rho_k^2-\rho_{k-1}^2)V_k\mathrm{d}x, \quad (3.2)\\
&\int_\Omega(\eta_k^2-\eta_{k-1}^2)G_k\mathrm{d}x\\
=&-\varepsilon^2\int_\Omega(\eta_k^2-\eta_{k-1}^2)\frac{(\eta_k)_{xx}}{\eta_k}\mathrm{d}x+\frac{\beta}{\beta-1}\int_\Omega(\eta_k^2-\eta_{k-1}^2)\eta_k^{2(\beta-1)}\mathrm{d}x+\int_\Omega(\rho_k^2-\rho_{k-1}^2)V_k\mathrm{d}x. \quad (3.3)
\end{aligned}$$

Next, we will treat the right sides of (3.2) and (3.3) term by term. Firstly, using integration by parts, we have

$$\begin{aligned}
&-\varepsilon^2\int_\Omega(\rho_k^2-\rho_{k-1}^2)\frac{(\rho_k)_{xx}}{\rho_k}\mathrm{d}x\\
=&\,\varepsilon^2\int_\Omega|(\rho_k)_x|^2\mathrm{d}x-\varepsilon^2\int_\Omega\Big(\frac{\rho_{k-1}^2}{\rho_k}\Big)_x(\rho_k)_x\mathrm{d}x\\
=&\,\varepsilon^2\int_\Omega|(\rho_k)_x|^2\mathrm{d}x-\varepsilon^2\int_\Omega|(\rho_{k-1})_x|^2\mathrm{d}x+\varepsilon^2\int_\Omega\Big|(\rho_{k-1})_x-\frac{\rho_{k-1}}{\rho_k}(\rho_k)_x\Big|^2\mathrm{d}x\\
\ge&\,\varepsilon^2\int_\Omega|(\rho_k)_x|^2\mathrm{d}x-\varepsilon^2\int_\Omega|(\rho_{k-1})_x|^2\mathrm{d}x.
\end{aligned}$$

Similarly, we have

$$-\varepsilon^2\int_\Omega(\eta_k^2-\eta_{k-1}^2)\frac{(\eta_k)_{xx}}{\eta_k}\mathrm{d}x\ge\varepsilon^2\int_\Omega|(\eta_k)_x|^2\mathrm{d}x-\varepsilon^2\int_\Omega|(\eta_{k-1})_x|^2\mathrm{d}x.$$

In view of the convexity of s^γ $(\gamma>1)$, we have

$$\begin{aligned}
\frac{\alpha}{\alpha-1}\int_\Omega(\rho_k^2-\rho_{k-1}^2)\rho_k^{2(\alpha-1)}\mathrm{d}x&\ge\frac{1}{\alpha-1}\int_\Omega\rho_k^{2\alpha}\mathrm{d}x-\frac{1}{\alpha-1}\int_\Omega\rho_{k-1}^{2\alpha}\mathrm{d}x,\\
\frac{\beta}{\beta-1}\int_\Omega(\eta_k^2-\eta_{k-1}^2)\eta_k^{2(\beta-1)}\mathrm{d}x&\ge\frac{1}{\beta-1}\int_\Omega\eta_k^{2\beta}\mathrm{d}x-\frac{1}{\beta-1}\int_\Omega\eta_{k-1}^{2\beta}\mathrm{d}x.
\end{aligned}$$

At last, by the Poisson equation, we have

$$\begin{aligned}
&-\int_\Omega(\rho_k^2-\rho_{k-1}^2)V_k\mathrm{d}x+\int_\Omega(\eta_k^2-\eta_{k-1}^2)V_k\mathrm{d}x\\
=&-\int_\Omega((V_k)_{xx}-(V_{k-1})_{xx})V_k\mathrm{d}x\\
=&\int_\Omega((V_k)_x-(V_{k-1})_x)(V_k)_x\mathrm{d}x\\
=&\,\frac12\int_\Omega|(V_k)_x|^2\mathrm{d}x-\frac12\int_\Omega|(V_{k-1})_x|^2\mathrm{d}x+\frac12\int_\Omega|(V_k)_x-(V_{k-1})_x|^2\mathrm{d}x\\
\ge&\,\frac12\int_\Omega|(V_k)_x|^2\mathrm{d}x-\frac12\int_\Omega|(V_{k-1})_x|^2\mathrm{d}x.
\end{aligned}$$

Thus, the above treatments yield (3.1).

It is easy to derive the following estimates from Lemma 3.1.

Corollary 3.1 *There exists a positive constant independent of τ and ε such that*

$$\|\rho_\tau\|_{L^\infty(0,T;L^{2\alpha}(\Omega))} + \|\eta_\tau\|_{L^\infty(0,T;L^{2\beta}(\Omega))} \le C, \tag{3.4}$$

$$\|\varepsilon(\rho_\tau)_x\,;\,\varepsilon(\eta_\tau)_x\|_{L^\infty(0,T;L^2(\Omega))} \le C, \tag{3.5}$$

$$\|\rho_\tau(F_\tau)_x\,;\,\eta_\tau(G_\tau)_x\|_{L^2(Q_T)} \le C. \tag{3.6}$$

Furthermore, we have

$$\|\rho_\tau^2(F_\tau)_x\|_{L^2(0,T;L^{\frac{2\alpha}{1+\alpha}}(\Omega))} + \|\eta_\tau^2(G_\tau)_x\|_{L^2(0,T;L^{\frac{2\beta}{1+\beta}}(\Omega))} \le C.$$

It is convenient to rewrite the approximate system (2.1) in the following form:

$$\begin{aligned}
\frac{\rho_k^2-\rho_{k-1}^2}{\tau} &= -\frac{\varepsilon^2}{2}(\rho_k^2(\ln\rho_k^2)_{xx})_{xx} + ((\rho_k^{2\alpha})_x - \rho_k^2(V_k)_x)_x,\\
\frac{\eta_k^2-\eta_{k-1}^2}{\tau} &= -\frac{\varepsilon^2}{2}(\eta_k^2(\ln\eta_k^2)_{xx})_{xx} + ((\eta_k^{2\beta})_x + \eta_k^2(V_k)_x)_x,\\
(V_k)_{xx} &= \rho_k^2 - \eta_k^2 - C(x).
\end{aligned} \tag{3.7}$$

Since both ρ_k and η_k are strictly positive, the boundary conditions are equivalent to

$$\begin{aligned}
&(\rho_k)_x = (\rho_k^2(\ln\rho_k^2)_{xx})_x = (V_k)_x = 0, \quad \text{on } \partial\Omega,\\
&(\eta_k)_x = (\eta_k^2(\ln\eta_k^2)_{xx})_x = 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{3.8}$$

Lemma 3.2 *Under the same assumptions as in Lemma* 3.1, *it holds that*

$$\begin{aligned}
&\int_\Omega((\rho_k^2-\ln\rho_k^2)+(\eta_k^2-\ln\eta_k^2))\mathrm{d}x + 2\varepsilon^2\tau\int_\Omega(|(\ln\rho_k)_{xx}|^2+|(\ln\eta_k)_{xx}|^2)\mathrm{d}x\\
&+\tau4\alpha\int_\Omega|\rho_k^{\alpha-2}(\rho_k)_x|^2\mathrm{d}x + \tau4\beta\int_\Omega|\eta_k^{\beta-2}|(\eta_k)_x|^2\mathrm{d}x + \tau\int_\Omega(\rho_k^2-\eta_k^2)(\ln\rho_k^2-\ln\eta_k^2)\mathrm{d}x\\
\le&\int_\Omega((\rho_{k-1}^2-\ln\rho_{k-1}^2)+(\eta_{k-1}^2-\ln\eta_{k-1}^2))\mathrm{d}x + 2\tau\|C(x)\|_{L^\infty}\int_\Omega(|\ln\rho_k|+|\ln\eta_k|)\mathrm{d}x.
\end{aligned} \tag{3.9}$$

Proof We multiply $(3.7)_1$ and $(3.7)_2$ by $(1-\frac{1}{\rho_k^2})$ and $(1-\frac{1}{\eta_k^2})$, respectively, and integrate them over Ω. By the inequality $x-1\ge\ln x$ ($\forall\, x>0$), we get

$$\begin{aligned}
&\int_\Omega\frac{\rho_k^2-\rho_{k-1}^2}{\tau}\Big(1-\frac{1}{\rho_k^2}\Big)\mathrm{d}x + \int_\Omega\frac{\eta_k^2-\eta_{k-1}^2}{\tau}\Big(1-\frac{1}{\eta_k^2}\Big)\mathrm{d}x\\
=&\frac{1}{\tau}\int_\Omega\Big(\rho_k^2-\rho_{k-1}^2+\frac{\rho_{k-1}^2}{\rho_k^2}-1\Big)\mathrm{d}x + \frac{1}{\tau}\int_\Omega\Big(\eta_k^2-\eta_{k-1}^2+\frac{\eta_{k-1}^2}{\eta_k^2}-1\Big)\mathrm{d}x\\
\ge&\frac{1}{\tau}\int_\Omega((\rho_k^2-\rho_{k-1}^2)-(\ln\rho_k^2-\ln\rho_{k-1}^2))\mathrm{d}x + \frac{1}{\tau}\int_\Omega((\eta_k^2-\eta_{k-1}^2)-(\ln\eta_k^2-\ln\eta_{k-1}^2))\mathrm{d}x.
\end{aligned}$$

Using integration by parts and the boundary condition (3.8), we have

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\int_\Omega(\rho_k^2(\ln\rho_k^2)_{xx})_{xx}\Big(1-\frac{1}{\rho_k^2}\Big)\mathrm{d}x + \frac{\varepsilon^2}{2}\int_\Omega(\eta_k^2(\ln\eta_k^2)_{xx})_{xx}\Big(1-\frac{1}{\eta_k^2}\Big)\mathrm{d}x\\
=&2\varepsilon^2\int_\Omega(|(\ln\rho_k)_{xx}|^2+|(\ln\eta_k)_{xx}|^2)\mathrm{d}x - 4\varepsilon^2\int_\Omega((\ln\rho_k)_{xx}|(\ln\rho_k)_x|^2+(\ln\eta_k)_{xx}|(\ln\eta_k)_x|^2)\mathrm{d}x\\
=&2\varepsilon^2\int_\Omega(|(\ln\rho_k)_{xx}|^2+|(\ln\eta_k)_{xx}|^2)\mathrm{d}x - \frac{4}{3}\varepsilon^2\int_\Omega[(((\ln\rho_k)_x)^3)_x+(((\ln\eta_k)_x)^3)_x]\mathrm{d}x\\
=&2\varepsilon^2\int_\Omega(|(\ln\rho_k)_{xx}|^2+|(\ln\eta_k)_{xx}|^2)\mathrm{d}x,
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega(\rho_k^{2\alpha})_{xx}\Big(1-\frac{1}{\rho_k^2}\Big)\mathrm{d}x-\int_\Omega(\eta_k^{2\beta})_{xx}\Big(1-\frac{1}{\eta_k^2}\Big)\mathrm{d}x\\
&=4\alpha\int_\Omega\rho_k^{2\alpha-4}|(\rho_k)_x|^2\mathrm{d}x+4\beta\int_\Omega\eta_k^{2\beta-4}|(\eta_k)_x|^2\mathrm{d}x\\
&=4\alpha\int_\Omega|\rho_k^{\alpha-2}(\rho_k)_x|^2\mathrm{d}x+4\beta\int_\Omega|\eta_k^{\beta-2}(\eta_k)_x|^2\mathrm{d}x.
\end{aligned}
$$

In view of integration by parts and Poisson equation $(3.7)_3$, we have

$$
\begin{aligned}
&\int_\Omega(\rho_k^2(V_k)_x)_x\Big(1-\frac{1}{\rho_k^2}\Big)\mathrm{d}x-\int_\Omega(\eta_k^2(V_k)_x)_x\Big(1-\frac{1}{\eta_k^2}\Big)\mathrm{d}x\\
&=-2\int_\Omega(V_k)_x(\ln\rho_k)_x\mathrm{d}x+2\int_\Omega(V_k)_x(\ln\eta_k)_x\mathrm{d}x\\
&=\int_\Omega(V_k)_{xx}(\ln\rho_k^2-\ln\eta_k^2)\mathrm{d}x\\
&\geq\int_\Omega(\rho_k^2-\eta_k^2)(\ln\rho_k^2-\ln\eta_k^2)\mathrm{d}x-\|C(x)\|_{L^\infty(\Omega)}\int_\Omega(|\ln\rho_k^2|+|\ln\eta_k^2|)\mathrm{d}x.
\end{aligned}
$$

Putting all the above inequalities together, we can prove (3.9).

For the subsequent entropy estimates, it is appropriate to use the following equivalent form of (3.7):

$$
\begin{aligned}
&\frac{\rho_k^2-\rho_{k-1}^2}{\tau}=-\varepsilon^2\Big(\rho_k^2\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x\Big)_x+((\rho_k^{2\alpha})_x-\rho_k^2(V_k)_x)_x,\\
&\frac{\eta_k^2-\eta_{k-1}^2}{\tau}=-\varepsilon^2\Big(\eta_k^2\Big(\frac{(\eta_k)_{xx}}{\eta_k}\Big)_x\Big)_x+((\eta_k^{2\beta})_x+\eta_k^2(V_k)_x)_x,\\
&(V_k)_{xx}=\rho_k^2-\eta_k^2-C(x).
\end{aligned}
\tag{3.10}
$$

Lemma 3.3 *Under the same assumptions as in Lemma* 3.1*, it holds that*

$$
\begin{aligned}
&\int_\Omega(\rho_k^2(\ln\rho_k^2-1)+1)\mathrm{d}x+\int_\Omega(\eta_k^2(\ln\eta_k^2-1)+1)\mathrm{d}x\\
&+\tau2\varepsilon^2\int_\Omega(|(\rho_k)_{xx}|^2+|(\eta_k)_{xx}|^2)\mathrm{d}x+\tau\frac{32}{3}\varepsilon^2\int_\Omega(|(\sqrt{\rho_k})_x|^4+|(\sqrt{\eta_k})_x|^4)\mathrm{d}x\\
&+\tau\frac{4}{\alpha}\int_\Omega|(\rho_k^\alpha)_x|^2\mathrm{d}x+\tau\frac{4}{\beta}\int_\Omega|(\rho_k^\beta)_x|^2\mathrm{d}x+\tau\int_\Omega(\rho_k^2-\eta_k^2)^2\mathrm{d}x\\
\leq&\int_\Omega(\rho_{k-1}^2(\ln\rho_{k-1}^2-1)+1)\mathrm{d}x+\int_\Omega(\eta_{k-1}^2(\ln\eta_{k-1}^2-1)+1)\mathrm{d}x\\
&+\tau\|C(x)\|_{L^\infty(\Omega)}\int_\Omega(\rho_k^2+\eta_k^2)\mathrm{d}x.
\end{aligned}
\tag{3.11}
$$

Proof We multiply $(3.10)_1$ and $(3.10)_2$ by $\ln\rho_k^2$ and $\ln\eta_k^2$, respectively, and integrate them over Ω. In the same way as in [4], we have

$$
\begin{aligned}
&\int_\Omega\frac{\rho_k^2-\rho_{k-1}^2}{\tau}\ln\rho_k^2\mathrm{d}x+\int_\Omega\frac{\eta_k^2-\eta_{k-1}^2}{\tau}\ln\eta_k^2\mathrm{d}x\\
\geq&\frac{1}{\tau}\int_\Omega(\rho_k^2(\ln\rho_k^2-1)+1)\mathrm{d}x-\frac{1}{\tau}\int_\Omega(\rho_{k-1}^2(\ln\rho_{k-1}^2-1)+1)\mathrm{d}x\\
&+\frac{1}{\tau}\int_\Omega(\eta_k^2(\ln\eta_k^2-1)+1)\mathrm{d}x-\frac{1}{\tau}\int_\Omega(\eta_{k-1}^2(\ln\eta_{k-1}^2-1)+1)\mathrm{d}x.
\end{aligned}
$$

Using integration by parts and boundary conditions, we have

$$\begin{aligned}\varepsilon^2\int_\Omega\Big(\rho_k^2\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x\Big)_x\ln\rho_k^2\mathrm{d}x&=-2\varepsilon^2\int_\Omega\Big(\frac{(\rho_k)_{xx}}{\rho_k}\Big)_x(\rho_k(\rho_k)_x)\mathrm{d}x\\&=2\varepsilon^2\int_\Omega((\rho_k)_{xx})^2\mathrm{d}x+2\varepsilon^2\int_\Omega\frac{(\rho_k)_{xx}((\rho_k)_x)^2}{\rho_k}\mathrm{d}x\\&=2\varepsilon^2\int_\Omega((\rho_k)_{xx})^2\mathrm{d}x+\frac{32}{3}\varepsilon^2\int_\Omega|(\sqrt{\rho_k})_x|^4\mathrm{d}x.\end{aligned}$$

Similarly, we have

$$\int_\Omega\varepsilon^2\Big(\eta_k^2\Big(\frac{(\eta_k)_{xx}}{\eta_k}\Big)_x\Big)_x\ln\eta_k^2\mathrm{d}x=2\varepsilon^2\int_\Omega((\eta_k)_{xx})^2\mathrm{d}x+\frac{32}{3}\varepsilon^2\int_\Omega|(\sqrt{\eta_k})_x|^4\mathrm{d}x.$$

The diffusion terms are treated as follows:

$$\begin{aligned}-\int_\Omega(\rho_k^{2\alpha})_{xx}\ln\rho_k^2\mathrm{d}x-\int_\Omega(\eta_k^{2\beta})_{xx}\ln\eta_k^2\mathrm{d}x&=4\alpha\int_\Omega\rho_k^{2\alpha-2}|(\rho_k)_x|^2\mathrm{d}x+4\beta\int_\Omega\eta_k^{2\beta-2}|(\eta_k)_x|^2\mathrm{d}x\\&=\frac{4}{\alpha}\int_\Omega|(\rho_k^\alpha)_x|^2\mathrm{d}x+\frac{4}{\beta}\int_\Omega|(\eta_k^\beta)_x|^2\mathrm{d}x.\end{aligned}$$

At last, we treat the drift terms. In view of integration by parts and Poisson equation $(3.10)_3$, we have

$$\begin{aligned}\int_\Omega(\rho_k^2(V_k)_x)_x\ln\rho_k^2\mathrm{d}x-\int_\Omega(\eta_k^2(V_k)_x)_x\ln\eta_k^2\mathrm{d}x&=-\int_\Omega(V_k)_x(\rho_k^2)_x\mathrm{d}x+\int_\Omega(V_k)_x(\eta_k^2)_x\mathrm{d}x\\&=\int_\Omega(V_k)_{xx}(\rho_k^2-\eta_k^2)\mathrm{d}x\\&=\int_\Omega(\rho_k^2-\eta_k^2-C(x))(\rho_k^2-\eta_k^2)\mathrm{d}x\\&\geq\int_\Omega(\rho_k^2-\eta_k^2)^2\mathrm{d}x-\|C(x)\|_{L^\infty(\Omega)}\int_\Omega(\rho_k^2+\eta_k^2)\mathrm{d}x.\end{aligned}$$

The above treatments yield the estimate (3.11).

From Lemmas 3.2 and 3.3, we get the following corollary.

Corollary 3.2 *There exists a positive constant C, independent of τ and ε, such that*

$$\|\rho_\tau\,;\,\eta_\tau\|_{L^\infty(0,T;L^2(\Omega))}\leq C,\tag{3.12}$$
$$\|\varepsilon(\rho_\tau)_x\,;\,\varepsilon(\eta_\tau)_x\|_{L^\infty(0,T;L^2(\Omega))}\leq C,\tag{3.13}$$
$$\|\ln\rho_\tau\,;\,\ln\eta_\tau\|_{L^\infty(0,T;L^1(\Omega))}\leq C,\tag{3.14}$$
$$\|(\rho_\tau^\alpha)_x\,;\,(\eta_\tau^\beta)_x\|_{L^2(Q_T)}\leq C,\tag{3.15}$$
$$\|(\rho_\tau^{\alpha-1})_x\,;\,(\eta_\tau^{\beta-1})_x\|_{L^2(Q_T)}\leq C,\tag{3.16}$$

and

$$\varepsilon^2\|(\ln\rho_\tau)_{xx}\,;\,(\ln\eta_\tau)_{xx}\|_{L^2(Q_T)}^2\leq C,\tag{3.17}$$
$$\varepsilon^2\|(\rho_\tau)_{xx}\,;\,(\eta_\tau)_{xx}\|_{L^2(Q_T)}^2\leq C,\tag{3.18}$$
$$\varepsilon^{\frac12}\|(\sqrt{\rho_\tau})_x\,;\,(\sqrt{\eta_\tau})_x\|_{L^4(Q_T)}\leq C.\tag{3.19}$$

Proof It is a direct consequence of (3.9), (3.11), Gronwall inequality, and the inequalities

$$x \le x(\ln x + 1) + 3, \quad |\ln x| \le x - \ln x \quad \text{for any } x > 0. \tag{3.20}$$

Remark 3.1 The positive constant C in Corollary 3.2 actually depends on the initial data ρ_0, η_0 and $C(x)$ in some norm.

Remark 3.2 In [4, 6], for isothermal pressure model, the uniform estimates on $\|(\rho_\tau)_{xx}\|_{L^2(Q_T)}$ and $\|(\eta_\tau)_{xx}\|_{L^2(Q_T)}$ in ε have been obtained. Here, we only get the estimates such as (3.18). But it is already sufficient for us to perform the semiclassical limit.

Furthermore, we have a uniform upper bound on potential V_x.

Corollary 3.3 *There exists a positive constant C, independent of τ and ε, such that*

$$\|(V_\tau)_x\|_{L^\infty(Q_T)} \le C. \tag{3.21}$$

Proof (3.12) implies

$$\|(V_\tau)_{xx}\|_{L^\infty(0,T;L^1(\Omega))} \le C$$

in view of Poisson equation $(2.1)_5$. Furthermore, by Poincaré inequality, we have

$$\|(V_\tau)_x\|_{L^\infty(0,T;W^{1,1}(\Omega))} \le C.$$

Hence Sobolev embedding $W^{1,1}(\Omega) \to L^\infty(\Omega)$ (in 1-dimension) yields (3.21).

Furthermore, if $\alpha, \beta \in (1, 3]$, we can get the following uniform estimates.

Corollary 3.4 *If $\alpha, \beta \in (1, 2]$, there exists a positive constant C independent of τ and ε such that*

$$\|(\rho_\tau)_x\,;\,(\eta_\tau)_x\|_{L^2(Q_T)} \le C, \tag{3.22}$$

and

$$\|\rho_\tau^2\|_{L^2(0,T;W^{1,\frac{2\alpha}{1+\alpha}}(\Omega))} \le C, \quad \|\eta_\tau^2\|_{L^2(0,T;W^{1,\frac{2\beta}{1+\beta}}(\Omega))} \le C. \tag{3.23}$$

If $\alpha, \beta \in (2, 3]$, there exists a positive constant C independent of τ and ε such that

$$\|\rho_\tau^2\,;\,\eta_\tau^2\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{3.24}$$

Proof Using (3.15) and (3.16), we have for $1 < \alpha \le 2$,

$$\begin{aligned}
\int_0^T\!\!\int_\Omega |(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t &\le \int_0^T\!\!\int_{\Omega\cap\{\rho_\tau<1\}} |(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{\Omega\cap\{\rho_\tau\ge1\}} |(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_0^T\!\!\int_{\Omega\cap\{\rho_\tau<1\}} \rho_\tau^{2\alpha-4}|(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_{\Omega\cap\{\rho_\tau\ge1\}} \rho_\tau^{2\alpha-2}|(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_0^T\!\!\int_\Omega \rho_\tau^{2\alpha-4}|(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t + \int_0^T\!\!\int_\Omega \rho_\tau^{2\alpha-2}|(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t \\
&\le C.
\end{aligned}$$

The similar estimate holds for $\|(\eta_\tau)_x\|_{L^2(Q_T)}$. Thus (3.22) is proved. By (3.4), (3.22) and Hölder inequality, we obtain

$$\|(\rho_\tau^2)_x\|_{L^2(0,T;L^{\frac{2\alpha}{1+\alpha}}(\Omega))} \leq C, \quad \|(\eta_\tau^2)_x\|_{L^2(0,T;L^{\frac{2\beta}{1+\beta}}(\Omega))} \leq C. \tag{3.25}$$

Moreover, in view of

$$\frac{2\alpha}{1+\alpha} < \alpha \quad \text{and} \quad \frac{2\beta}{1+\beta} < \beta \quad \text{for } \alpha,\ \beta > 1,$$

(3.4) and (3.25) imply (3.23).

For $2 < \alpha \leq 3$, it holds that

$$\begin{aligned}\int_0^T\int_\Omega \rho_\tau^2|(\rho_\tau)_x|^2 \mathrm{d}x\mathrm{d}t &\leq \int_0^T\int_{\Omega\cap\{\rho_\tau<1\}} \rho_\tau^2|(\rho_\tau)_x|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega\cap\{\rho_\tau\geq 1\}} \rho_\tau^2|(\rho_\tau)_x|^2\mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T\int_{\Omega\cap\{\rho_\tau<1\}} \rho_\tau^{2\alpha-4}|(\rho_\tau)_x|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_{\Omega\cap\{\rho_\tau\geq 1\}} \rho_\tau^{2\alpha-2}|(\rho_\tau)_x|^2\mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T\int_\Omega \rho_\tau^{2\alpha-4}|(\rho_\tau)_x|^2\mathrm{d}x\mathrm{d}t + \int_0^T\int_\Omega \rho_\tau^{2\alpha-2}|(\rho_\tau)_x|^2\mathrm{d}x\mathrm{d}t \\ &\leq C.\end{aligned}$$

The similar estimate holds for $\|(\eta_\tau^2)_x\|_{L^2(Q_T)}$. Thus (3.24) is proved in view of (3.4).

4 Existence of Solutions

In this section, we will prove the existence of weak solutions for any fixed ε. The constant C in this section will be independent of τ, but may depend on ε.

For the following arguments, we also need some bound on the time differences. For this purpose, we define

$$\partial_t^\tau \rho_\tau^2(x,t) = \frac{\rho_k^2 - \rho_{k-1}^2}{\tau}, \quad \partial_t^\tau \eta_\tau^2(x,t) = \frac{\eta_k^2 - \eta_{k-1}^2}{\tau}$$

for $x \in \Omega,\ t \in ((k-1)\tau, k\tau]$.

From Corollary 3.2, we have the boundedness of (ρ_τ) in $L^2(0,T;H^2(\Omega))$, which implies that we can choose a subsequences, again denoted by (ρ_τ), such that

$$\rho_\tau \rightharpoonup \rho \quad \text{weakly in } L^2(0,T;H^2(\Omega)). \tag{4.1}$$

Since $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ (in 1-dimension), we have

$$\|\rho_\tau\|_{L^\infty(Q_T)} \leq \|\rho_\tau\|_{L^\infty(0,T;H^1(\Omega))} \leq C.$$

Thus we obtain

$$\|\rho_\tau^2(F_\tau)_x\|_{L^2(0,T;L^2(\Omega))} \leq \|\rho_\tau\|_{L^\infty(Q_T)}\|\rho_\tau(F_\tau)_x\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{4.2}$$

which yields

$$\|\partial_t^\tau \rho_\tau^2\|_{L^2(0,T;(H^1(\Omega))')} \leq C.$$

Also, by Poincaré inequality, we have

$$\|\rho_\tau^2\|_{L^\infty(0,T;H^1(\Omega))} \le C(\|\rho_\tau(\rho_\tau)_x\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_\tau^2\|_{L^\infty(0,T;L^1(\Omega))}) \le C.$$

Since

$$H^1(\Omega) \hookrightarrow\hookrightarrow C^{0,\gamma}(\overline{\Omega})$$

in 1-dimension for $\gamma \in (0, \frac{1}{2})$, we deduce from Aubin-Lions lemma that

$$\rho_\tau^2 \to \rho^2 \quad \text{in } C([0,T]; C^{0,\gamma}(\overline{\Omega})). \tag{4.3}$$

Furthermore, we have

$$\rho_\tau^{2\alpha-2} \to \rho^{2\alpha-2} \quad \text{strongly in } C([0,T]; C^{0,\gamma}(\Omega)). \tag{4.4}$$

By Gagliardo-Nirenberg's inequality, we have

$$\|((\rho_\tau)_x)^2\|_{L^2(Q_T)} = \|(\rho_\tau)_x\|_{L^4(Q_T)}^2 \le C\|\rho_\tau\|_{L^\infty(Q_T)}\|\rho_\tau\|_{L^2(0,T;H^2(\Omega))} \le C.$$

So we obtain

$$\|(\rho_\tau^2)_{xx}\|_{L^2(Q_T)} \le 2\|\rho_\tau\|_{L^\infty(Q_T)}\|(\rho_\tau)_{xx}\|_{L^2(Q_T)} + 2\|((\rho_\tau)_x)^2\|_{L^2(Q_T)} \le C,$$

which means that

$$\|\rho_\tau^2\|_{L^2(0,T;H^2(\Omega))} \le C.$$

We use again Aubin-Lions lemma to show that

$$\rho_\tau^2 \to \rho^2 \quad \text{strongly in } L^2(0,T; H^1(\Omega)). \tag{4.5}$$

So (4.4) and (4.5) imply

$$\rho_\tau^{2\alpha-2}(\rho_\tau^2)_x \to \rho^{2\alpha-2}(\rho^2)_x \quad \text{strongly in } L^2(0,T; L^2(\Omega)).$$

Next, we will prove that

$$(\rho_\tau)_x \to \rho_x \quad \text{strongly in } L^6(0,T; L^2(\Omega)). \tag{4.6}$$

Indeed, by Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned}\int_0^T \|(\rho_\tau)_x - \rho_x\|_{L^2(\Omega)}^6 \mathrm{d}t &\le C\int_0^T \|\rho_\tau - \rho\|_{L^\infty(\Omega)}^4 \|\rho_\tau - \rho\|_{H^2(\Omega)}^2 \mathrm{d}t \\ &\le C\|\rho_\tau - \rho_\varepsilon\|_{L^\infty(Q_T)}^4 \int_0^T \|\rho_\tau - \rho\|_{H^2(\Omega)}^2 \mathrm{d}t,\end{aligned}$$

which implies (4.6) in view of (4.3). So (4.1) and (4.6) yield

$$(\rho_\tau)_{xx}(\rho_\tau)_x \rightharpoonup (\rho)_{xx}\rho_x \quad \text{weakly in } L^{\frac{3}{2}}(0,T; L^1(\Omega)).$$

From Corollary 3.3, we have

$$(V_\tau)_x \rightharpoonup E \quad \text{weakly-}* \text{ in } L^\infty(Q_T).$$

(4.2) implies

$$\rho_\tau^2(F_\tau)_x \rightharpoonup J \quad \text{weakly in } L^2(Q_T).$$

The similar convergence arguments could be performed for (η_τ) and $(\eta_\tau^2(G_\tau)_x)$. We denote the weak limit of (η_τ) by η in $L^2(0,T;H^2)$. Also, we have

$$\eta_\tau^2(G_\tau)_x \rightharpoonup K \quad \text{weakly in } L^2(Q_T).$$

Corollary 4.1 *The function* (ρ,η,J,K,E) *obtained is the solution of* (1.2) *in the sense of* (1.12)–(1.16).

5 Semiclassical Limit

In this section, we will prove Theorem 1.2. Since the solutions obtained in Corollary 4.1 depend on ε, we denote them by $(\rho_\varepsilon,\ \eta_\varepsilon,\ J_\varepsilon, K_\varepsilon,\ E_\varepsilon)$ for clarity. For convenience, we first discuss the case that $\alpha,\beta\in(1,2]$. By weak convergence and the uniform estimates obtained in Section 3, it is easy to get the following uniform estimates:

$$\|E_\varepsilon\|_{L^\infty(Q_T)} \le C, \tag{5.1}$$
$$\|(\rho_\varepsilon)_x\ ;\ (\eta_\varepsilon)_x\|_{L^2(Q_T)} \le C, \tag{5.2}$$
$$\|\rho_\varepsilon\|_{L^\infty(0,T;L^{2\alpha}(\Omega))} + \|\eta_\varepsilon\|_{L^\infty(0,T;L^{2\beta}(\Omega))} \le C, \tag{5.3}$$
$$\|\rho_\varepsilon^2\|_{L^2(0,T;W^{1,\frac{2\alpha}{1+\alpha}}(\Omega))} + \|\eta_\varepsilon^2\|_{L^2(0,T;W^{1,\frac{2\beta}{1+\beta}}(\Omega))} \le C, \tag{5.4}$$
$$\|J_\varepsilon\|_{L^2(0,T;L^{\frac{2\alpha}{1+\alpha}}(\Omega))} + \|K_\varepsilon\|_{L^2(0,T;L^{\frac{2\beta}{1+\beta}}(\Omega))} \le C, \tag{5.5}$$
$$\|\varepsilon(\rho_\varepsilon)_{xx}\,;\,\varepsilon(\eta_\varepsilon)_{xx}\|_{L^2(Q_T)} \le C, \tag{5.6}$$
$$\|\varepsilon^{\frac12}(\sqrt{\rho_\varepsilon})_x\,;\,\varepsilon^{\frac12}(\sqrt{\eta_\varepsilon})_x\|_{L^4(Q_T)} \le C, \tag{5.7}$$

where C is independent of ε. So there exists a subsequence, again denoted by $(\rho_\varepsilon,\eta_\varepsilon,J_\varepsilon,K_\varepsilon,E_\varepsilon)$, such that

$$\rho_\varepsilon^2 \rightharpoonup n \qquad \text{weakly in } L^2(0,T;W^{1,\frac{2\alpha}{1+\alpha}}(\Omega)), \tag{5.8}$$
$$\eta_\varepsilon^2 \rightharpoonup p \qquad \text{weakly in } L^2(0,T;W^{1,\frac{2\beta}{1+\beta}}(\Omega)), \tag{5.9}$$
$$J_\varepsilon \rightharpoonup J \qquad \text{weakly in } L^2(0,T;L^{\frac{2\alpha}{1+\alpha}}(\Omega)), \tag{5.10}$$
$$K_\varepsilon \rightharpoonup K \qquad \text{weakly in } L^2(0,T;L^{\frac{2\beta}{1+\beta}}(\Omega)), \tag{5.11}$$
$$E_\varepsilon \rightharpoonup E \qquad \text{weakly-}*\text{ in } L^\infty(Q_T). \tag{5.12}$$

From (5.5), (1.12) and (1.13), we have

$$\|\partial_t\rho_\varepsilon^2\|_{L^2(0,T;(W^{1,\frac{2\alpha}{1+\alpha}}(\Omega))')} + \|\partial_t\eta_\varepsilon^2\|_{L^2(0,T;(W^{1,\frac{2\beta}{1+\beta}}(\Omega))')} \le C.$$

Observing that

$$W^{1,\frac{2\alpha}{1+\alpha}}(\Omega) \hookrightarrow\hookrightarrow C^{0,\gamma_1}(\overline{\Omega}) \quad \text{and} \quad W^{1,\frac{2\beta}{1+\beta}}(\Omega) \hookrightarrow\hookrightarrow C^{0,\gamma_2}(\overline{\Omega})$$

for some $0 < \gamma_1, \gamma_2 < 1$, we deduce from Aubin-Lions lemma the following strong convergence:

$$\rho_\varepsilon^2 \to n \quad \text{strongly in } L^2(0,T;C^{0,\gamma_1}(\Omega)),$$
$$\eta_\varepsilon^2 \to p \quad \text{strongly in } L^2(0,T;C^{0,\gamma_2}(\Omega)).$$

Thus, by (5.12), we obtain

$$E_\varepsilon \rho_\varepsilon^2 \rightharpoonup En \quad \text{weakly in } L^2(0,T;C^{0,\gamma_1}(\Omega)),$$
$$E_\varepsilon \eta_\varepsilon^2 \rightharpoonup Ep \quad \text{weakly in } L^2(0,T;C^{0,\gamma_2}(\Omega)).$$

Furthermore, by Lebesgue dominated convergence theorem and the assumption $\alpha, \beta \in (1,2]$, we have

$$\rho_\varepsilon^{2\alpha-2} \to n^{\alpha-1} \quad \text{strongly in } L^2(0,T;C^{0,\gamma_1}(\Omega)),$$
$$\eta_\varepsilon^{2\beta-2} \to p^{\beta-1} \quad \text{strongly in } L^2(0,T;C^{0,\gamma_2}(\Omega)).$$

Hence, we obtain

$$\rho_\varepsilon^{2\alpha-2}(\rho_\varepsilon^2)_x \rightharpoonup n^{\alpha-1} n_x \quad \text{weakly in } L^2(0,T;L^{\frac{2\alpha}{1+\alpha}}),$$
$$\eta_\varepsilon^{2\beta-2}(\eta_\varepsilon^2)_x \rightharpoonup p^{\beta-1} p_x \quad \text{weakly in } L^2(0,T;L^{\frac{2\beta}{1+\beta}}).$$

Next, we have to show that the quantum term would disappear in some sense as $\varepsilon \to 0$. By (5.2), (5.3), (5.6) and (5.7), we have

$$\|\varepsilon^2(\rho_\varepsilon)_{xx}\rho_\varepsilon\|_{L^2(0,T;L^{\frac{2\alpha}{1+\alpha}}(\Omega))} \le \varepsilon\|\varepsilon(\rho_\varepsilon)_{xx}\|_{L^2(Q_T)}\|\rho_\varepsilon\|^2_{L^\infty(0,T;L^{2\alpha}(\Omega))} \le \varepsilon C \to 0,$$
$$\|\varepsilon^2(\rho_\varepsilon)_{xx}(\rho_\varepsilon)_x\|_{L^1(Q_T)} \le \varepsilon\|\varepsilon(\rho_\varepsilon)_{xx}\|_{L^2(Q_T)}\|(\rho_\varepsilon)_x\|_{L^2(Q_T)} \le \varepsilon C \to 0,$$

as $\varepsilon \to 0^+$.

Similarly, we have

$$\|\varepsilon^2(\eta_\varepsilon)_{xx}\eta_\varepsilon\|_{L^2(0,T;L^{\frac{2\beta}{1+\beta}}(\Omega))} \to 0, \quad \|\varepsilon^2(\eta_\varepsilon)_{xx}(\eta_\varepsilon)_x\|_{L^1(Q_T)} \to 0.$$

Now, we can pass to the limit in (1.12)–(1.16) and conclude that the limit satisfies (1.24)–(1.28).

For $\alpha, \beta \in (2,3]$, we get the following uniform estimates from (3.24):

$$\|\rho_\varepsilon^2\|_{L^2(0,T;H^1(\Omega))} + \|\eta_\varepsilon^2\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{5.13}$$

And with the help of (5.6) and (5.7), we have

$$\begin{aligned}\|\varepsilon^2(\rho_\varepsilon)_{xx}(\rho_\varepsilon)_x\|_{L^1(Q_T)} &= 2\|\varepsilon^2(\rho_\varepsilon)_{xx}(\sqrt{\rho_\varepsilon})_x\sqrt{\rho_\varepsilon}\|_{L^1(Q_T)} \\ &\le 2\varepsilon^{\frac{1}{2}}\|\varepsilon(\rho_\varepsilon)_{xx}\|_{L^2(Q_T)}\|\varepsilon^{\frac{1}{2}}(\sqrt{\rho_\varepsilon})_x\|_{L^4(Q_T)}\|\sqrt{\rho_\varepsilon}\|_{L^4(Q_T)} \\ &\le \varepsilon^{\frac{1}{2}}C \to 0,\end{aligned}$$

as $\varepsilon \to 0^+$. Similarly we have

$$\|\varepsilon^2(\eta_\varepsilon)_{xx}(\eta_\varepsilon)_x\|_{L^1(Q_T)} \to 0.$$

Thus, the previous discussion on $\alpha, \beta \in (1,2]$ could be repeated to finish the proof.

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