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A Cheeger-Müller Theorem for Symmetric Bilinear Torsions**

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Abstract The authors establish a Cheeger-Müller type theorem for the complex valued analytic torsion introduced by Burghelea and Haller for flat vector bundles carrying non-degenerate symmetric bilinear forms. As a consequence, they prove the Burghelea-Haller conjecture in full generality, which gives an analytic interpretation of (the square of) the Turaev torsion.

Keywords Analytic torsion, Symmetric bilinear form, Cheeger-Müller theorem, Bismut-Zhang theorem
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1 Introduction

Let F be a unitary flat vector bundle on a closed Riemannian manifold M. In [28], Ray and Singer defined an analytic torsion associated to (M, F) and proved that it does not depend on the Riemannian metric on M. Moreover, they conjectured that this analytic torsion coincides with the classical Reidemeister torsion defined using a triangulation on M (cf. [23]). This conjecture was later proved in the celebrated papers of Cheeger [13] and Müller [24]. Müller generalized this result in [25] to the case where F is a unimodular flat vector bundle on M. In [4], inspired by the considerations of Quillen [26], Bismut and Zhang reformulated the above Cheeger-Müller theorem as an equality between the Reidemeister and Ray-Singer metrics defined on the determinant of cohomology, and proved an extension of it to the case of general flat vector bundles over M. The method used in [4] is different from those of Cheeger and Müller in that it makes use of a deformation by Morse functions introduced by Witten [35] on the de Rham complex.

On the other hand, Turaev generalizes the concept of Reidemeister torsion to a complex valued invariant whose absolute value provides the original Reidemeister torsion, with the help of the so-called Euler structure (cf. [34, 15]). It is natural to ask whether there exists an analytic interpretation of this Turaev torsion.

Recently, there appear two groups of papers dealing with explicitly this question. On one hand, Braverman and Kappeler [6, 7] define what they call "refined analytic torsion" for flat vector bundles over odd dimensional manifolds, and show that it equals to the Turaev torsion

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up to a multiplication by a complex number of absolute value one. On the other hand, Burghelea and Haller [10, 11], following a suggestion of Müller, define a generalized analytic torsion associated to a nondegenerate symmetric bilinear form on a flat vector bundle over an arbitrary dimensional manifold and make an explicit conjecture between this generalized analytic torsion and the Turaev torsion.

Both Braverman-Kappeler and Burghelea-Haller deal with the analysis of determinants of non-self-adjoint Laplacians.

In this paper, we will follow the approach of Burghelea and Haller, which is closer in spirit to the approach developed by Bismut-Zhang in [4, 5].

Let F be a flat complex vector bundle over an oriented closed manifold M. Let $\det H^*(M,F)$ be the determinant line of the cohomology with coefficient F.

We make the assumption that F admits a smooth fiberwise nondegenerate symmetric bilinear form. (In general, this might not exist. However, as indicated by Burghelea and Haller [11], we can form a direct sum of copies of F to make such a symmetric bilinear form exists at least on the direct sum.)

Following Farber-Turaev [15] and Burghelea-Haller [10, 11], one constructs naturally a (nondegenerate) symmetric bilinear form on $\det H^*(M,F)$. This resembles closely with the construction of the Ray-Singer metric in [4], where one replaces the symmetric bilinear form by a Hermitian metric on F. The main difference is that while the Ray-Singer metric is a real valued function on elements in $\det H^*(M,F)$, the analytically induced symmetric bilinear form generally takes complex values on elements in $\det H^*(M,F)$.

The main purpose of this paper is to generalize the main result in [4] to the current situation. That is to say, we establish an explicit comparison result between the above analytically induced symmetric bilinear form on $\det H^*(M,F)$ and another one, which is of Reidemeister type, constructed through a combinatorial way. We will state this result in Theorem 3.1.

We will prove this result by the same method as in [4], that is, by making use of the Witten deformation (cf. [35]) of the de Rham complex by a Morse function. However, since we are going to deal with complex valued torsion which arises from non-self-adjoint Laplacians (the non-self-adjoint property comes from the fact that we are dealing with symmetric bilinear forms instead of Hermitian metrics), we should take care at each step when we proceed the analytical arguments in [4]. In particular, instead of generalizing each step in [4] to the non-self-adjoint case, we will make full use of the results in [4] and see what else one needs to do in the current case. It is remarkable that everything fits at last to give the desired result.

The idea of using the Witten deformation to study symmetric bilinear torsions was mentioned before in [10]. Moreover, an important anomaly formula for the analytically constructed symmetric bilinear forms on $\det H^*(M,F)$ has been proved in [11].

A direct consequence of our main result is that if M is of vanishing Euler characteristic and we consider the Euler structure (introduced in [34]) on M, then we can prove the Burghelea-Haller conjecture (cf. [11, Conjecture 5.1]) identifying a modified version of the above analytic symmetric bilinear form on $\det H^*(M,F)$ with (the square of) the Turaev torsion, which is also interpreted as a symmetric bilinear form on $\det H^*(M,F)$.

We should mention that independently and almost at the same time of our preprint of this paper (cf. [33]), Burghelea and Haller [12] proved their conjecture, up to sign, in the case where

M is of odd dimension. The method they use is different from ours.

The rest of this paper is organized as follows. In Section 2, we recall the basic definitions of various torsions associated with nondegenerate symmetric bilinear forms on a flat vector bundle, we also state an anomaly formula for the analytic torsion associated with nondegenerate symmetric bilinear forms on a flat vector bundle. In Section 3, we state the main result of this paper and provides a proof of it based on several intermediate technical results. Sections 4 to 9 are devoted to the proofs of the intermediate results stated in Section 3. In the final Section 10, we apply the main result proved in Section 3 to prove the Burghelea-Haller conjecture (cf. [11, Conjecture 5.1]) on the analytic interpretation of (the square of) the Turaev torsion.

Since we will make substantial use of the results in [4], we will refer to [4] for related definitions and notations directly when there will be no confusion.

2 Symmetric Bilinear Torsions Associated to the de Rham and Thom-Smale Complexes

In this section, for a nondegenerate bilinear symmetric form on a complex flat vector bundle over an oriented closed manifold, we define two naturally associated symmetric bilinear forms on the determinant of the cohomology $H^*(M,F)$ with coefficient F. One is constructed in a combinatorial way through the Thom-Smale complex associated to a Morse function, and the other one is constructed in an analytic way through the de Rham complex. An anomaly formula essentially due to Burghelea-Haller [11] of the later will also be recalled.

2.1 Symmetric bilinear torsion of a finite dimensional complex

Let (C^*,∂) be a finite cochain complex

$$(C^*,\partial): 0 \longrightarrow C^0 \xrightarrow{\partial_0} C^1 \xrightarrow{\partial_1} \cdots \xrightarrow{\partial_{n-1}} C^n \longrightarrow 0, \tag{2.1}$$

where each C^i, $0 \le i \le n$, is a finite dimensional complex vector space.

Let

$$H^*(C^*,\partial) = \bigoplus_{i=0}^{n} H^i(C^*,\partial) \tag{2.2}$$

be the cohomology of (C^*,∂).

Let

$$\det(C^*,\partial) = \bigotimes_{i=0}^{n} (\det C^i)^{(-1)^i}, \tag{2.3}$$

$$\det H^*(C^*,\partial) = \bigotimes_{i=0}^{n} (\det H^i(C^*,\partial))^{(-1)^i} \tag{2.4}$$

be the determinant lines of (C^*,∂) and $H^*(C^*,\partial)$ respectively.

It is well-known that there is a canonical isomorphism (cf. [20] and [2, Section 1a)])

$$\det(C^*,\partial) \simeq \det H^*(C^*,\partial). \tag{2.5}$$

Let each C^i, $0 \le i \le n$, admit a nondegenerate symmetric bilinear form b_i. Then by (2.3) they induce canonically a symmetric bilinear form $b_{\det(C^*,\partial)}$ on $\det(C^*,\partial)$, which in turn, via (2.5), induces a symmetric bilinear form $b_{\det H^*(C^*,\partial)}$ on $\det H^*(C^*,\partial)$.

Definition 2.1 (cf. [15, 10, 11]) *We call $b_{\det H^*(C^*,\partial)}$ the symmetric bilinear torsion on* $\det H^*(C^*,\partial)$.

Remark 2.1 If (C^*,∂) is acyclic, that is, $H^*(C^*,\partial)=\{0\}$, then $b_{\det H^*(C^*,\partial)}$ is identified as a complex number.

Let A_i, $0\le i\le n$, be an automorphism of C^i. Then it induces a symmetric bilinear form b_i' on C^i defined by

$$b_i'(x,y)=b_i(A_ix,A_iy). \tag{2.6}$$

Let $b'_{\det H^*(C^*,\partial)}$ be the associated symmetric bilinear torsion on $\det H^*(C^*,\partial)$.

The following anomaly result is obvious.

Proposition 2.1 *The following identity holds:*

$$\frac{b'_{\det H^*(C^*,\partial)}}{b_{\det H^*(C^*,\partial)}}=\prod_{i=0}^{n}(\det(A_i)^2)^{(-1)^i}. \tag{2.7}$$

2.2 Milnor symmetric bilinear torsion of the Thom-Smale complex

Let M be a closed smooth manifold, with $\dim M=n$. For simplicity, we make the assumption that M is oriented (the non-orientable case can be treated in exactly the same way, with obvious modifications).

Let (F,∇^F) be a complex flat vector bundle over M carrying the flat connection ∇^F. We make the assumption that F carries a nondegenerate symmetric bilinear form b^F.

Let (F^*,∇^{F^*}) be the dual complex flat vector bundle of (F,∇^F) carrying the dual flat connection ∇^{F^*}.

Let $f:M\to\mathbf{R}$ be a Morse function. Let g^{TM} be a Riemannian metric on TM such that the corresponding gradient vector field $-X=-\nabla f\in\Gamma(TM)$ satisfies the Smale transversality conditions (cf. [32]), that is, the unstable cells (of $-X$) intersect transversally with the stable cells.

Set

$$B=\{x\in M;X(x)=0\}. \tag{2.8}$$

For any $x\in B$, let $W^u(x)$ (resp. $W^s(x)$) denote the unstable (resp. stable) cell at x, with respect to $-X$. We also choose an orientation O_x^- (resp. O_x^+) on $W^u(x)$ (resp. $W^s(x)$).

Let x, $y\in B$ satisfy the Morse index relation $\mathrm{ind}(y)=\mathrm{ind}(x)-1$. Then $\Gamma(x,y)=W^u(x)\cap W^s(y)$ consists of a finite number of integral curves γ of $-X$. Moreover, for each $\gamma\in\Gamma(x,y)$, by using the orientations chosen above, one can define a number $n_\gamma(x,y)=\pm1$ as in [4, (1.28)].

If $x\in B$, let $[W^u(x)]$ be the complex line generated by $W^u(x)$. Set

$$C_*(W^u,F^*)=\bigoplus_{x\in B}[W^u(x)]\otimes F_x^*, \tag{2.9}$$

$$C_i(W^u,F^*)=\bigoplus_{\substack{x\in B\\ \mathrm{ind}(x)=i}}[W^u(x)]\otimes F_x^*. \tag{2.10}$$

If $x \in B$, the flat vector bundle F^* is canonically trivialized on $W^u(x)$. In particular, if $x, y \in B$ satisfy $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$, and if $\gamma \in \Gamma(x, y)$, $f^* \in F_x^*$, let $\tau_\gamma(f^*)$ be the parallel transport of $f^* \in F_x^*$ into F_y^* along γ with respect to the flat connection ∇^{F^*}.

Clearly, for any $x \in B$, there is only a finite number of $y \in B$, satisfying together that $\mathrm{ind}(y) = \mathrm{ind}(x) - 1$ and $\Gamma(x, y) \neq \emptyset$.

If $x \in B$, $f^* \in F_x^*$, set

$$\partial(W^u(x) \otimes f^*) = \sum_{\substack{y \in B \\ \mathrm{ind}(y) = \mathrm{ind}(x) - 1}} \sum_{\gamma \in \Gamma(x,y)} n_\gamma(x, y) W^u(y) \otimes \tau_\gamma(f^*). \tag{2.11}$$

Then ∂ maps $C_i(W^u, F^*)$ into $C_{i-1}(W^u, F^*)$. Moreover, one has

$$\partial^2 = 0. \tag{2.12}$$

That is, $(C_*(W^u, F^*), \partial)$ forms a chain complex. We call it the Thom-Smale complex associated to $(M, F, -X)$.

If $x \in B$, let $[W^u(x)]^*$ be the dual line to $W^u(x)$. Let $(C^*(W^u, F), \partial)$ be the complex which is dual to $(C_*(W^u, F^*), \partial)$. For $0 \leq i \leq n$, one has

$$C^i(W^u, F) = \bigoplus_{\substack{x \in B \\ \mathrm{ind}(x) = i}} [W^u(x)]^* \otimes F_x. \tag{2.13}$$

Let $W^u(x)^* \in [W^u(x)]^*$ be such that $\langle W^u(x), W^u(x)^* \rangle = 1$.

We now introduce a symmetric bilinear form on each $[W^u(x)]^* \otimes F_x$ such that for any $f,\ f' \in F_x$,

$$\langle W^u(x)^* \otimes f, W^u(x)^* \otimes f' \rangle = \langle f, f' \rangle_{b^{F_x}}. \tag{2.14}$$

For any $0 \leq i \leq n$, let $C^i(W^u, F)$ carry the symmetric bilinear form obtained from those defined in (2.14) so that the splitting (2.13) is orthogonal with respect to it. One verifies that this symmetric bilinear form is nondegenerate on $C^i(W^u, F)$.

Definition 2.2 *The symmetric bilinear torsion on the determinant line of the cohomology of the Thom-Smale cochain complex* $(C^*(W^u, F), \partial)$*, in the sense of Definition* 2.1*, is called the Milnor symmetric bilnear torsion associated to* $(M, F, b^F, -X)$*, and is denoted by* $b^{\mathcal{M}}_{(M,F,b^F,-X)}$.

From the anomaly formula (2.7), one deduces easily the following result.

Proposition 2.2 *If* b_1^F *is another nondegenerate symmetric bilinear form on the flat vector bundle* F *over* M *and* $b^{\mathcal{M}}_{(M,F,b_1^F,-X)}$ *denotes the corresponding symmetric bilinear torsion on* $\det H^*(C^*(W^u, F), \partial)$*, then the following anomaly formula holds:*

$$b^{\mathcal{M}}_{(M,F,b_1^F,-X)} = b^{\mathcal{M}}_{(M,F,b^F,-X)} \prod_{x \in B} \det((b^{F|_x})^{-1} b_1^{F|_x})^{(-1)^{\mathrm{ind}(x)}}. \tag{2.15}$$

2.3 Ray-Singer symmetric bilinear torsion of the de Rham complex

We continue the discussion of the previous subsection. However, we do not use the Morse function and make transversality assumptions.

For any $0 \le i \le n$, denote

$$\Omega^i(M,F) = \Gamma(\Lambda^i(T^*M)\otimes F), \quad \Omega^*(M,F) = \bigoplus_{i=0}^{n} \Omega^i(M,F). \tag{2.16}$$

Let d^F denote the natural exterior differential on $\Omega^*(M,F)$ induced from ∇^F which maps each $\Omega^i(M,F)$, $0 \le i \le n$, into $\Omega^{i+1}(M,F)$.

Let g^F be a Hermitian metric on F. The Riemannian metric g^{TM} and g^F determine a natural inner product (that is, a pre-Hilbert space structure) on $\Omega^*(M,F)$ (cf. [4, (2.2)] and [5, (2.3)]).

On the other hand g^{TM} and the symmetric bilinear form b^F determine together a symmetric bilinear form on $\Omega^*(M,F)$ such that if $u = \alpha f$, $v = \beta g \in \Omega^*(M,F)$ such that $\alpha,\ \beta \in \Omega^*(M)$, $f,\ g \in \Gamma(F)$, then

$$\langle u, v\rangle_b = \int_M (\alpha \wedge *\beta) b^F(f,g), \tag{2.17}$$

where $*$ is the Hodge star operator (cf. [36]).

Consider the de Rham complex

$$(\Omega^*(M,F), d^F)\ :\ 0 \ \to\ \Omega^0(M,F)\ \stackrel{d^F}{\to}\ \Omega^1(M,F)\ \to\ \cdots\ \stackrel{d^F}{\to}\ \Omega^n(M,F)\ \to\ 0. \tag{2.18}$$

Let $d_b^{F*} : \Omega^*(M,F) \to \Omega^*(M,F)$ denote the formal adjoint of d^F with respect to the symmetric bilinear form in (2.17). That is, for any $u,\ v \in \Omega^*(M,F)$, one has

$$\langle d^F u, v\rangle_b = \langle u, d_b^{F*} v\rangle_b. \tag{2.19}$$

Set

$$D_b = d^F + d_b^{F*}, \quad D_b^2 = (d^F + d_b^{F*})^2 = d_b^{F*} d^F + d^F d_b^{F*}. \tag{2.20}$$

Then the Laplacian D_b^2 preserves the $\mathbf{Z}$-grading of $\Omega^*(M,F)$.

As was pointed out in [10] and [11], D_b^2 has the same principal symbol as the usual Hodge Laplacian (constructed using the inner product on $\Omega^*(M,F)$ induced from (g^{TM}, g^F)) studied for example in [4].

We collect some well-known facts concerning D_b^2 as in [11, Proposition 4.1], where the reference [30] is indicated.

Proposition 2.3 *The following properties hold for the Laplacian D_b^2:*

(i) *The spectrum of D_b^2 is discrete. For every $\theta > 0$ all but finitely many points of the spectrum are contained in the angle $\{z \in \mathbf{C} \mid -\theta < \arg(z) < \theta\}$;*

(ii) *If λ is in the spectrum of D_b^2, then the image of the associated spectral projection is finite dimensional and contains smooth forms only. We refer to this image as the (generalized) λ-eigen space of D_b^2 and denote it by $\Omega^*_{\{\lambda\}}(M,F)$. There exists $N_\lambda \in \mathbf{N}$ such that*

$$(D_b^2 - \lambda)^{N_\lambda}|_{\Omega^*_{\{\lambda\}}(M,F)} = 0. \tag{2.21}$$

We have a D_b^2-invariant $\langle\ ,\ \rangle_b$-orthogonal decomposition

$$\Omega^*(M,F) = \Omega^*_{\{\lambda\}}(M,F) \bigoplus \Omega^*_{\{\lambda\}}(M,F)^\perp. \tag{2.22}$$

The restriction of $D_b^2 - \lambda$ *to* $\Omega^*_{\{\lambda\}}(M,F)^{\perp}$ *is invertible;*

(iii) *The decomposition* (2.22) *is invariant under* d^F *and* d_b^{F*}*;*

(iv) *For* $\lambda \neq \mu$*, the eigen spaces* $\Omega^*_{\{\lambda\}}(M,F)$ *and* $\Omega^*_{\{\mu\}}(M,F)$ *are* $\langle\, ,\, \rangle_b$*-orthogonal to each other.*

For any $a \geq 0$, set

$$\Omega^*_{[0,a]}(M,F) = \bigoplus_{0\leq|\lambda|\leq a} \Omega^*_{\{\lambda\}}(M,F). \tag{2.23}$$

Let $\Omega^*_{[0,a]}(M,F)^{\perp}$ denote the $\langle\, ,\, \rangle_b$-orthogonal complement to $\Omega^*_{[0,a]}(M,F)$.

By [11, (29)] and Proposition 2.3, one sees that $(\Omega^*_{[0,a]}(M,F), d^F)$ forms a finite dimensional complex whose cohomology equals to that of $(\Omega^*(M,F), d^F)$. Moreover, the symmetric bilinear form $\langle\, ,\, \rangle_b$ clearly induces a nondegenerate symmetric bilinear form on each $\Omega^i_{[0,a]}(M,F)$ with $0 \leq i \leq n$. By Definition 2.1 one then gets a symmetric bilinear torsion $b_{\det H^*(\Omega^*_{[0,a]}(M,F),d^F)}$ on $\det H^*(\Omega^*_{[0,a]}(M,F), d^F) = \det H^*(\Omega^*(M,F), d^F)$.

For any $0 \leq i \leq n$, let $D^2_{b,i}$ be the restriction of D_b^2 on $\Omega^i(M,F)$. Then it is shown in [11] (cf. [30, Theorem 13.1]) that for any $a \geq 0$, the following regularized zeta determinant is well-defined:

$$\det{}'(D^2_{b,(a,+\infty),i}) = \exp\Big(- \frac{\partial}{\partial s}\Big|_{s=0} \mathrm{Tr}[(D^2_{b,i}|_{\Omega^*_{[0,a]}(M,F)^{\perp}})^{-s}]\Big). \tag{2.24}$$

Proposition 2.4 (cf. [11, Proposition 4.7]) *The symmetric bilinear form on* $\det H^*(\Omega^*(M, F), d^F)$ *defined by*

$$b_{\det H^*(\Omega^*_{[0,a]}(M,F),d^F)} \prod_{i=0}^{n} (\det{}'(D^2_{b,(a,+\infty),i}))^{(-1)^i i} \tag{2.25}$$

does not depend on the choice of $a \geq 0$.

Definition 2.3 *The symmetric bilinear form defined by* (2.25) *is called the Ray-Singer symmetric bilinear torsion on* $\det H^*(\Omega^*(M,F), d^F)$ *and is denoted by* $b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}$.

2.4 An anomaly formula for the Ray-Singer symmetric bilinear torsion

We continue the discussion of the above subsection.

Let $\theta(F, b^F) \in \Omega^1(M)$ be the Kamber-Tondeur form defined by (cf. [11, (4)])

$$\theta(F, b^F) = \mathrm{Tr}[(b^F)^{-1}\nabla^F b^F]. \tag{2.26}$$

Then $\theta(F, b^F)$ is a closed one form on M whose cohomology class depends only on the homotopy class of b^F (cf. [11]).

Let ∇^{TM} denote the Levi-Civita connection associated to the Riemannian metric g^{TM} on TM. Let $R^{TM} = (\nabla^{TM})^2$ be the curvature of ∇^{TM}. Let $\mathrm{e}(TM, \nabla^{TM}) \in \Omega^n(M)$ be the associated Euler form defined by (cf. [4, (3.17)] and [36, Chapter 3])

$$\mathrm{e}(TM, \nabla^{TM}) = \mathrm{Pf}\Big(\frac{R^{TM}}{2\pi}\Big). \tag{2.27}$$

Let g'^{TM} be another Riemannian metric on TM and ∇'^{TM} be the associated Levi-Civita connection. Let $\widetilde{\mathrm{e}}(TM, \nabla^{TM}, \nabla'^{TM})$ be the Chern-Simons class of $n-1$ smooth forms on M, which is defined modulo exact $n-1$ forms, such that

$$d\,\widetilde{\mathrm{e}}(TM, \nabla^{TM}, \nabla'^{TM}) = \mathrm{e}(TM, \nabla'^{TM}) - \mathrm{e}(TM, \nabla^{TM}) \tag{2.28}$$

(cf. [4, (4.10)]). Of course, if n is odd,

$$\widetilde{\mathrm{e}}(TM, \nabla^{TM}, \nabla'^{TM}) = 0. \tag{2.29}$$

Let b'^F be another nondegenerate symmetric bilinear form on F.

Let $b^{\mathrm{RS}}_{(M,F,g'^{TM},b'^F)}$ denote the Ray-Singer symmetric bilinear torsion associated to g'^{TM} and b'^F. Then the complex number

$$\frac{b^{\mathrm{RS}}_{(M,F,g'^{TM},b'^F)}}{b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}} \in \mathbf{C}^*$$

is well-defined.

We can now state the anomaly formula, of which an equivalent form has been proved in [11, Theorem 4.2], for the Ray-Singer symmetric bilinear torsion as follows.

Theorem 2.1 *If b^F, b'^F lie in the same homotopy class of nondegenerate symmetric bilinear forms on F, then the following identity holds:*

$$\frac{b^{\mathrm{RS}}_{(M,F,g'^{TM},b'^F)}}{b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}} = \exp\Big(\int_M \log(\det((b^F)^{-1}b'^F))\mathrm{e}(TM, \nabla^{TM})\Big) \cdot \exp\Big(-\int_M \theta(F, b'^F)\widetilde{\mathrm{e}}(TM, \nabla^{TM}, \nabla'^{TM})\Big). \tag{2.30}$$

In particular, if $\dim M = n$ *is odd, then*

$$\frac{b^{\mathrm{RS}}_{(M,F,g'^{TM},b'^F)}}{b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}} = 1. \tag{2.31}$$

Remark 2.2 Since b^F, b'^F lie in the same homotopy class, one sees that $\log(\det((b^F)^{-1}b'^F))$ is a well-defined univalent function on M.

Remark 2.3 For an alternate approach to the above anomaly formula, compare with Remark 6.1.

3 Comparison Between the Ray-Singer and Milnor Symmetric Bilinear Torsions

In this section, we prove the main result of this paper, which is an explicit comparison result between the Ray-Singer and Milnor symmetric bilinear torsions introduced in the last section.

The form of the result we will state formally looks very similar to a theorem of Bismut-Zhang proved in [4, Theorem 0.2], if one replaces the Hermitian metrics there by the symmetric bilinear forms. This similarity also reflects in the proof of the main result here, where we will use as in [4] the Witten deformation [35] of the de Rham complex by Morse functions. Moreover, we will make use the analytic techniques developed in [4, 5], some of which go back to the paper of Bismut-Lebeau [3].

Still, since we will deal with non-self-adjoint operators, we have to generalize many of the techniques in [4, 5] to the current situation. We will point out the differences in due context.

3.1 A Cheeger-Müller theorem for symmetric bilinear torsions

We assume that we are in the same situation as in Sections 2.2–2.4. By a simple argument of Helffer-Sjöstrand [19, Proposition 5.1] (cf. [4, Section 7b)]), we may and we will assume that g^{TM} there satisfies the following property without altering the Thom-Smale cochain complex $(C^*(W^u,F),\partial)$:

$(*)$ For any $x\in B$, there is a system of coordinates $y=(y^1,\cdots,y^n)$ centered at x such that near x,

$$g^{TM}=\sum_{i=1}^{n}|\mathrm{d}y^i|^2,\quad f(y)=f(x)-\frac{1}{2}\sum_{i=1}^{\mathrm{ind}(x)}|y^i|^2+\frac{1}{2}\sum_{i=\mathrm{ind}(x)+1}^{n}|y^i|^2. \tag{3.1}$$

By a result of Laudenbach [21], $\{W^u(x):x\in B\}$ forms a CW decomposition of M.

For any $x\in B$, F is canonically trivialized over each cell $W^u(x)$.

Let P_∞ be the de Rham map defined by

$$\alpha\in\Omega^*(M,F)\to P_\infty\alpha=\sum_{x\in B}W^u(x)^*\int_{W^u(x)}\alpha\in C^*(W^u,F). \tag{3.2}$$

By the Stokes theorem, one has

$$\partial P_\infty=P_\infty d^F. \tag{3.3}$$

Moreover, it is shown in [21] that P_∞ is a $\mathbf{Z}$-graded quasi-isomorphism, inducing a canonical isomorphism

$$P_\infty^H:H^*(\Omega^*(M,F),d^F)\to H^*(C^*(W^u,F),\partial), \tag{3.4}$$

which in turn induces a natural isomorphism between the determinant lines,

$$P_\infty^{\det H}:\det H^*(\Omega^*(M,F),d^F)\to\det H^*(C^*(W^u,F),\partial). \tag{3.5}$$

Now let h^{TM} be an arbitrary smooth metric on TM.

By Definition 2.3, one has an associated Ray-Singer symmetric bilinear torsion $b^{\mathrm{RS}}_{(M,F,h^{TM},b^F)}$ on $\det H^*(\Omega^*(M,F),d^F)$. From (3.5), one gets a well-defined symmetric bilinear form

$$P_\infty^{\det H}(b^{\mathrm{RS}}_{(M,F,h^{TM},b^F)}) \tag{3.6}$$

on $\det H^*(C^*(W^u,F),\partial)$.

On the other hand, by Definition 2.2, one has a well-defined Milnor symmetric bilinear torsion $b^{\mathcal{M}}_{(M,F,b^F,-X)}$ on $\det H^*(C^*(W^u,F),\partial)$, where $X=\nabla f$ is the gradient vector field of f associated to g^{TM}.

Let $\psi(TM,\nabla^{TM})$ be the Mathai-Quillen current (cf. [22]) over TM, associated to h^{TM}, defined in [4, Definition 3.6]. As indicated in [4, Remark 3.8], the pull-back current $X^*\psi(TM,\nabla^{TM})$ is well-defined over M.

The main result of this paper, which generalizes [4, Theorem 0.2] to the case where F admits a nondegenerate symmetric bilinear form, can be stated as follows.

Theorem 3.1 *The following identity in* $\mathbf{C}$ *holds:*

$$\frac{P_\infty^{\det H}(b^{\mathrm{RS}}_{(M,F,h^{TM},b^F)})}{b^{\mathcal{M}}_{(M,F,b^F,-X)}} = \exp\Big(-\int_M \theta(F,b^F)X^*\psi(TM,\nabla^{TM})\Big). \tag{3.7}$$

Remark 3.1 By proceeding similarly as in [4, Section 7b], in order to prove (3.7), we may well assume that $h^{TM} = g^{TM}$. Moreover, we may assume that b^F, as well as the Hermitian metric g^F on F, are flat on an open neighborhood of the zero set B of X. From now on, we will make these assumptions.

3.2 Some intermediate results

We assume that the assumptions made in Remark 3.1 hold.

For any $T \in \mathbf{R}$, let b^F_T be the deformed symmetric bilinear form on F defined by

$$b^F_T(u,v) = \mathrm{e}^{-2Tf}b^F(u,v). \tag{3.8}$$

Let $d^{F*}_{b_T}$ be the associated formal adjoint in the sense of (2.19). Set

$$D_{b_T} = d^F + d^{F*}_{b_T}, \quad D^2_{b_T} = (d^F + d^{F*}_{b_T})^2 = d^{F*}_{b_T}d^F + d^F d^{F*}_{b_T}. \tag{3.9}$$

Let $\Omega^*_{[0,1],T}(M,F)$ be defined as in (2.23) with respect to $D^2_{b_T}$, and let $\Omega^*_{[0,1],T}(M,F)^\perp$ be the corresponding $\langle\ ,\ \rangle_{b_T}$-orthogonal complement.

Let $P^{[0,1]}_T$ be the orthogonal projection from $\Omega^*(M,F)$ to $\Omega^*_{[0,1],T}(M,F)$ with respect to the inner product determined by g^{TM} and $g^F_T = \mathrm{e}^{-2Tf}g^F$. Set $P^{(1,+\infty)}_T = \mathrm{Id} - P^{[0,1]}_T$.

Following [4, (7.13)–(7.15)], we introduce the notations

$$\begin{aligned}
\chi(F) &= \sum_{i=0}^{\dim M}(-1)^i \dim H^i(M,F) = \mathrm{rk}(F)\sum_{x\in B}(-1)^{\mathrm{ind}(x)}, \\
\chi'(F) &= \mathrm{rk}(F)\sum_{x\in B}(-1)^{\mathrm{ind}(x)}\mathrm{ind}(x) = \mathrm{rk}(F)\sum_{i=0}^{n}(-1)^i iM_i, \\
\mathrm{Tr}^B_s[f] &= \sum_{x\in B}(-1)^{\mathrm{ind}(x)}f(x),
\end{aligned} \tag{3.10}$$

where for any $0 \le i \le n$, M_i is the number of $x \in B$ of index i.

Let N be the number operator on $\Omega^*(M,F)$ acting on $\Omega^i(M,F)$ by multiplication by i.

We now state several intermediate results whose proofs will be given later in Sections 4 to 9.

Theorem 3.2 (compare with [4, Theorem 7.6]) *Let* $P^{[0,1]}_T$ *be the restriction of* P_∞ *on* $\Omega^*_{[0,1],T}(M,F)$*, and let* $P^{[0,1],\det H}_T$ *be the induced isomorphism on cohomology. Then the following identity holds:*

$$\lim_{T\to+\infty}\frac{P^{[0,1],\det H}_T(b_{\det H^*(\Omega^*_{[0,1],T}(M,F),d^F)})}{b^{\mathcal{M}}_{(M,F,b^F,-X)}}\Big(\frac{T}{\pi}\Big)^{\frac{n}{2}\chi(F)-\chi'(F)}\exp(2\,\mathrm{rk}(F)\mathrm{Tr}^B_s[f]T) = 1. \tag{3.11}$$

Theorem 3.3 (compare with [4, Theorem 7.8]) *For any $t > 0$,*

$$\lim_{T\to+\infty} \mathrm{Tr}_s[N\exp(-tD_{b_T}^2)P_T^{(1,+\infty)}] = 0. \tag{3.12}$$

Moreover, for any $d > 0$ there exist $c > 0$, $C > 0$ and $T_0 \geq 1$ such that for any $t \geq d$ and $T \geq T_0$,

$$|\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)P_T^{(1,+\infty)}]| \leq c\exp(-Ct). \tag{3.13}$$

Theorem 3.4 (compare with [4, Theorem 7.9]) *For $T \geq 0$ large enough, then*

$$\dim \Omega^i_{[0,1],T}(M,F) = \mathrm{rk}(F)\, M_i. \tag{3.14}$$

Also,

$$\lim_{T\to+\infty} \mathrm{Tr}[D_{b_T}^2 P_T^{[0,1]}] = 0. \tag{3.15}$$

For the next results, we will make use the same notation for Clifford multiplications and Berezin integrals as in [4, Section 4].

Theorem 3.5 (compare with [4, Theorem 7.10]) *As $t \to 0$, the following identity holds,*

$$\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)] = \begin{cases} \dfrac{n}{2}\chi(F) + O(t) \quad (\textit{if } n \textit{ is even}), \\ \mathrm{rk}(F)\displaystyle\int_M \int^B L\exp\Big(-\frac{\dot{R}^{TM}}{2}\Big)\frac{1}{\sqrt{t}} + O(\sqrt{t}\,) \quad (\textit{if } n \textit{ is odd}), \end{cases} \tag{3.16}$$

where L is originally defined in [4, (3.52)].

Theorem 3.6 (compare with [5, Theorem A.1]) *There exist $0 < \alpha \leq 1$, $C > 0$ such that for any $0 < t \leq \alpha$, $0 \leq T \leq \frac{1}{t}$, then*

$$\begin{aligned} \Big|\mathrm{Tr}_s[N\exp(-(tD_b + T\widehat{c}(\nabla f))^2)] - \frac{1}{t}\int_M \int^B L\exp(-B_{T^2})\mathrm{rk}(F) \\ - \frac{T}{2}\int_M \theta(F,b^F)\int^B \widehat{df}\exp(-B_{T^2}) - \frac{n}{2}\chi(F)\Big| \leq Ct. \end{aligned} \tag{3.17}$$

Theorem 3.7 (compare with [5, Theorem A.2]) *For any $T > 0$, the following identity holds:*

$$\lim_{t\to 0}\mathrm{Tr}_s\Big[N\exp\Big(-\Big(tD_b + \frac{T}{t}\widehat{c}(\nabla f)\Big)^2\Big)\Big] = \frac{1}{1-\mathrm{e}^{-2T}}((1+\mathrm{e}^{-2T})\chi'(F) - n\mathrm{e}^{-2T}\chi(F)). \tag{3.18}$$

Theorem 3.8 (compare with [5, Theorem A.3]) *There exist $\alpha \in (0,1]$, $c > 0$, $C > 0$ such that for any $t \in (0,\alpha]$, $T \geq 1$, then*

$$\Big|\mathrm{Tr}_s\Big[N\exp\Big(-\Big(tD_b + \frac{T}{t}\widehat{c}(\nabla f)\Big)^2\Big)\Big] - \chi'(F)\Big| \leq c\exp(-CT). \tag{3.19}$$

Clearly, we may and we will assume that the number $\alpha > 0$ in Theorems 3.7 and 3.9 have been chosen to be the same.

3.3 Proof of Theorem 3.1

First of all, by the anomaly formula (2.30), for any $T \geq 0$, one has

$$\frac{P_T^{[0,1],\det H}\big(b_{\det H^*(\Omega^*_{[0,1],T}(M,F),d^F)}\big)}{b^{\mathcal{M}}_{(M,F,b^F,-X)}}\prod_{i=0}^{n}(\det(D^2_{b_T}|_{\Omega^*_{[0,1],T}(M,F)^\perp\cap\Omega^i(M,F)}))^{(-1)^i i}$$
$$= \frac{P_\infty^{\det H}(b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)})}{b^{\mathcal{M}}_{(M,F,b^F,-X)}}\exp\Big(-2T\mathrm{rk}(F)\int_M f\mathrm{e}(TM,\nabla^{TM})\Big). \tag{3.20}$$

From now on, we will write $a \simeq b$ for $a,\ b \in \mathbf{C}$ if $\mathrm{e}^a = \mathrm{e}^b$. Thus, we can rewrite (3.20) as

$$\log\left(\frac{P_\infty^{\det H}(b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)})}{b^{\mathcal{M}}_{(M,F,b^F,-X)}}\right) \simeq \log\left(\frac{P_T^{[0,1],\det H}\big(b_{\det H^*(\Omega^*_{[0,1],T}(M,F),d^F)}\big)}{b^{\mathcal{M}}_{(M,F,b^F,-X)}}\right)$$
$$+\sum_{i=0}^{n}(-1)^i i\log(\det(D^2_{b_T}|_{\Omega^*_{[0,1],T}(M,F)^\perp\cap\Omega^i(M,F)}))$$
$$+2T\mathrm{rk}(F)\int_M f\mathrm{e}(TM,\nabla^{TM}). \tag{3.21}$$

Let $T_0 > 0$ be as in Theorem 3.3. For any $T \geq T_0$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) \geq n+1$, set

$$\theta_T(s) = \frac{1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\mathrm{Tr}_s[N\exp(-tD^2_{b_T})P_T^{(1,+\infty)}]\mathrm{d}t. \tag{3.22}$$

By (3.13), $\theta_T(s)$ is well-defined and can be extended to a meromorphic function which is holomorphic at $s = 0$ (cf. [30]). Moreover,

$$\sum_{i=0}^{n}(-1)^i i\log(\det(D^2_{b_T}|_{\Omega^*_{[0,1],T}(M,F)^\perp\cap\Omega^i(M,F)})) \simeq -\frac{\partial\theta_T(s)}{\partial s}\Big|_{s=0}. \tag{3.23}$$

Let $d = \alpha^2$ with α being as in Theorem 3.8. From (3.22) and Theorems 3.3–3.5, one finds

$$\frac{\partial\theta_T(s)}{\partial s}\Big|_{s=0} = \int_0^d\Big(\mathrm{Tr}_s[N\exp(-tD^2_{b_T})P_T^{(1,+\infty)}] - \frac{a_{-1}}{\sqrt{t}} - \frac{n}{2}\chi(F) + \chi'(F)\Big)\frac{\mathrm{d}t}{t}$$
$$+\int_d^{+\infty}\mathrm{Tr}_s[N\exp(-tD^2_{b_T})P_T^{(1,+\infty)}]\frac{\mathrm{d}t}{t} - \frac{2a_{-1}}{\sqrt{d}}$$
$$-(\Gamma'(1)-\log d)\Big(\frac{n}{2}\chi(F)-\chi'(F)\Big), \tag{3.24}$$

where we denote for simplicity that

$$a_{-1} = \mathrm{rk}(F)\int_M\int^B L\exp\Big(-\frac{\dot{R}^{TM}}{2}\Big). \tag{3.25}$$

Proposition 3.1 *One has*

$$\lim_{T\to+\infty}\int_d^{+\infty}\mathrm{Tr}_s[N\exp(-tD^2_{b_T})P_T^{(1,+\infty)}]\frac{\mathrm{d}t}{t} = 0. \tag{3.26}$$

Proof This follows from Theorem 3.3 directly.

Now we write

$$\begin{aligned}&\int_0^d \Big(\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)P_T^{(1,+\infty)}] - \frac{a_{-1}}{\sqrt{t}} - \frac{n}{2}\chi(F) + \chi'(F)\Big)\frac{\mathrm{d}t}{t}\\ &= \int_0^d \Big(\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)] - \frac{a_{-1}}{\sqrt{t}} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &\quad - \int_0^d (\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)P_T^{[0,1]}] - \chi'(F))\frac{\mathrm{d}t}{t}.\end{aligned} \tag{3.27}$$

From Theorem 3.4, one deduces that

$$\lim_{T\to+\infty}\int_0^d (\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)P_T^{[0,1]}] - \chi'(F))\frac{\mathrm{d}t}{t} = 0. \tag{3.28}$$

To study the first term in the right-hand side of (3.27), we observe first that for any $T \geq 0$,

$$\mathrm{e}^{-Tf}D_{b_T}^2\mathrm{e}^{Tf} = (D_b + T\widehat{c}(\nabla f))^2. \tag{3.29}$$

Thus, one has

$$\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)] = \mathrm{Tr}_s[N\exp(-t(D_b + T\widehat{c}(\nabla f))^2)]. \tag{3.30}$$

By (3.30), one writes

$$\begin{aligned}&\int_0^d \Big(\mathrm{Tr}_s[N\exp(-tD_{b_T}^2)] - \frac{a_{-1}}{\sqrt{t}} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &= 2\int_0^{\sqrt{d}} \Big(\mathrm{Tr}_s[N\exp(-(tD_b + tT\widehat{c}(\nabla f))^2)] - \frac{a_{-1}}{t} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &= 2\int_{\frac{1}{\sqrt{T}}}^{\sqrt{d}} \Big(\mathrm{Tr}_s[N\exp(-(tD_b + tT\widehat{c}(\nabla f))^2)] - \frac{a_{-1}}{t} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &\quad + 2\int_0^{\frac{1}{\sqrt{T}}} \Big(\mathrm{Tr}_s[N\exp(-(tD_b + tT\widehat{c}(\nabla f))^2)] - \frac{a_{-1}}{t} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &= 2\int_1^{\sqrt{dT}} \Big(\mathrm{Tr}_s\Big[N\exp\Big(-\Big(\frac{t}{\sqrt{T}}D_b + t\sqrt{T}\widehat{c}(\nabla f)\Big)^2\Big)\Big] - \frac{\sqrt{T}}{t}a_{-1} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &\quad + 2\int_0^{\frac{1}{\sqrt{T}}} \Big(\mathrm{Tr}_s[N\exp(-(tD_b + tT\widehat{c}(\nabla f))^2)] - \frac{a_{-1}}{t} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}.\end{aligned} \tag{3.31}$$

In view of Theorem 3.6, we write

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{T}}} \Big(\mathrm{Tr}_s[N\exp(-(tD_b + tT\widehat{c}(\nabla f))^2)] - \frac{a_{-1}}{t} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &= \int_0^{\frac{1}{\sqrt{T}}} \Big(\mathrm{Tr}_s[N\exp(-(tD_b + tT\widehat{c}(\nabla f))^2)] - \frac{1}{t}\int_M\int^B L\exp(-B_{(tT)^2})\mathrm{rk}(F)\\ &\quad - \frac{tT}{2}\int_M \theta(F,b^F)\int^B \widehat{df}\exp(-B_{(tT)^2}) - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\ &\quad + \int_0^{\frac{1}{\sqrt{T}}} \Big(\frac{1}{t}\int_M\int^B L\exp(-B_{(tT)^2})\mathrm{rk}(F) - \frac{a_{-1}}{t}\Big)\frac{\mathrm{d}t}{t}\\ &\quad + \int_0^{\frac{1}{\sqrt{T}}} \frac{tT}{2}\int_M \theta(F,b^F)\int^B \widehat{df}\exp(-B_{(tT)^2})\frac{\mathrm{d}t}{t}.\end{aligned} \tag{3.32}$$

By [4, Definitions 3.6, 3.12 and Theorem 3.18], one has, as $T\to+\infty$,

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{T}}}\frac{tT}{2}\int_M\theta(F,b^F)\int^B\widehat{df}\exp(-B_{(tT)^2})\frac{\mathrm{d}t}{t}\\&=\frac{1}{2}\int_0^{\sqrt{T}}\int_M\theta(F,b^F)\int^B\widehat{df}\exp(-B_{t^2})\mathrm{d}t\\&\to\frac{1}{2}\int_0^{+\infty}\int_M\theta(F,b^F)\int^B\widehat{df}\exp(-B_{t^2})\mathrm{d}t\\&=\frac{1}{2}\int_M\theta(F,b^F)(\nabla f)^*\psi(TM,\nabla^{TM}).\end{aligned}\tag{3.33}$$

By [4, (3.58)] we have, for any $T\geq 0$,

$$\begin{aligned}&\int_M\int^B(L\exp(-B_T)-L\exp(-B_0))\\&=-\sqrt{T}f\int_M\int^B(\exp(-B_T)-\exp(-B_0))\\&\quad+\int_M\frac{f}{2}\int_0^T\Big(\int^B(\exp(-B_t)-\exp(-B_0))\Big)\frac{\mathrm{d}t}{\sqrt{t}}.\end{aligned}\tag{3.34}$$

From (3.34), one deduces easily that

$$\lim_{T\to0^+}\frac{1}{\sqrt{T}}\int_M\int^B(L\exp(-B_T)-L\exp(-B_0))=0.\tag{3.35}$$

From [4, (3.54)], (3.35) and the integration by parts, we have

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{T}}}\Big(\frac{1}{t}\int_M\int^B L\exp(-B_{(tT)^2})\mathrm{rk}(F)-\frac{a_{-1}}{t}\Big)\frac{\mathrm{d}t}{t}\\&=-T\mathrm{rk}(F)\int_0^T\int_M\int^B(L\exp(-B_t)-L\exp(-B_0))\mathrm{d}\frac{1}{\sqrt{t}}\\&=-\sqrt{T}\,\mathrm{rk}(F)\int_M\int^B(L\exp(-B_T)-L\exp(-B_0))-T\mathrm{rk}(F)\int_M\int_0^T f\frac{\partial}{\partial t}\int^B\exp(-B_t)\mathrm{d}t\\&=-\sqrt{T}\,\mathrm{rk}(F)\int_M\int^B L\exp(-B_T)+\sqrt{T}a_{-1}-T\mathrm{rk}(F)\int_M f\int^B\exp(-B_T)\\&\quad+T\mathrm{rk}(F)\int_M f\int^B\exp(-B_0).\end{aligned}\tag{3.36}$$

From Theorems 3.6, 3.7, (3.35), [4, Theorem 3.20], [4, (7.72) and (7.73)] and the dominate convergence, one finds that as $T\to+\infty$,

$$\begin{aligned}&\int_0^{\frac{1}{\sqrt{T}}}\Big(\mathrm{Tr}_s[N\exp(-(tD_b+tT\widehat{c}(\nabla f))^2)]-\frac{1}{t}\int_M\int^B L\exp(-B_{(tT)^2})\mathrm{rk}(F)\\&\quad-\frac{tT}{2}\int_M\theta(F,b^F)\int^B\widehat{df}\exp(-B_{(tT)^2})-\frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\&=\int_0^1\Big(\mathrm{Tr}_s\Big[N\exp\Big(-\Big(\frac{t}{\sqrt{T}}D_b+t\sqrt{T}\widehat{c}(\nabla f)\Big)^2\Big)\Big]-\frac{\sqrt{T}}{t}\int_M\int^B L\exp(-B_{(t\sqrt{T})^2})\mathrm{rk}(F)\end{aligned}$$

$$
\begin{aligned}
&- \frac{t\sqrt{T}}{2}\int_M \theta(F,b^F)\int^B \widehat{df}\exp(-B_{(t\sqrt{T})^2}) - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\
&\to \int_0^1 \Big(\frac{1}{1-\mathrm{e}^{-2t^2}}((1+\mathrm{e}^{-2t^2})\chi'(F) - n\mathrm{e}^{-2t^2}\chi(F))\\
&\quad + \frac{\mathrm{rk}(F)}{2t^2}\sum_{x\in B}(-1)^{\mathrm{ind}(x)}(n-2\,\mathrm{ind}(x)) - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\
&= \frac{1}{2}\Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\int_0^1 \Big(\frac{1+\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}} - \frac{1}{t}\Big)\frac{\mathrm{d}t}{t}.
\end{aligned}
\tag{3.37}
$$

On the other hand, by Theorems 3.7, 3.8 and the dominate convergence, we have, as $T \to +\infty$,

$$
\begin{aligned}
&\int_1^{\sqrt{Td}} \Big(\mathrm{Tr}_s\Big[N\exp\Big(-\Big(\frac{t}{\sqrt{T}}D_b + t\sqrt{T}\widehat{c}(\nabla f)\Big)^2\Big)\Big] - \frac{\sqrt{T}}{t}a_{-1} - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t}\\
&= \int_1^{\sqrt{Td}} \Big(\mathrm{Tr}_s\Big[N\exp\Big(-\Big(\frac{t}{\sqrt{T}}D_b + t\sqrt{T}\widehat{c}(\nabla f)\Big)^2\Big)\Big] - \chi'(F)\Big)\frac{\mathrm{d}t}{t}\\
&\quad + \frac{1}{2}\chi'(F)\log(Td) + a_{-1}\sqrt{T}\Big(\frac{1}{\sqrt{Td}} - 1\Big) - \frac{n}{4}\chi(F)\log(Td)\\
&= \int_1^{+\infty} \Big(\frac{1}{1-\mathrm{e}^{-2t^2}}((1+\mathrm{e}^{-2t^2})\chi'(F) - n\mathrm{e}^{-2t^2}\chi(F)) - \chi'(F)\Big)\frac{\mathrm{d}t}{t}\\
&\quad + \frac{1}{2}\Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\log(Td) + \frac{a_{-1}}{\sqrt{d}} - \sqrt{T}a_{-1} + o(1)\\
&= \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\int_1^{+\infty}\frac{\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\frac{\mathrm{d}t}{t} + \frac{1}{2}\Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\log(Td)\\
&\quad + \frac{a_{-1}}{\sqrt{d}} - \sqrt{T}a_{-1} + o(1).
\end{aligned}
\tag{3.38}
$$

Combining (3.11), (3.21), (3.23)–(3.28), (3.31)–(3.33) and (3.36)–(3.38), one deduces, by setting $T \to +\infty$, that

$$
\begin{aligned}
&\log\Big(\frac{P_\infty^{\det H}(b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)})}{b^{\mathcal{M}}_{(M,F,b^F,-X)}}\Big)\\
&\simeq -2\,\mathrm{rk}(F)\mathrm{Tr}_s^B[f]T + \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\log T\\
&\quad - \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\log\pi - \int_M \theta(F,b^F)(\nabla f)^*\psi(TM,\nabla^{TM})\\
&\quad + 2\sqrt{T}\,\mathrm{rk}(F)\int_M\int^B L\exp(-B_T) - 2\sqrt{T}\,a_{-1} + 2\,T\mathrm{rk}(F)\int_M f\int^B \exp(-B_T)\\
&\quad - 2\,T\mathrm{rk}(F)\int_M f\int^B \exp(-B_0) - \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\int_0^1\Big(\frac{1+\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}} - \frac{1}{t}\Big)\frac{\mathrm{d}t}{t}\\
&\quad - \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\int_1^{+\infty}\frac{2\,\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\frac{\mathrm{d}t}{t} - \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\log(Td) - \frac{2a_{-1}}{\sqrt{d}}\\
&\quad + 2\sqrt{T}a_{-1} + 2\,T\mathrm{rk}(F)\int_M f\,\mathrm{e}(TM,\nabla^{TM}) + \frac{2a_{-1}}{\sqrt{d}}\\
&\quad - (\Gamma'(1) - \log d)\Big(\chi'(F) - \frac{n}{2}\chi(F)\Big) + o(1)
\end{aligned}
$$

$$
\begin{aligned}
&= 2T\mathrm{rk}(F)\int_M f\Big(\int^B \exp(-B_T) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\Big)\\
&\quad - \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)\Big(\int_0^1 \Big(\frac{1+\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}} - \frac{1}{t}\Big)\frac{\mathrm{d}t}{t} + \int_1^{+\infty}\frac{2\,\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\frac{\mathrm{d}t}{t}\Big)\\
&\quad - \Big(\chi'(F) - \frac{n}{2}\chi(F)\Big)(\log\pi + \Gamma'(1)) + 2\sqrt{T}\,\mathrm{rk}(F)\int_M\int^B L\exp(-B_T)\\
&\quad - \int_M \theta(F, b^F)(\nabla f)^*\psi(TM, \nabla^{TM}) + o(1).
\end{aligned}
\tag{3.39}
$$

By [4, Theorem 3.20] and [4, (7.72)], one has

$$
\lim_{T\to+\infty} 2T\mathrm{rk}(F)\int_M f\Big(\int^B \exp(-B_T) - \sum_{x\in B}(-1)^{\mathrm{ind}(x)}\delta_x\Big) = -\Big(\chi'(F) - \frac{n}{2}\chi(F)\Big), \tag{3.40}
$$

$$
\lim_{T\to+\infty} 2\sqrt{T}\,\mathrm{rk}(F)\int_M\int^B L\exp(-B_T) = 2\Big(\chi'(F) - \frac{n}{2}\chi(F)\Big). \tag{3.41}
$$

On the other hand, by [4, (7.93)], one has

$$
\int_0^1 \Big(\frac{1+\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}} - \frac{1}{t}\Big)\frac{\mathrm{d}t}{t} + \int_1^{+\infty}\frac{2\,\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}}\frac{\mathrm{d}t}{t} = 1 - \log\pi - \Gamma'(1). \tag{3.42}
$$

From (3.39)–(3.42), we get (3.7), which completes the proof of Theorem 3.1.

Remark 3.2 We have used the strategy outlined in [5, Appendix] to prove Theorem 3.1, instead of using that in [4, Section 7]. In particular, we avoid the explicit use of [4, Theorem 3.9] which is crucial in [4, Section 7], though we still make use of the variation formulas (cf. [4, (3.54) and (3.58)]).

Remark 3.3 By Theorem 3.6, one deduces that

$$
\begin{aligned}
&\lim_{T\to+\infty}\int_0^1 \Big(\mathrm{Tr}_s\Big[N\exp\Big(-\Big(\frac{t}{\sqrt{T}}D_b + t\sqrt{T}\widehat{c}(\nabla f)\Big)^2\Big)\Big]\\
&\quad - \frac{\sqrt{T}}{t}\int_M\int^B L\exp(-B_{(t\sqrt{T})^2})\mathrm{rk}(F)\\
&\quad - \frac{t\sqrt{T}}{2}\int_M \theta(F, b^F)\int^B \widehat{df}\exp(-B_{(t\sqrt{T})^2}) - \frac{n}{2}\chi(F)\Big)\frac{\mathrm{d}t}{t} = 0.
\end{aligned}
\tag{3.43}
$$

Combining this with (3.37), one gets

$$
\int_0^1 \Big(\frac{1+\mathrm{e}^{-2t}}{1-\mathrm{e}^{-2t}} - \frac{1}{t}\Big)\frac{\mathrm{d}t}{t} = 0. \tag{3.44}
$$

4 Asymptotics of the Symmetric Bilinear Torsion of the Witten Complex

In this section, we prove Theorems 3.2 and 3.4.

We make the same assumptions and use the same notations as in Section 3.

4.1 Some formulas related to D_b

Recall that b^F is a nondegenerate symmetric bilinear form on a complex flat vector bundle F over an oriented closed Riemannian manifold M. Then it determines a nondegenerate symmetric bilinear form $\langle\, ,\, \rangle_b$ on $\Omega^*(M,F)$ (cf. (2.17)).

Recall that the formal adjoint d_b^{F*} of d^F with respect to the symmetric bilinear form $\langle\, ,\, \rangle_b$ has been defined in (2.19), and D_b is the operator defined by

$$D_b = d^F + d_b^{F*}. \tag{4.1}$$

Let

$$\omega_b^F = \omega_b\left(F, \nabla^F\right) = \left(b^F\right)^{-1} \nabla^F b^F \tag{4.2}$$

be defined as in [11].

Let $\nabla = \nabla^{\Lambda^*(T^*M)\otimes F}$ be the tensor product connection on $\Lambda^*(T^*M)\otimes F$ obtained from the Levi-Civita connection ∇^{TM} associated to g^{TM} and the flat connection ∇^F on F.

For any $X \in TM$, let $X^* \in T^*M$ corresponds to X via g^{TM}. Recall that

$$c(X) = X^* - i_X, \quad \widehat{c}(X) = X^* + i_X \tag{4.3}$$

denote the Clifford actions on $\Lambda^*(T^*M)$, where X^* and i_X are the exterior and interior multiplications respectively (cf. [4, Section 4]).

For any oriented orthonormal basis $e_1, \ldots, e_n$ of TM, set

$$c(\omega_b^F) = \sum_{i=1}^n c(e_i)\omega_b^F(e_i), \quad \widehat{c}(\omega_b^F) = \sum_{i=1}^n \widehat{c}(e_i)\omega_b^F(e_i). \tag{4.4}$$

With these definitions and notations one verifies easily that (cf. [11, (92)])

$$d^F + d_b^{F*} = \sum_{i=1}^n c(e_i)\nabla_{e_i} + \frac{1}{2}c(\omega_b^F) - \frac{1}{2}\widehat{c}(\omega_b^F). \tag{4.5}$$

Recall that g^F is a Hermitian metric on F. Together with g^{TM} it determines an inner product $\langle\, ,\, \rangle_g$ on $\Omega^*(M,F)$ (cf. [4, (2.2)] and [5, (2.3)]).

Let d_g^{F*} be the formal adjoint of d^F with respect to $\langle\, ,\, \rangle_g$.

Set as in [4] and [5]

$$\omega_g^F = \omega_g(F, \nabla^F) = (g^F)^{-1}\nabla^F g^F. \tag{4.6}$$

Then ω_g^F is a one form taking values in the self-adjoint elements in $\mathrm{End}(F)$. Moreover,

$$\nabla^{F,u} = \nabla^F + \frac{1}{2}\omega_g^F \tag{4.7}$$

is a Hermitian connection on F with respect to g^F (cf. [4, Section 4] and [5, Section 2]). Let ∇^u be the associated tensor product connection on $\Lambda^*(T^*M)\otimes F$.

By [4, (4.25)], one has

$$D_g := d^F + d_g^{F*} = \sum_{i=1}^n c(e_i)\nabla^u_{e_i} - \frac{1}{2}\widehat{c}(\omega_g^F) = \sum_{i=1}^n c(e_i)\nabla_{e_i} + \frac{1}{2}c(\omega_g^F) - \frac{1}{2}\widehat{c}(\omega_g^F). \tag{4.8}$$

From (4.5) and (4.8), one gets

$$d^F + d_b^{F*} = d^F + d_g^{F*} + \frac{1}{2}c(\omega_b^F) - \frac{1}{2}\widehat{c}(\omega_b^F) - \frac{1}{2}c(\omega_g^F) + \frac{1}{2}\widehat{c}(\omega_g^F). \tag{4.9}$$

Write ω_b^F as

$$\omega_b^F = \omega_{b,1}^F + \omega_{b,2}^F, \tag{4.10}$$

where $\omega_{b,1}^F$ (resp. $\omega_{b,2}^F$) takes values in self-adjoint (resp. skew-adjoint) elements (with respect to g^F) in $\mathrm{End}(F)$.

From (4.9), one gets the decomposition of D_b into self-adjoint and skew-adjoint parts (with respect to $\langle\, ,\, \rangle_g$) as follows:

$$\begin{aligned} d^F + d_b^{F*} = {} & \Big(d^F + d_g^{F*} + \frac{1}{2}\widehat{c}(\omega_g^F) - \frac{1}{2}\widehat{c}(\omega_{b,1}^F) + \frac{1}{2}c(\omega_{b,2}^F)\Big) \\ & + \Big(-\frac{1}{2}c(\omega_g^F) + \frac{1}{2}c(\omega_{b,1}^F) - \frac{1}{2}\widehat{c}(\omega_{b,2}^F)\Big). \end{aligned} \tag{4.11}$$

4.2 Witten deformation and some basic estimates

Let $f : M \to \mathbf{R}$ be a Morse function on M. We make the assumption that the Riemannian metric g^{TM} and f verify the condition (3.1). We also assume that g^F, like b^F, is flat near the set of critical points of f.

Following Witten [35], for any $T \in \mathbf{R}$, set

$$d_T^F = \mathrm{e}^{-Tf} d^F \mathrm{e}^{Tf}, \quad \delta_{b,T}^F = \mathrm{e}^{Tf} d_b^{F*} \mathrm{e}^{-Tf}, \quad \delta_{g,T}^F = \mathrm{e}^{Tf} d_g^{F*} \mathrm{e}^{-Tf}. \tag{4.12}$$

Set

$$\widetilde{D}_{b,T} = d_T^F + \delta_{b,T}^F = D_b + T\widehat{c}(df), \quad \widetilde{D}_{g,T} = d_T^F + \delta_{g,T}^F = D_g + T\widehat{c}(df). \tag{4.13}$$

Observe that the skew-adjoint part of $\widetilde{D}_{b,T}$ is the same as that of $\widetilde{D}_b$.

Let $\|\,\|_0$ be the L^2 norm on $\Omega^*(M,F)$ associated to $\langle\, ,\, \rangle_g$. For any $q > 0$, let $\|\,\|_q$ be a fixed q-Sobolev norm on $\Omega^*(M,F)$.

Proposition 4.1 *For any open neighborhood U of B, there exist $T_0 > 0$, $C > 0$, $c > 0$ such that for any $s \in \Omega^*(M,F)$ with $\mathrm{supp}(s) \subset M \setminus U$ and $T \geq T_0$, one has*

$$\|\widetilde{D}_{b,T} s\|_0^2 \geq C(\|s\|_1^2 + (T-c)\|s\|_0^2). \tag{4.14}$$

Proof From (4.11) and (4.13), one sees that the formal adjoint $\widetilde{D}_{b,T}^*$ of $\widetilde{D}_{b,T}$ is given by

$$\begin{aligned} \widetilde{D}_{b,T}^* = {} & \Big(D_g + T\widehat{c}(df) + \frac{1}{2}\widehat{c}(\omega_g^F) - \frac{1}{2}\widehat{c}(\omega_{b,1}^F) + \frac{1}{2}c(\omega_{b,2}^F)\Big) \\ & - \Big(-\frac{1}{2}c(\omega_g^F) + \frac{1}{2}c(\omega_{b,1}^F) - \frac{1}{2}\widehat{c}(\omega_{b,2}^F)\Big). \end{aligned} \tag{4.15}$$

For simplicity, we denote

$$\begin{aligned} A^F &= \frac{1}{2}\widehat{c}(\omega_g^F) - \frac{1}{2}\widehat{c}(\omega_{b,1}^F) + \frac{1}{2}c(\omega_{b,2}^F), \\ B^F &= -\frac{1}{2}c(\omega_g^F) + \frac{1}{2}c(\omega_{b,1}^F) - \frac{1}{2}\widehat{c}(\omega_{b,2}^F). \end{aligned} \tag{4.16}$$

Then one computes

$$\begin{aligned}\widetilde{D}_{b,T}^{*}\widetilde{D}_{b,T} &= (D_g + A^F)^2 + (D_g + A^F)B^F - B^F(D_g + A^F) - (B^F)^2 \\ &\quad + T([D_g + A^F, \widehat{c}(df)] + \widehat{c}(df)B^F - B^F\widehat{c}(df)) + T^2|df|^2, \end{aligned} \tag{4.17}$$

where by [,] we denote the super bracket in the sense of Quillen [27].

Since it is easy to check (cf. [4, (5.17)]) that

$$[D_g, \widehat{c}(df)] = \sum_{i=1}^{n} c(e_i)\widehat{c}(\nabla_{e_i}^{TM}\nabla f) - \omega_g^F(\nabla f), \tag{4.18}$$

where $\nabla f \in \Gamma(TM)$ is the gradient vector field of f with respect to g^{TM}, is of order zero, the coefficient of T in the right-hand side of (4.17) is of order zero.

Also, it is clear that there is $c_0 > 0$ such that for any $x \in M \setminus U$,

$$|df(x)| \geq c_0. \tag{4.19}$$

From (4.17) and (4.19), one gets Proposition 4.1 easily, as

$$\|\widetilde{D}_{b,T}s\|_0^2 = \langle \widetilde{D}_{b,T}s, \widetilde{D}_{b,T}s\rangle = \langle \widetilde{D}_{b,T}^{*}\widetilde{D}_{b,T}s, s\rangle. \tag{4.20}$$

Proposition 4.2 *For any $c > 0$, there exists $T_c > 0$ such that for any $T \geq T_c$, $z \in \mathbf{C}$ with $|z| = c$, $z \notin \mathrm{Spec}(\widetilde{D}_{b,T}^2)$.*

Proof For any $p \in B$, let $\mathbf{y} = (y_1, \cdots, y_n)$ be the coordinate system of p as in (3.1), in an open ball U_p of radius $4a$, around p. We also assume that both b^F and g^F are flat on each U_p, $p \in B$. The existence of $a > 0$ is clear.

By (4.9), one then has

$$D_b = D_g \quad \text{on } U_B = \bigcup_{p\in B} U_p. \tag{4.21}$$

Let $\gamma : \mathbf{R} \to [0, 1]$ be a smooth function such that $\gamma(x) = 1$ if $|x| \leq a$, while $\gamma(x) = 0$ if $|x| \geq 2a$.

For any $T > 0$ and $p \in B$, set

$$\begin{aligned}\alpha_{p,T} &= \int_{U_p} \gamma(|\mathbf{y}|)^2 \exp(-T|\mathbf{y}|^2)\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n, \\ \rho_{p,T} &= \frac{\gamma(|\mathbf{y}|)}{\sqrt{\alpha_{p,T}}} \exp\Big(-\frac{T|\mathbf{y}|^2}{2}\Big)\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^{n_f(p)},\end{aligned} \tag{4.22}$$

where $n_f(p) = \mathrm{ind}(p)$ is the Morse index of f at p. Then $\rho_{p,T} \in \Omega^{n_f(p)}(M)$ is of unit length with compact support contained in U_p.

Set

$$E_T = \bigoplus_{p\in B}\{\rho_{p,T} \otimes h_p : p \in B,\ h_p \in F_p\}. \tag{4.23}$$

Let $E_T^\perp$ be the orthogonal complement to E_T in $L^2(\Omega^*(M,F))$ with respect to $\langle\, ,\rangle_g$, with $L^2(\Omega^*(M,F))$ being the L^2 completion of $\Omega^*(M,F)$. Then one has the orthogonal decomposition

$$L^2\left(\Omega^*(M,F)\right) = E_T \oplus E_T^\perp. \tag{4.24}$$

Let p_T, $p_T^\perp$ be the orthogonal projections from $L^2(\Omega^*(M,F))$ onto E_T, $E_T^\perp$ respectively.

Following [3, Section 9b)] (cf. [36, (5.19)]), set

$$\begin{aligned} \widetilde{D}_{b,T,1} &= p_T\widetilde{D}_{b,T}p_T, \quad \widetilde{D}_{b,T,2} = p_T\widetilde{D}_{b,T}p_T^\perp, \\ \widetilde{D}_{b,T,3} &= p_T^\perp\widetilde{D}_{b,T}p_T, \quad \widetilde{D}_{b,T,4} = p_T^\perp\widetilde{D}_{b,T}p_T^\perp. \end{aligned} \tag{4.25}$$

From (4.17), (4.20), (4.21), (4.25) and proceed as in [3, Section 9] and [36, Proof of Proposition 5.6], one can prove in the same way that there exist $T_0 > 0$, $C > 0$ such that for any $T \geq T_0$, one has

$$\widetilde{D}_{b,T,1} = 0, \tag{4.26}$$

$$\|\widetilde{D}_{b,T,2}s\|_0 \leq \frac{\|s\|_0}{T}, \quad \|\widetilde{D}_{b,T,3}s'\|_0 \leq \frac{\|s'\|_0}{T} \tag{4.27}$$

for any $s \in E_T^\perp \cap \mathbf{H}^1(M,F)$, $s' \in E_T$, where $\mathbf{H}^1(M,F)$ is the Sobolev space with respect to the Sobolev norm $\|\,\|_1$ on $\Omega^*(M,F)$, and

$$\|\widetilde{D}_{b,T,4}s\|_0 \geq C\sqrt{T}\|s\|_0 \tag{4.28}$$

for any $s \in E_T^\perp \cap \mathbf{H}^1(M,F)$.

Now for any $\lambda \in \mathbf{C}$, $T \geq T_0$ and $s \in \Omega^*(M,F)$, by (4.26)–(4.28), we have (cf. [36, (5.26)])

$$\begin{aligned} \|(\lambda - \widetilde{D}_{b,T})s\|_0 &\geq \frac{1}{2}\|\lambda p_T s - \widetilde{D}_{b,T,2}p_T^\perp s\|_0 + \frac{1}{2}\|\lambda p_T^\perp s - \widetilde{D}_{b,T,3}s - \widetilde{D}_{b,T,4}p_T^\perp s\|_0 \\ &\geq \frac{1}{2}\Big(\Big(|\lambda| - \frac{1}{T}\Big)\|p_T s\|_0 + \Big(C\sqrt{T} - |\lambda| - \frac{1}{T}\Big)\|p_T^\perp s\|_0\Big). \end{aligned} \tag{4.29}$$

From (4.29), one sees easily that there exist $C_0 > 0$, $T_0' \geq T_0$ such that for any $T \geq T_0'$ and $\lambda \in \mathbf{C}$ with $|\lambda|^2 = c$, one has

$$\|(\lambda^2 - \widetilde{D}_{b,T}^2)s\|_0 = \|(\lambda + \widetilde{D}_{b,T})(\lambda - \widetilde{D}_{b,T})s\|_0 \geq C_0\|s\|_0, \tag{4.30}$$

from which Proposition 4.2 follows.

From now on, we take $c = 1$, $T_{c=1}$ as in Proposition 4.2 and assume $T \geq T_1$.

Let $\widetilde{\Omega}^*_{[0,1],T}(M,F)$ be defined as in (2.23) with respect to $\widetilde{D}_{b,T}$. Let $\widetilde{P}_T^{[0,1]}$ be the orthogonal projection from $L^2(\Omega^*(M,F))$ onto $\widetilde{\Omega}^*_{[0,1],T}(M,F)$.

For any $p \in B$, let $[W^u(p)]^*$ admit a Hermitian metric such that $|W^u(p)^*| = 1$. Let $[W^u(p)]^* \otimes F_p$ carry the tensor product metric from the above one with g^{F_p}. Let $C^*(W^u,F)$ carry a Hermitian metric through the orthogonal direct sum of the Hermitian metrics on $[W^u(p)]^* \otimes F_p$'s.

Let $J_T : C^*(W^u,F) \to \Omega^*(M,F)$ be the isometry defined by that for any $p \in B$, $h \in F_p$ and $\mathbf{y}$ the coordinate system as above in U_p,

$$J_T\left(W^u(p)^* \otimes h\right)(\mathbf{y}) = \rho_{p,T} \otimes h. \tag{4.31}$$

From (4.11) and (4.21), one can proceed in exactly that same way as in [4, Theorem 8.8] and [5, Theorem 6.7] to get the following result.

Theorem 4.1 *There exists $c > 0$ such that as $T \to +\infty$, for any $s \in C^*(W^u, F)$,*

$$(\widetilde{P}_T^{[0,1]} J_T - J_T)s = O(\mathrm{e}^{-cT})s \quad \text{uniformly on } M. \tag{4.32}$$

4.3 Proof of Theorem 3.4

From Theorem 4.1, one gets immediately that

$$\dim \widetilde{\Omega}^*_{[0,1],T}(M, F) \geq \# B. \tag{4.33}$$

By (4.21) and proceeding as in [36, Proof of Proposition 5.5], one sees that indeed, (4.33) holds in equality.

Since $\widetilde{P}_T^{[0,1]}$ preserves the **Z**-grading of $\Omega^*(M, F)$ (as $\widetilde{D}^2_{b,T}$ does), by applying (4.32) in each grade and by (4.33) with equality, one then gets, for any $0 \leq i \leq n$,

$$\dim \widetilde{\Omega}^i_{[0,1],T}(M, F) = \mathrm{rk}(F) M_i = \mathrm{rk}(F) \cdot \#\{p \in B : \mathrm{ind}(p) = i\}. \tag{4.34}$$

On the other hand, since the number c in Proposition 4.2 can be chosen arbitrarily small, one sees that when $T \to +\infty$, one has

$$\mathrm{Tr}[\widetilde{D}^2_{b,T} \widetilde{P}_T^{[0,1]}] \to 0. \tag{4.35}$$

Now consider the isomorphism $r_T : \Omega^*(M, F) \to \Omega^*(M, F)$ defined by $r_T(s) = \mathrm{e}^{Tf} s$. Then it induces a map preserving the corresponding symmetric bilinear forms, as well as the inner products,

$$\begin{aligned} r_T &: (\Omega^*(M, F), \langle\ ,\ \rangle_b) \mapsto (\Omega^*(M, F), \langle\ ,\ \rangle_{b_T}), \\ r_T &: (\Omega^*(M, F), \langle\ ,\ \rangle_g) \mapsto (\Omega^*(M, F), \langle\ ,\ \rangle_{g_T}), \end{aligned} \tag{4.36}$$

with $\langle , \rangle_{g_T}$ obtained from g^{TM} and $g^F_T = \mathrm{e}^{-2Tf} g^F$ (cf. [4, (5.1)]). Moreover, one verifies directly that

$$r_T \widetilde{D}_{b,T} = D_{b_T} r_T. \tag{4.37}$$

From (4.34)–(4.37), one gets Theorem 3.4 immediately.

4.4 Proof of Theorem 3.2

We still assume that $T \geq T_{c=1}$, where $T_{c=1}$ verifies Proposition 4.2.

Let $e_T : C^*(W^u, F) \to \Omega^*_{[0,1],T}(M, F)$ be defined by

$$e_T = r_T \widetilde{P}_T^{[0,1]} J_T. \tag{4.38}$$

Recall that $C^*(W^u, F)$ carries a symmetric bilinear form determined in (2.13) and (2.14), while $\Omega^*_{[0,1],T}(M, F)$ carries the induced symmetric bilinear form $\langle , \rangle_{b_T}$. Let $e_T^{\#}$ be the adjoint of e_T with respect to these two symmetric bilinear forms.

Proposition 4.3 *There exists $c > 0$ such that as $T \to +\infty$,*

$$e_T^{\#} e_T = 1 + O(\mathrm{e}^{-cT}). \tag{4.39}$$

In particular, when $T > 0$ is large enough, $e_T : C^*(W^u, F) \to \Omega^*_{[0,1],T}(M, F)$ is a $\mathbf{Z}$-graded isomorphism.

Proof By the definition of e_T and $e_T^{\#}$, one has that for any $s, s' \in C^*(W^u, F)$,

$$\langle e_T^{\#} e_T s, s' \rangle_{b_T} = \langle e_T s, e_T s' \rangle_{b_T} = \langle \widetilde{P}_T^{[0,1]} J_T s, \widetilde{P}_T^{[0,1]} J_T s' \rangle_b. \tag{4.40}$$

On the other hand, from (4.22) and (4.31), one sees directly that

$$\langle J_T s, J_T s' \rangle_b = \langle s, s' \rangle_b. \tag{4.41}$$

From Theorem 4.1, (4.40), and (4.41), one gets (4.39).

From Theorem 3.4 and (4.39), one sees that when $T > 0$ is large enough, e_T is an isomorphism.

Recall that the quasi-isomorphism $P_\infty : (\Omega^*(M, F), d^F) \to (C^*(W^u, F), \partial)$ has been defined in (3.2). Let $P_{\infty,T} : \Omega^*_{[0,1],T}(M, F) \to C^*(W^u, F)$ be the restriction of P_∞ on $\Omega^*_{[0,1],T}(M, F)$.

By (3.3), one has

$$\partial P_{\infty,T} = P_{\infty,T} d^F. \tag{4.42}$$

By Theorem 4.1 and (4.42), one can proceed in exactly the same way as in [5, Proof of Theorem 6.11] (cf. [36, Section 6.4]), to get the following analogue of [5, Theorem 6.11].

Proposition 4.4 *There exists $c > 0$ such that as $T \to +\infty$, one has*

$$P_{\infty,T} e_T = \mathrm{e}^{T\mathcal{F}} \Big(\frac{\pi}{T}\Big)^{\frac{N}{2} - \frac{n}{4}} (1 + O(\mathrm{e}^{-cT})), \tag{4.43}$$

where $\mathcal{F}$ acts on $[W^u(p)]^* \otimes F_p$ with $p \in B$ by multiplication by $f(p)$, and N is the number operator acting on $[W^u(p)]^* \otimes F_p$ with $p \in B$ by multiplication by $\mathrm{ind}(p)$. In particular, for $T > 0$ large enough, $P_{\infty,T} e_T \in \mathrm{End}(C^*(W^u, F))$ is one to one.

From (4.42) and Propositions 4.3, 4.4, one sees that when $T > 0$ is large enough,

$$P_{\infty,T} : (\Omega^*_{[0,1],T}(M, F), d^F) \to (C^*(W^u, F), \partial) \tag{4.44}$$

is a cochain isomorphism.

From Proposition 2.2 and (4.44), one finds

$$\frac{P_T^{[0,1],\det H}(b_{\det H^*(\Omega^*_{[0,1],T}(M,F),d^F)})}{b^{\mathcal{M}}_{(M,F,b^F,-X)}} = \prod_{i=0}^{n} \det(P^{\#}_{\infty,T} P_{\infty,T}|_{\Omega^i_{[0,1],T}(M,F)})^{(-1)^{i+1}}, \tag{4.45}$$

where $P^{\#}_{\infty,T}$ is the adjoint of $P_{\infty,T}$ with respect to the symmetric bilinear forms $\langle\ ,\ \rangle_b$.

From Propositions 4.3 and 4.4, one deduces that as $T \to +\infty$,

$$\begin{aligned}\det(P^{\#}_{\infty,T} P_{\infty,T}|_{\Omega^i_{[0,1],T}(M,F)}) &= \det(e_T e_T^{\#} P^{\#}_{\infty,T} P_{\infty,T}|_{\Omega^i_{[0,1],T}(M,F)}) \cdot \det{}^{-1}(e_T e_T^{\#}|_{\Omega^i_{[0,1],T}(M,F)}) \\ &= \det((P_{\infty,T} e_T)^{\#} P_{\infty,T} e_T|_{C^i(W^u,F)}) \cdot \det{}^{-1}(e_T^{\#} e_T|_{C^i(W^u,F)}) \\ &= \det\Big((1 + O(\mathrm{e}^{-cT}))^{\#} \Big(\frac{\pi}{T}\Big)^{N-n/2} \mathrm{e}^{2T\mathcal{F}} (1 + O(\mathrm{e}^{-cT}))|_{C^i(W^u,F)}\Big) \\ &\quad \cdot \det{}^{-1}((1 + O(\mathrm{e}^{-cT}))|_{C^i(W^u,F)}).\end{aligned} \tag{4.46}$$

From (4.45) and (4.46), one gets (3.11) immediately.

The proof of Theorem 3.2 is completed.

5 Proof of Theorem 3.3

In this section we prove Theorem 3.3.

In view of (4.36), we may restate Theorem 3.3 as follows.

Theorem 5.1 *For any $t > 0$,*

$$\lim_{T\to+\infty} \mathrm{Tr}_s[N \exp(-t\widetilde{D}_{b,T}^2)\widetilde{P}_T^{(1,+\infty)}] = 0, \tag{5.1}$$

where $\widetilde{P}_T^{(1,+\infty)} = \mathrm{Id} - \widetilde{P}_T^{[0,1]}$. Moreover, for any $d > 0$, there exist $c > 0$, $C > 0$ and $T_0 \geq 1$ such that for any $t \geq d$ and $T \geq T_0$,

$$|\mathrm{Tr}_s[N \exp(-t\widetilde{D}_{b,T}^2)\widetilde{P}_T^{(1,+\infty)}]| \leq c \exp(-Ct). \tag{5.2}$$

Set

$$c_{b,g} = 1 + 2 \max_{x\in M} \Big\{ \Big| \Big(-\frac{1}{2}c(\omega_g^F) + \frac{1}{2}c(\omega_{b,1}^F) - \frac{1}{2}\widehat{c}(\omega_{b,2}^F) \Big)(x) \Big| \Big\}. \tag{5.3}$$

By the decomposition formula (4.11) and by (4.13), one sees that for any $\lambda \in \mathbf{C}$ with $|\mathrm{Im}(\lambda)| = c_{b,g}$, $\lambda - \widetilde{D}_{b,T}$ is invertible.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$ be the union of two contours defined by

$$\Gamma_1 = \{x \pm \sqrt{-1}\, c_{b,g} :\ 2 \leq x \leq +\infty\} \cup \{2 + \sqrt{-1}\, y :\ -c_{b,g} \leq y \leq c_{b,g}\},$$
$$\Gamma_2 = \{x \pm \sqrt{-1}\, c_{b,g} :\ -\infty \leq x \leq -2\} \cup \{-2 + \sqrt{-1}\, y :\ -c_{b,g} \leq y \leq c_{b,g}\}.$$

We orient Γ anti-clockwise.

By Proposition 4.2, one sees that there exists $T_0 > 0$ such that for any $T \geq T_0$,

$$\mathrm{Tr}_s[N \exp(-t\widetilde{D}_{b,T}^2)\widetilde{P}_T^{(1,+\infty)}] = \frac{1}{2\pi\sqrt{-1}} \mathrm{Tr}_s\Big[N \int_\Gamma \frac{\mathrm{e}^{-t\lambda^2}}{\lambda - \widetilde{D}_{b,T}} \mathrm{d}\lambda\Big]. \tag{5.4}$$

Let $C > 0$ be the constant verifying (4.28). Following [3, (9.113)], for any $T \geq 1$, set

$$U_T = \Big\{\lambda \in \mathbf{C} : 1 \leq |\lambda| \leq \frac{C\sqrt{T}}{4}\Big\}. \tag{5.5}$$

From (4.26)–(4.28), (5.4) and (5.5), one can proceed as in [3, Section 9e] to show that there exists $T_1 \geq T_0$ such that for any $T \geq T_1$, $\lambda \in U_T$, $\lambda - \widetilde{D}_{b,T}$ is invertible. Moreover, for any integer $p \geq n+2$, there exists $C' > 0$ such that if $T \geq T_1$, $\lambda \in U_T$, the following analogue of [3, (9.142)] holds:

$$|\mathrm{Tr}_s[N(\lambda - \widetilde{D}_{b,T})^{-p}] - \lambda^{-p}\chi'(F)| \leq \frac{C'}{\sqrt{T}}(1 + |\lambda|)^{p+1}. \tag{5.6}$$

From (5.6), one can proceed as in [3, Sections 9g, 9h], with an obvious modification, to complete the proof of Theorem 5.1.

6 Proof of Theorem 3.5

In this section, we provide a proof of Theorem 3.5, which computes the asymptotics, as $t \to 0$, of $\mathrm{Tr}_s[N \exp(-tD_{b_T}^2)]$ for fixed $T \geq 0$.

Since $T \geq 0$ is fixed, we may well assume that $T = 0$.

One way to prove Theorem 3.5 is to apply the method developed in [11, Sections 7 and 8], which deals directly with the operator D_b^2. Here we will prove it as an application of the corresponding result for D_g^2 established in [4, Theorem 7.10]. The basic idea is very simple: we use Duhamel principle to express the heat operator of D_b^2 by using the heat operator of D_g^2, then one can use the results for D_g^2 to obtain the required results for D_b^2 (Indeed, this idea will also be used in later sections for other local index estimates as well).

Set

$$\omega^F = \omega_g^F - \omega_b^F. \tag{6.1}$$

From (6.1), one can rewrite (4.9) as

$$d^F + d_b^{F*} = d^F + d_g^{F*} + \frac{1}{2}\widehat{c}(\omega^F) - \frac{1}{2}c(\omega^F). \tag{6.2}$$

From (6.2), one sees that

$$B_{b,g} := D_b^2 - D_g^2 = (d^F + d_b^{F*})^2 - (d^F + d_g^{F*})^2 \tag{6.3}$$

is a differential operator of first order.

By Duhamel principle, one deduces that for any $t > 0$,

$$\begin{aligned} \mathrm{e}^{-tD_b^2} = {} & \mathrm{e}^{-tD_g^2} + \sum_{k=1}^{n} (-1)^k t^k \int_{\Delta_k} \mathrm{e}^{-t_1 tD_g^2} B_{b,g} \mathrm{e}^{-t_2 tD_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{k+1} tD_g^2} \mathrm{d}t_1 \cdots \mathrm{d}t_k \\ & + (-1)^{n+1} t^{n+1} \int_{\Delta_{n+1}} \mathrm{e}^{-t_1 tD_g^2} B_{b,g} \mathrm{e}^{-t_2 tD_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{n+1} tD_g^2} B_{b,g} \mathrm{e}^{-t_{n+2} tD_b^2} \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1}, \end{aligned} \tag{6.4}$$

where Δ_k, $1 \leq k \leq n+1$, is the k-simplex defined by $t_1 + \cdots + t_{k+1} = 1$, $t_1 \geq 0, \cdots, t_{k+1} \geq 0$.

Proposition 6.1 *As $t \to 0^+$, one has*

$$t^{n+1} \int_{\Delta_{n+1}} \mathrm{Tr}_s[N \mathrm{e}^{-t_1 tD_g^2} B_{b,g} \mathrm{e}^{-t_2 tD_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{n+2} tD_b^2}] \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1} \to 0. \tag{6.5}$$

Proof For any $r > 0$, let $\| \ \|_r$ denote the Schatten norm defined for any linear operator A by

$$\|A\|_r = (\mathrm{Tr}[(A^* A)^{\frac{r}{2}}])^{\frac{1}{r}}. \tag{6.6}$$

Recall the basic properties of $\| \ \|_r$ (cf. [31]) that

(i) If A is of trace class, then

$$|\mathrm{Tr}[A]| \leq \|A\|_1, \quad \|A\| \leq \|A\|_1. \tag{6.7}$$

(ii) For any $r > 0$ and compact operator A and any bounded operator B,

$$\|AB\|_r \leq \|B\| \, \|A\|_r, \quad \|BA\|_r \leq \|B\| \, \|A\|_r. \tag{6.8}$$

(iii) (Hölder inequality) For any $p,\ q,\ r > 0$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

$$\|AB\|_r \leq \|A\|_p \|B\|_q. \tag{6.9}$$

Lemma 6.1 *For any $r > 0$, one has as $t \to 0^+$ that*

$$\|\exp(-tD_b^2)\|_r = O\Big(\frac{1}{t^{\frac{n}{2}}}\Big). \tag{6.10}$$

Proof Since $B_{b,g}$ is of order one, by [14, Lemma 2.8] and [16, Lemma 1], there exists a (fixed) constant $C > 0$ such that for any $u > 0$, $t > 0$ with $ut \leq 1$,

$$\|\exp(-utD_g^2)B_{b,g}\|_{u^{-1}} \leq C(ut)^{-\frac{1}{2}}\Big(\mathrm{Tr}\Big[\exp\Big(-\frac{tD_g^2}{2}\Big)\Big]\Big)^u. \tag{6.11}$$

From (6.8), (6.9) and (6.11), one sees that for any $k \geq 1$ and $(t_1, \cdots, t_{k+1}) \in \Delta_k \setminus \{t_1 \cdots t_{k+1} = 0\}$,

$$\begin{aligned}
&\|\mathrm{e}^{-t_1 t D_g^2} B_{b,g} \mathrm{e}^{-t_2 t D_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{k+1} t D_g^2}\|_1 \\
&\leq \|\mathrm{e}^{-t_1 t D_g^2} B_{b,g}\|_{t_1^{-1}} \cdots \|\mathrm{e}^{-t_k t D_g^2} B_{b,g}\|_{t_k^{-1}} \|\mathrm{e}^{-t_{k+1} t D_g^2}\|_{t_{k+1}^{-1}} \\
&\leq C^k t^{-\frac{k}{2}} (t_1 \cdots t_k)^{-\frac{1}{2}} \big(\mathrm{Tr}\big[\mathrm{e}^{-\frac{tD_g^2}{2}}\big]\big)^{t_1 + \cdots + t_k} (\mathrm{Tr}[\mathrm{e}^{-tD_g^2}])^{t_{k+1}} \\
&\leq C^k t^{-\frac{k}{2}} (t_1 \cdots t_k)^{-\frac{1}{2}} \mathrm{Tr}\big[\mathrm{e}^{-\frac{tD_g^2}{2}}\big].
\end{aligned} \tag{6.12}$$

Thus for any $k \geq 1$, $t > 0$, one has

$$\begin{aligned}
&\Big\| t^k \int_{\Delta_k} \mathrm{e}^{-t_1 t D_g^2} B_{b,g} \mathrm{e}^{-t_2 t D_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{k+1} t D_g^2} \mathrm{d}t_1 \cdots \mathrm{d}t_k \Big\|_1 \\
&\leq (2C\sqrt{t}\,)^k \mathrm{Tr}\big[\mathrm{e}^{-\frac{tD_g^2}{2}}\big] \int_{\Delta_k} \mathrm{d}\sqrt{t_1} \cdots \mathrm{d}\sqrt{t_k} \\
&\leq (2C\sqrt{t}\,)^k \mathrm{Tr}\big[\mathrm{e}^{-\frac{tD_g^2}{2}}\big].
\end{aligned} \tag{6.13}$$

From (6.4) and (6.13), one sees that at least for $0 < t \leq \min\{1, \frac{1}{8C^2}\}$, one has

$$\mathrm{e}^{-tD_b^2} = \mathrm{e}^{-tD_g^2} + \sum_{k=1}^{+\infty} (-1)^k t^k \int_{\Delta_k} \mathrm{e}^{-t_1 t D_g^2} B_{b,g} \mathrm{e}^{-t_2 t D_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{k+1} t D_g^2} \mathrm{d}t_1 \cdots \mathrm{d}t_k. \tag{6.14}$$

From (6.6), (6.13) and (6.14), one gets (6.10) easily.

The proof of Lemma 6.1 is completed.

From (6.8)–(6.10) and proceeding as in (6.12) and (6.13), one deduces that when $t > 0$ is small enough,

$$\Big| t^{n+1} \int_{\Delta_{n+1}} \mathrm{Tr}_s[N \mathrm{e}^{-t_1 t D_g^2} B_{b,g} \mathrm{e}^{-t_2 t D_g^2} \cdots B_{b,g} \mathrm{e}^{-t_{n+2} t D_b^2}] \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1} \Big| = O(t^{\frac{1}{2}}), \tag{6.15}$$

which completes the proof of Proposition 6.1.

To compute the local index contribution to other terms in (6.4), we give the following formula for $B_{b,g}$.

Theorem 6.1 *The following identity holds:*

$$\begin{aligned} D_b^2 = D_g^2 &+ \frac{1}{2}\sum_{i,\,j=1}^{n} c(e_i)\widehat{c}(e_j)(\nabla^u_{e_i}\omega^F(e_j)) - \frac{1}{2}\sum_{\substack{i,\,j=1\\ i\neq j}}^{n} c(e_i)c(e_j)(\nabla^u_{e_i}\omega^F(e_j)) \\ &+ \frac{1}{2}\sum_{i=1}^{n}(\nabla^u_{e_i}\omega^F(e_i)) + \sum_{i=1}^{n}\omega^F(e_i)\nabla^u_{e_i} + \frac{1}{4}(\widehat{c}(\omega^F) - c(\omega^F))^2 \\ &- \frac{1}{4}[\widehat{c}(\omega^F) - c(\omega^F), \widehat{c}(\omega^F_g)]. \end{aligned} \tag{6.16}$$

Proof From (4.8) and (6.3), one has

$$\begin{aligned} D_b^2 - D_g^2 &= (d^F + d_b^{F*})^2 - (d^F + d_g^{F*})^2 \\ &= \Big(d^F + d_g^{F*} + \frac{1}{2}\widehat{c}(\omega^F) - \frac{1}{2}c(\omega^F)\Big)^2 - (d^F + d_g^{F*})^2 \\ &= \frac{1}{2}[d + d_g^{F*}, \widehat{c}(\omega^F) - c(\omega^F)] + \frac{1}{4}(\widehat{c}(\omega^F) - c(\omega^F))^2 \\ &= \frac{1}{2}\Big[\sum_{i=1}^{n} c(e_i)\nabla^u_{e_i} - \frac{1}{2}\widehat{c}(\omega^F_g), \widehat{c}(\omega^F) - c(\omega^F)\Big] + \frac{1}{4}(\widehat{c}(\omega^F) - c(\omega^F))^2. \end{aligned} \tag{6.17}$$

Now we compute (cf. [4, (4.33)])

$$\Big[\sum_{i=1}^{n} c(e_i)\nabla^u_{e_i}, \widehat{c}(\omega^F)\Big] = \sum_{i,j=1}^{n} c(e_i)\widehat{c}(e_j)(\nabla^u_{e_i}\omega^F(e_j)), \tag{6.18}$$

$$\Big[\sum_{i=1}^{n} c(e_i)\nabla^u_{e_i}, c(\omega^F)\Big] = \sum_{\substack{i,j=1\\ i\neq j}}^{n} c(e_i)c(e_j)(\nabla^u_{e_i}\omega^F(e_j)) - \sum_{i=1}^{n}(\nabla^u_{e_i}\omega^F(e_i)) - 2\omega^F(e_i)\nabla^u_{e_i}. \tag{6.19}$$

From (6.17)–(6.19), we get (6.16).

To compute the local index, let $a > 0$ be the injectivity radius of (M, g^{TM}). Take $x \in M$ and let $e_1, \cdots, e_n$ be an orthonormal basis of T_xM. We identify the open ball $B^{T_xM}(0, \frac{a}{2})$ with the open ball $B^M(x, \frac{a}{2})$ in M using geodesic coordinates. Then $y \in T_xM$, $|y| \leq \frac{a}{2}$, represents an element of $B^M(0, \frac{a}{2})$. For $y \in T_xM$, $|y| \leq \frac{a}{2}$, we identify T_yM, F_y to T_xM, F_x by parallel transport along the geodesic $t \in [0,1] \to ty$ with respect to the connections ∇^{TM}, $\nabla^{F,u}$ respectively.

Let $\Gamma^{TM,x}$, $\Gamma^{F,u,x}$ be the connection forms for ∇^{TM}, ∇^F in the considered trivialization of TM. By [1, Proposition 4.7], one has

$$\begin{aligned} \Gamma^{TM,x}_y &= \frac{1}{2}R^{TM}_x(y, \cdot) + O(|y|^2), \\ \Gamma^{F,u,x}_y &= O(|y|). \end{aligned} \tag{6.20}$$

Following [4, (4.20)], for any $t > 0$, we introduce the Getzler rescaling

$$c_t(e_i) = \frac{e_i}{t^{\frac14}} \wedge - t^{\frac14} i_{e_i}, \quad \widehat{c}_t(e_i) = \frac{\widehat{e}_i}{t^{\frac14}} \wedge + t^{\frac14} i_{\widehat{e}_i}, \quad y \to \sqrt{t}\, y, \tag{6.21}$$

where we have written $e_i^*\wedge$ in [4, (4.20)] as $e_i\wedge$ for the sake of simplicity.

From (6.3), (6.16), one verifies easily that under the Getzler rescaling G_t defined in (6.21), one has, as $t \to 0^+$,

$$G_t(tB_{b,g}) = \sqrt{t}\Big(\frac{1}{2}\sum_{i,\,j=1}^{n} e_i \wedge \widehat{e}_j(\nabla^u_{e_i}\omega^F(e_j)) - \frac{1}{2}\sum_{i,\,j=1}^{n} e_i \wedge e_j(\nabla^u_{e_i}\omega^F(e_j)) + \sum_{i=1}^{n}\omega^F(e_i)\frac{\partial}{\partial y^i} + \frac{1}{4}(\widehat{\omega^F} - \omega^F)^2 - \frac{1}{4}[\widehat{\omega^F} - \omega^F, \widehat{\omega^F_g}]\Big) + O(t). \tag{6.22}$$

On the other hand, by [4, (11.1)], one has

$$G_t(N) = \frac{1}{2\sqrt{t}}\sum_{i=1}^{n} e_i \wedge \widehat{e}_i + O(1) = \frac{1}{\sqrt{t}}L + O(1). \tag{6.23}$$

From (6.22), (6.23) and proceeding as in [4, Section 4], and [17, 18], one deduces that for any $1 < k \le n$, $(t_1, \cdots, t_{k+1}) \in \Delta_k$,

$$\lim_{t\to 0^+} t^k \mathrm{Tr}_s[N\mathrm{e}^{-t_1tD_g^2}B_{b,g}\mathrm{e}^{-t_2tD_g^2}\cdots B_{b,g}\mathrm{e}^{-t_{k+1}tD_g^2}] = 0, \tag{6.24}$$

while for $k = 1$, $0 \le t_1 \le 1$, one has

$$\begin{aligned}\lim_{t\to 0^+} t\mathrm{Tr}_s[N\mathrm{e}^{-t_1tD_g^2}B_{b,g}\mathrm{e}^{-(1-t_1)tD_g^2}] &= \lim_{t\to 0^+} t\mathrm{Tr}_s[NB_{b,g}\mathrm{e}^{-tD_g^2}] \\ &= \frac{1}{2}\int_M\int^B \mathrm{Tr}\Big[\sum_{i,\,j=1}^{n} e_i \wedge \widehat{e}_j(\nabla^u_{e_i}\omega^F(e_j)) \\ &\quad + \frac{1}{2}[\omega^F, \widehat{\omega^F_g} - \widehat{\omega^F}]\Big]L\exp\Big(-\frac{\dot{R}^{TM}}{2}\Big).\end{aligned} \tag{6.25}$$

Now it is clear that

$$\mathrm{Tr}[\omega^F, \widehat{\omega^F_g} - \widehat{\omega^F}] = 0, \tag{6.26}$$

while by [4, (4.73)] and using the notation in [4, Section 4] one has

$$\sum_{i,\,j=1}^{n} e_i \wedge \widehat{e}_j \mathrm{Tr}[(\nabla^u_{e_i}\omega^F(e_j))] = \nabla^{TM}\varphi\mathrm{Tr}[\omega^F], \tag{6.27}$$

from which, by [4, (3.10)] and [4, (3.53)], one gets

$$\begin{aligned}&\int_M\int^B \sum_{i,\,j=1}^{n} e_i \wedge \widehat{e}_j \mathrm{Tr}[(\nabla^u_{e_i}\omega^F(e_j))]L\exp\Big(-\frac{\dot{R}^{TM}}{2}\Big) \\ &= \int_M\int^B \nabla^{TM}\Big((\varphi\mathrm{Tr}[\omega^F])L\exp\Big(-\frac{\dot{R}^{TM}}{2}\Big)\Big) = 0.\end{aligned} \tag{6.28}$$

From (6.25), (6.26) and (6.28), one gets, for any $0 \le t_1 \le 1$,

$$\lim_{t\to 0^+} t\mathrm{Tr}_s[N\mathrm{e}^{-t_1tD_g^2}B_{b,g}\mathrm{e}^{-(1-t_1)tD_g^2}] = 0. \tag{6.29}$$

From (6.4), (6.5), (6.24), (6.29) and [4, Theorem 7.10], one gets (3.16).

The proof of Theorem 3.5 is completed.

Remark 6.1 The method developed in this section, combined with the method in [4, Section 4], can be used to give an alternate proof of Theorem 2.1.

7 Proof of Theorem 3.6

We first restate Theorem 3.6 as follows.

Theorem 7.1 *There exist* $0 < d \leq 1$, $C > 0$ *such that for any* $0 < t \leq d$, $0 \leq T \leq \frac{1}{t}$,

$$\Big|\mathrm{Tr}_s[N\exp(-(tD_b + T\widehat{c}(\nabla f))^2)] - \frac{1}{t}\int_M \int^B L\exp(-B_{T^2})\mathrm{rk}(F) - \frac{T}{2}\int_M \theta(F, b^F)\int^B \widehat{df}\exp(-B_{T^2}) - \frac{n}{2}\chi(F)\Big| \leq Ct. \tag{7.1}$$

Set, in view of (4.6),

$$\theta(F, g^F) = \mathrm{Tr}[\omega_g^F] = \mathrm{Tr}[(g^F)^{-1}\nabla^F g^F]. \tag{7.2}$$

By [5, Theorem A.1], one has, under the same conditions as in Theorem 7.1,

$$\Big|\mathrm{Tr}_s[N\exp(-(tD_g + T\widehat{c}(\nabla f))^2)] - \frac{1}{t}\int_M \int^B L\exp(-B_{T^2})\mathrm{rk}(F) - \frac{T}{2}\int_M \theta(F, g^F)\int^B \widehat{df}\exp(-B_{T^2}) - \frac{n}{2}\chi(F)\Big| \leq C't \tag{7.3}$$

for some constant $C' > 0$.

Thus, in order to prove (7.1), one need only to prove that under the conditions of Theorem 7.1, there exists constant $C'' > 0$ such that

$$\Big|\mathrm{Tr}_s[N\exp(-(tD_b + T\widehat{c}(\nabla f))^2)] - \mathrm{Tr}_s[N\exp(-(tD_g + T\widehat{c}(\nabla f))^2)] - \frac{T}{2}\int_M (\theta(F, b^F) - \theta(F, g^F))\int^B \widehat{df}\exp(-B_{T^2})\Big| \leq C''t. \tag{7.4}$$

For $t > 0$, $T \geq 0$, set

$$A_{b,t,T} = tD_b + T\widehat{c}(\nabla f), \quad A_{g,t,T} = tD_g + T\widehat{c}(\nabla f), \tag{7.5}$$

$$C_{t,T} = A_{b,t,T}^2 - A_{g,t,T}^2. \tag{7.6}$$

Then by (6.2) and (6.3) one has

$$\begin{aligned} C_{t,T} &= (tD_b + T\widehat{c}(\nabla f))^2 - (tD_g + T\widehat{c}(\nabla f))^2 = t^2 B_{b,g} + tT[D_b - D_g, \widehat{c}(\nabla f)] \\ &= t^2 B_{b,g} + \frac{tT}{2}[\widehat{c}(\omega^F) - c(\omega^F), \widehat{c}(\nabla f)] = t^2 B_{b,g} + tT\omega^F(\nabla f). \end{aligned} \tag{7.7}$$

By (7.6) and the Duhamel principle, one has

$$\begin{aligned} \mathrm{e}^{-A_{b,t,T}^2} &= \mathrm{e}^{-A_{g,t,T}^2} + \sum_{k=1}^n (-1)^k \int_{\Delta_k} \mathrm{e}^{-t_1 A_{g,t,T}^2} C_{t,T} \mathrm{e}^{-t_2 A_{g,t,T}^2} \cdots C_{t,T} \mathrm{e}^{-t_{k+1} A_{g,t,T}^2} \mathrm{d}t_1 \cdots \mathrm{d}t_k \\ &\quad + (-1)^{n+1} \int_{\Delta_{n+1}} \mathrm{e}^{-t_1 A_{g,t,T}^2} C_{t,T} \mathrm{e}^{-t_2 A_{g,t,T}^2} \cdots C_{t,T} \mathrm{e}^{-t_{n+2} A_{b,t,T}^2} \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1}. \end{aligned} \tag{7.8}$$

Lemma 7.1 *There exists* $C_0 > 0$ *such that for any* $T \geq 0$, $s \in \Omega^*(M, F)$, *one has*

$$\|B_{b,g}s\|_0^2 \leq C_0(\|s\|_0^2 + \|(D_g + T\widehat{c}(\nabla f))s\|_0^2). \tag{7.9}$$

Proof Since both b^F and g^F by assumption are flat near the set B of critical points of the Morse function f, by (6.3) and (6.16) we find that there exists $\delta > 0$ such that

$$B_{b,g} = 0 \tag{7.10}$$

on $\bigcup_{x\in B} B_x^M(2\delta)$, where for each $x \in B$, $B_x^M(2\delta) \subset M$ is the ball of radius 2δ centered at x.

Let $\psi \geq 0$ be a function on M such that $\mathrm{supp}(\psi) \subset M \setminus \bigcup_{x\in B} B_x^M(\delta)$ while $\psi \equiv 1$ on $M \setminus \bigcup_{x\in B} B_x^M(\frac{3}{2}\delta)$. Then by (7.10) and the standard elliptic estimate, there exists $C_1 > 0$ such that for any $s \in \Omega^*(M,F)$,

$$\|B_{b,g}s\|_0^2 = \|B_{b,g}(\psi s)\|_0^2 \leq C_1(\|\psi s\|_0^2 + \|D_g(\psi s)\|_0^2). \tag{7.11}$$

Also, by (4.18) and (4.19) it is clear that there exists $C_2 > 0$ such that for any $T \geq 0$ and $y \in M \setminus \bigcup_{x\in B} B_x^M(\delta)$,

$$T[D_g, \widehat{c}(\nabla f)] + T^2|\nabla f|^2 \geq -C_2. \tag{7.12}$$

From (7.11) and (7.12), one deduces that there exists $C_3 > 0$ such that for any $T \geq 0$ and any $s \in \Omega^*(M,F)$, one has

$$\begin{aligned}\|D_g(\psi s)\|_0^2 &\leq C_2\|\psi s\|_0^2 + \langle (D_g + T\widehat{c}(\nabla f))^2(\psi s), \psi s\rangle_g \\ &= C_2\|\psi s\|_0^2 + \|(D_g + T\widehat{c}(\nabla f))(\psi s)\|_0^2 \\ &\leq C_3(\|s\|_0^2 + \|(D_g + T\widehat{c}(\nabla f))s\|_0^2).\end{aligned} \tag{7.13}$$

From (7.11) and (7.13), one gets (7.9).

By (7.5), Lemma 7.1 and proceeding as in [14, Lemma 2.8] and [16, Lemma 1], one finds that there exists $C_4 > 0$ such that for any $t > 0,\ u > 0$ verifying $ut^2 \leq 1$ and $T \geq 0$,

$$\|\exp(-uA_{g,t,T}^2)B_{b,g}\|_{u^{-1}} \leq C_4 u^{-\frac{1}{2}} t^{-1}\Big(\mathrm{Tr}\Big[\exp\Big(-\frac{A_{g,t,T}^2}{2}\Big)\Big]\Big)^u. \tag{7.14}$$

Similarly, as

$$\omega^F = 0 \tag{7.15}$$

on $\bigcup_{x\in B} B_x^M(2\delta)$, one deduces that there exists $C_5 > 0$ such that for any $u > 0$, $t > 0$, $T \geq 0$,

$$\|\exp(-uA_{g,t,T}^2)T\omega^F(\nabla f)\|_{u^{-1}} \leq C_5 u^{-\frac{1}{2}}\Big(\mathrm{Tr}\Big[\exp\Big(-\frac{A_{g,t,T}^2}{2}\Big)\Big]\Big)^u. \tag{7.16}$$

From (6.8), (6.9), (7.7), (7.14), (7.16) and proceeding as in (6.12), one sees that for any $k \geq 1$ and $t > 0$, $t_i > 0$ for $1 \leq i \leq k+1$ with $\sum_{i=1}^{k+1} t_i = 1$, one has

$$\|\mathrm{e}^{-t_1 A_{g,t,T}^2} C_{t,T} \mathrm{e}^{-t_2 A_{g,t,T}^2} \cdots C_{t,T} \mathrm{e}^{-t_{k+1} A_{g,t,T}^2}\|_1 \leq (C_4 + C_5)^k t^k (t_1 \cdots t_{t_k})^{-\frac{1}{2}} \mathrm{Tr}[\mathrm{e}^{-\frac{A_{g,t,T}^2}{2}}]. \tag{7.17}$$

From (7.17) and proceeding as in (6.13), one has, for any $k \geq 1$ and $t > 0$,

$$\Big\|\int_{\Delta_k} \mathrm{e}^{-t_1 A^2_{g,t,T}} C_{t,T} \mathrm{e}^{-t_2 A^2_{g,t,T}} \cdots C_{t,T} \mathrm{e}^{-t_{k+1} A^2_{g,t,T}} \mathrm{d}t_1 \cdots \mathrm{d}t_k \Big\|_1 \leq (2(C_4+C_5)t)^k \mathrm{Tr}[\mathrm{e}^{-\frac{A^2_{g,t,T}}{2}}]. \quad (7.18)$$

From (7.8) and (7.18), one sees that at least for $0 < t \leq \min\{1, \frac{1}{4(C_4+C_5)}\}$ and $T \geq 0$ with $tT \leq 1$, one has

$$\mathrm{e}^{-A^2_{b,t,T}} = \mathrm{e}^{-A^2_{g,t,T}} + \sum_{k=1}^{+\infty} (-1)^k \int_{\Delta_k} \mathrm{e}^{-t_1 A^2_{g,t,T}} C_{t,T} \mathrm{e}^{-t_2 A^2_{g,t,T}} \cdots C_{t,T} \mathrm{e}^{-t_{k+1} A^2_{g,t,T}} \mathrm{d}t_1 \cdots \mathrm{d}t_k. \quad (7.19)$$

From (6.6), (7.18), (7.19) and [4, (12.34)], one finds that for any $0 < t \leq \min\{1, \frac{1}{4(C_4+C_5)}\}$ and $T \geq 0$ with $tT \leq 1$, one has that for any $r > 0$, there exists $C_6 > 0$ such that

$$\| \exp(-A^2_{b,t,T}) \|_r \leq \frac{C_6}{t^n}. \quad (7.20)$$

From (7.14), (7.16) and (7.20), one can proceed as in (6.12) and (6.15) to see that there exists $C_7 > 0$ such that for any $t > 0$ small enough and $T \in [0, \frac{1}{t}]$,

$$\Big| \int_{\Delta_{n+1}} \mathrm{Tr}_s[N \mathrm{e}^{-t_1 A^2_{g,t,T}} C_{t,T} \mathrm{e}^{-t_2 A^2_{g,t,T}} \cdots C_{t,T} \mathrm{e}^{-t_{n+2} A^2_{b,t,T}}] \mathrm{d}t_1 \cdots \mathrm{d}t_{n+1} \Big| \leq C_7 t. \quad (7.21)$$

Now for any $x \in M$, we introduce the coordinates and identification around x as in Section 6, and use the Getzler rescaling introduced in (6.21), with t there replaced by t^2 here. By using (7.7), one has

$$G_{t^2}(C_{t,T}) = G_{t^2}(t^2 B_{b,g}) + tT \omega^F(\nabla f). \quad (7.22)$$

From (6.21)–(6.29), (7.22) and proceeding as in [4, Section 13], one deduces that there exist $C_8 > 0$, $0 < d \leq 1$ such that for any $1 < k \leq n$, $0 < t \leq d$, $T \geq 0$ with $tT \leq 1$,

$$\Big| \int_{\Delta_k} \mathrm{Tr}_s[N \mathrm{e}^{-t_1 A^2_{g,t,T}} C_{t,T} \mathrm{e}^{-t_2 A^2_{g,t,T}} \cdots C_{t,T} \mathrm{e}^{-t_{k+1} A^2_{g,t,T}}] \mathrm{d}t_1 \cdots \mathrm{d}t_k \Big| \leq C_8 t, \quad (7.23)$$

while for $k = 1$ one has for any $0 < t \leq d$, $T \geq 0$ with $tT \leq 1$ and $0 \leq t_1 \leq 1$,

$$\Big| \mathrm{Tr}_s[N \mathrm{e}^{-t_1 A^2_{g,t,T}} C_{t,T} \mathrm{e}^{-(1-t_1) A^2_{g,t,T}}] - T \int_M \int^B \mathrm{Tr}[\omega^F(\nabla f)] L \exp(-B_{T^2} t) \Big| \leq C_8 t. \quad (7.24)$$

Now from [4, (3.9), (3.52)–(3.53)], (2.26), (4.2), (6.1) and (7.2), one deduces that

$$\begin{aligned}
&\int_M \int^B \mathrm{Tr}[\omega^F(\nabla f)] L \exp(-B_{T^2}) \\
&= \int_M \int^B i_{\nabla f}(\mathrm{Tr}[\omega^F]) L \exp(-B_{T^2}) \\
&= \int_M \int^B \mathrm{Tr}[\omega^F] i_{\nabla f}(L) \exp(-B_{T^2}) + \int_M \int^B \mathrm{Tr}[\omega^F] L\, i_{\nabla f}(\exp(-B_{T^2})) \\
&= \frac{1}{2} \int_M \int^B \mathrm{Tr}[\omega^F] \widehat{\nabla f} \exp(-B_{T^2}) - \frac{1}{2} \int_M \int^B \mathrm{Tr}[\omega^F] L\, \nabla^{TM}(\exp(-B_{T^2})) \\
&= \frac{1}{2} \int_M \mathrm{Tr}[\omega^F] \int^B \widehat{\nabla f} \exp(-B_{T^2}) - \frac{1}{2} \int_M \mathrm{Tr}[\omega^F] \int^B \nabla^{TM}(L(\exp(-B_{T^2}))) \\
&= \frac{1}{2} \int_M (\theta(F, g^F) - \theta(F, b^F)) \int^B \widehat{\nabla f} \exp(-B_{T^2}).
\end{aligned} \quad (7.25)$$

From (7.8), (7.21) and (7.23)–(7.25), one gets (7.4), which completes the proof of Theorem 7.1.

8 Proof of Theorem 3.7

In view of (3.18) and [5, Theorem A.2], in order to prove Theorem 3.7, we need only to prove that for any $T > 0$,

$$\lim_{t\to 0^+}(\mathrm{Tr}_s[N\exp(-A^2_{b,t,\frac{T}{t}})] - \mathrm{Tr}_s[N\exp(-A^2_{g,t,\frac{T}{t}})]) = 0. \tag{8.1}$$

First of all, by (7.18), there exists $0 < C_0 \le 1$ such that when $0 < t \le C_0$, one has

$$\Big\| \int_{\Delta_k} \mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}} \mathrm{d}t_1\cdots \mathrm{d}t_k \Big\|_1 \le \Big(\frac{t}{2C_0}\Big)^k \mathrm{Tr}[\mathrm{e}^{-\frac{A^2_{g,t,\frac{T}{t}}}{2}}]. \tag{8.2}$$

Thus we have the absolute convergent expansion formula

$$\mathrm{e}^{-A^2_{b,t,\frac{T}{t}}} - \mathrm{e}^{-A^2_{g,t,\frac{T}{t}}} = \sum_{k=1}^{+\infty}(-1)^k \int_{\Delta_k} \mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}} \mathrm{d}t_1\cdots \mathrm{d}t_k. \tag{8.3}$$

Since $T > 0$ is fixed, by [4, (12.34) and (15.22)], there exists $C_1 > 0$ such that for $0 < t \le C_0$,

$$\mathrm{Tr}[\mathrm{e}^{-\frac{A^2_{g,t,\frac{T}{t}}}{2}}] \le \frac{C_1}{t^n}. \tag{8.4}$$

From (8.2) and (8.4), one sees that

$$\sum_{k=n}^{+\infty}(-1)^k \int_{\Delta_k} \mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}} \mathrm{d}t_1\cdots \mathrm{d}t_k \tag{8.5}$$

is uniformly absolute convergent for $0 < t \le C_0$.

Let $\psi \ge 0$ be the function on M defined in Section 7. Then by definition one has

$$C_{t,\frac{T}{t}} = \psi C_{t,\frac{T}{t}} = C_{t,\frac{T}{t}}\psi = \psi C_{t,\frac{T}{t}}\psi. \tag{8.6}$$

From (8.3) one sees that for each $k \ge 1$ and any $T > 0$, $0 < t \le C_0$ and $(t_1,\cdots,t_{k+1}) \in \Delta_k$, since $\sum\limits_{i=1}^{k+1} t_i = 1$, there is $j \in [1, k+1]$ such that $t_j \ge \frac{1}{k+1}$. We here deal with the case where $j = k+1$; the other cases can be dealt with similarly.

From (6.8), (6.9), (7.7), (7.14), (7.16), (8.6) and proceeding as in (6.12), one has, for any $(t_1,\cdots,t_{k+1}) \in \Delta_k \setminus \{t_1\cdots t_{k+1} = 0\}$,

$$\begin{aligned}
&|\mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}]| \\
&= |\mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \psi \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}]| \\
&\le C_2 \|\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}}\|_{t_1^{-1}} \|\mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}}\|_{t_2^{-1}} \cdots \|\mathrm{e}^{-t_k A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}}\|_{t_k^{-1}} \|\psi \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}\|_{t_{k+1}^{-1}} \\
&\le C_3 t^k (t_1\cdots t_k)^{-\frac{1}{2}} \mathrm{Tr}[\mathrm{e}^{-\frac{A^2_{g,t,\frac{T}{t}}}{2}}] \|\psi \mathrm{e}^{-\frac{t_{k+1}}{2} A^2_{g,t,\frac{T}{t}}}\|
\end{aligned} \tag{8.7}$$

for some positive constants $C_2 > 0,\ C_3 > 0$.

From (8.4), (8.7) and the assumption that $t_{k+1} \geq \frac{1}{k+1}$, one gets

$$\begin{aligned}&\Big|\int_{\Delta_k} \mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}] \mathrm{d}t_1 \cdots \mathrm{d}t_k\Big| \\ &\leq C_4 t^{k-n} \|\psi \mathrm{e}^{-\frac{1}{2(k+1)} A^2_{g,t,\frac{T}{t}}}\| \end{aligned} \tag{8.8}$$

for some constant $C_4 > 0$.

By (8.8) one need to estimate

$$\begin{aligned}\|\psi \mathrm{e}^{-\frac{1}{2(k+1)} A^2_{g,t,\frac{T}{t}}}\| &= \|\psi \mathrm{e}^{-\frac{1}{2(k+1)} A^2_{g,t,\frac{T}{t}}} (\psi \mathrm{e}^{-\frac{1}{2(k+1)} A^2_{g,t,\frac{T}{t}}})^*\|^{\frac{1}{2}} \\ &= \|\psi \mathrm{e}^{-\frac{1}{k+1} A^2_{g,t,\frac{T}{t}}} \psi\|^{\frac{1}{2}} \\ &\leq \sqrt{\int_M \mathrm{Tr}[\psi(x) S_{\frac{t}{\sqrt{k+1}}, \frac{1}{\sqrt{k+1}}\frac{T}{t}}(x,x)\psi(x)] \mathrm{d}\,\mathrm{vol}_x}, \end{aligned} \tag{8.9}$$

where as in [4, Section 14], $S_{t,\frac{T}{t}}(x,y)$ for $x, y \in M$ denotes the kernel of $\exp(-A^2_{g,t,\frac{T}{t}})$ with respect to the Riemannian volume $\mathrm{d}\,\mathrm{vol}_{g^{TM}}$.

Now since $\mathrm{Supp}(\psi) \subset M \setminus \bigcup_{x\in B} B_x(\delta)$, by [4, Proposition 14.1], one sees that there exist $C_5, C_6 > 0$ such that

$$\int_M \mathrm{Tr}[\psi(x) S_{\frac{t}{\sqrt{k+1}}, \frac{1}{\sqrt{k+1}}\frac{T}{t}}(x,x)\psi(x)] \mathrm{d}\,\mathrm{vol}_x \leq C_5 \exp\Big(-\frac{C_6}{t^2}\Big). \tag{8.10}$$

From (8.3), (8.5), (8.8)–(8.10) and the dominate convergence, we get (8.1), which completes the proof of Theorem 3.7.

9 Proof of Theorem 3.8

In view of (3.19) and [5, Theorem A.3], in order to prove Theorem 3.8, we need only to prove that there exist $c > 0$, $C > 0$, $0 < C_0 \leq 1$ such that for any $0 < t \leq C_0$, $T \geq 1$,

$$|\mathrm{Tr}_s[N \exp(-A^2_{b,t,\frac{T}{t}})] - \mathrm{Tr}_s[N \exp(-A^2_{g,t,\frac{T}{t}})]| \leq c \exp(-CT). \tag{9.1}$$

First of all, one can choose $C_0 > 0$ small enough so that for any $0 < t \leq C_0$, $T > 0$, by (8.3), we have the absolute convergent expansion formula

$$\mathrm{e}^{-A^2_{b,t,\frac{T}{t}}} - \mathrm{e}^{-A^2_{g,t,\frac{T}{t}}} = \sum_{k=1}^{+\infty} (-1)^k \int_{\Delta_k} \mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}} \mathrm{d}t_1 \cdots \mathrm{d}t_k, \tag{9.2}$$

from which one has

$$\begin{aligned}&\mathrm{Tr}_s[N \exp(-A^2_{b,t,\frac{T}{t}})] - \mathrm{Tr}_s[N \exp(-A^2_{g,t,\frac{T}{t}})] \\ &= \sum_{k=1}^{+\infty} (-1)^k \int_{\Delta_k} \mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}] \mathrm{d}t_1 \cdots \mathrm{d}t_k. \end{aligned} \tag{9.3}$$

Thus, in order to prove (9.1), we need only to prove

$$\begin{aligned}&\sum_{k=1}^{+\infty}\Big|\int_{\Delta_k}\mathrm{Tr}_s[N\mathrm{e}^{-t_1A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\mathrm{e}^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}\mathrm{e}^{-t_{k+1}A^2_{g,t,\frac{T}{t}}}]\mathrm{d}t_1\cdots\mathrm{d}t_k\Big|\\&=\sum_{k=1}^{+\infty}\Big|\int_{\Delta_k}\mathrm{Tr}_s[N\mathrm{e}^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\mathrm{e}^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}]\mathrm{d}t_1\cdots\mathrm{d}t_k\Big|\\&\le c\exp(-CT).\end{aligned}\tag{9.4}$$

Let $\psi\ge 0$ be the function on M defined in Section 7. By (8.6), we have for any $t>0$, $T\ge 1$, $(t_1,\cdots,t_{k+1})\in\Delta_k\setminus\{t_1\cdots t_{k+1}=0\}$,

$$\begin{aligned}&\mathrm{Tr}_s[N\mathrm{e}^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\mathrm{e}^{-t_2A^2_{g,t,\frac{T}{t}}}\cdots C_{t,\frac{T}{t}}]\\&=\mathrm{Tr}_s[N\psi\mathrm{e}^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\psi\mathrm{e}^{-t_2A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\cdots\psi\mathrm{e}^{-t_kA^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}].\end{aligned}\tag{9.5}$$

We first state a refinement of the estimates (6.11), (7.14) and (7.16).

Lemma 9.1 *There exists $C_1>0$ such that for any $0<u\le 1$, $0<t\le 1$, $T\ge 1$, one has*

$$\|\psi\mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\|_{u^{-1}}\le C_1u^{-\frac12}t(\mathrm{Tr}[\mathrm{e}^{-\frac12A^2_{g,t,\frac{T}{t}}}])^u\|\psi\mathrm{e}^{-\frac{u}{4}A^2_{g,t,\frac{T}{t}}}\|.\tag{9.6}$$

Proof From (7.9), (7.15) and (7.16), one sees that there exists a constant $C_2>0$ such that

$$C^*_{t,\frac{T}{t}}C_{t,\frac{T}{t}}\le C_2t^2(1+A^2_{g,t,\frac{T}{t}}).\tag{9.7}$$

From (6.8) and (9.7), one gets

$$\begin{aligned}\|\psi\mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\|_{u^{-1}}&\le\|\psi\mathrm{e}^{-\frac{3u}{4}A^2_{g,t,\frac{T}{t}}}\|_{u^{-1}}\|\mathrm{e}^{-\frac{u}{4}A^2_{g,t,\frac{T}{t}}}C_{t,\frac{T}{t}}\|\\&\le C_3u^{-\frac12}t\|\psi\mathrm{e}^{-\frac{u}{4}A^2_{g,t,\frac{T}{t}}}\|\|\mathrm{e}^{-\frac{u}{2}A^2_{g,t,\frac{T}{t}}}\|_{u^{-1}}\Big\|\mathrm{e}^{-\frac{u}{4}A^2_{g,t,\frac{T}{t}}}\Big(1+\frac{u}{4}A^2_{g,t,\frac{T}{t}}\Big)^{\frac12}\Big\|\\&\le C_4u^{-\frac12}t(\mathrm{Tr}[\mathrm{e}^{-\frac12A^2_{g,t,\frac{T}{t}}}])^u\|\psi\mathrm{e}^{-\frac{u}{4}A^2_{g,t,\frac{T}{t}}}\|\end{aligned}\tag{9.8}$$

for some positive constants $C_3>0$, $C_4>0$.

The proof of Lemma 9.1 is completed.

Lemma 9.2 *There exist $0<c_1\le 1$, $C_5>0$, $C_6>0$ such that for any $0<u\le 1$, $0<t\le c_1$, $T\ge 1$, one has*

$$\|\psi\mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}}\|\le C_5\exp(-C_6uT).\tag{9.9}$$

Proof From (4.13) and (7.5), one has

$$A^2_{g,t,\frac{T}{t}}=t^2\Big(D_g+\frac{T}{t^2}\widehat{c}(\nabla f)\Big)^2=t^2\widetilde{D}^2_{g,\frac{T}{t^2}}.\tag{9.10}$$

Since $T\ge 1$, it is known (cf. [29]) that there exist $0<c_1\le 1$, $c_2>0$, $c_3>0$ such that for any $0<t\le c_1$, the spectrum of $\widetilde{D}^2_{g,\frac{T}{t^2}}$ splits into two parts:

$$\mathrm{Spec}(\widetilde{D}^2_{g,\frac{T}{t^2}})\subset\Big[0,\exp\Big(-\frac{c_2T}{t^2}\Big)\Big]\bigcup\Big[\frac{c_3T}{t^2},+\infty\Big).\tag{9.11}$$

For $0 < t \le c_1$ and $T \ge 1$ let $Q^{[0,1]}_{\frac{T}{t^2}}$ denote the orthogonal projection from $L^2(\Omega^*(M,F))$ to the direct sum of the eigenspaces of $\widetilde{D}^2_{g,\frac{T}{t^2}}$ corresponding to the eigenvalues lying in $[0,1]$. Let $Q^{[1,+\infty)}_{\frac{T}{t^2}} = \mathrm{Id} - Q^{[0,1]}_{\frac{T}{t^2}}$. Then it is known that (cf. [4, (7.20)]) $\mathrm{Im}(Q^{[0,1]}_{\frac{T}{t^2}})$ is a finite dimensional space.

Now we write

$$\begin{aligned}\|\psi \mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}}\| &= \|\psi \mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}}(Q^{[0,1]}_{\frac{T}{t^2}} + Q^{[1,+\infty)}_{\frac{T}{t^2}})\| \\ &\le \|\psi \mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}} Q^{[0,1]}_{\frac{T}{t^2}}\| + \|\psi \mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}} Q^{[1,+\infty)}_{\frac{T}{t^2}}\|. \end{aligned} \tag{9.12}$$

From (9.10) and (9.11), one sees that

$$\|\psi \mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}} Q^{[1,+\infty)}_{\frac{T}{t^2}}\| \le \|\mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}} Q^{[1,+\infty)}_{\frac{T}{t^2}}\| \le \exp(-c_3 uT). \tag{9.13}$$

From (9.11) one has

$$\|\psi \mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}} Q^{[0,1]}_{\frac{T}{t^2}}\| \le \|\psi(\mathrm{e}^{-uA^2_{g,t,\frac{T}{t}}} - \mathrm{Id}) Q^{[0,1]}_{\frac{T}{t^2}}\| + \|\psi Q^{[0,1]}_{\frac{T}{t^2}}\| \le \|\psi Q^{[0,1]}_{\frac{T}{t^2}}\| + C_7 \exp\Big(-\frac{c_4 T}{t^2}\Big) \tag{9.14}$$

for some positive constants $c_4 > 0$, $C_7 > 0$.

For any $T > 0$, let J_T be the map defined in (4.31) where we assume without loss of generality that the radius $4a$ there verifies $4a \le \delta$. Then one has

$$\psi J_{\frac{T}{t^2}} = 0. \tag{9.15}$$

By (9.15) and [4, Theorem 8.8] and [5, Theorem 6.7], an analogue of which has been proved in Theorem 4.1, one sees that there exist $C_8 > 0$, $c_5 > 0$ such that

$$\|\psi Q^{[0,1]}_{\frac{T}{t^2}} J_{\frac{T}{t^2}}\| \le C_8 \exp\Big(-\frac{c_5 T}{t^2}\Big). \tag{9.16}$$

From (9.16) one deduces easily that there exist $C_9 > 0$, $c_6 > 0$ such that

$$\|\psi Q^{[0,1]}_{\frac{T}{t^2}}\| \le C_9 \exp\Big(-\frac{c_6 T}{t^2}\Big). \tag{9.17}$$

From (9.12)–(9.14) and (9.17), one gets (9.9).

The proof of Lemma 9.2 is completed.

From Lemmas 9.1 and 9.2, one deduces that for any $0 < t \le \min\{C_0,\ c_1\}$, $T \ge 1$ and $(t_1, \cdots, t_{k+1}) \in \Delta_k \setminus \{t_1 \cdots t_{k+1} = 0\}$,

$$\begin{aligned} &\|\psi \mathrm{e}^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \psi \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \cdots \psi \mathrm{e}^{-t_k A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}}\|_1 \\ &\le \|\psi \mathrm{e}^{-(t_1+t_{k+1})A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}}\|_{(t_1+t_{k+1})^{-1}} \cdots \|\psi \mathrm{e}^{-t_k A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}}\|_{t_k^{-1}} \\ &\le (C_1 C_5 t)^k ((t_1+t_{k+1}) t_2 \cdots t_k)^{-\frac{1}{2}} \mathrm{Tr}[\mathrm{e}^{-\frac{1}{2}A^2_{g,t,\frac{T}{t}}}] \exp\Big(-\frac{C_6 T}{4}\Big). \end{aligned} \tag{9.18}$$

By [4, (15.22)], one sees that there exists $C_{10} > 0$ such that for any $0 < t \le \min\{C_0,\ c_1\}$, $T \ge 1$,

$$\mathrm{Tr}[\mathrm{e}^{-\frac{1}{2}A^2_{g,t,\frac{T}{t}}}] \le C_{10} \frac{T^{\frac{n}{2}}}{t^n}. \tag{9.19}$$

From (9.5), (9.18) and (9.19), one sees that there exists $C_{11} > 0$ such that for any $k \geq 1$,

$$\Big|\int_{\Delta_k} \mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}]\mathrm{d}t_1 \cdots \mathrm{d}t_k\Big|$$
$$\leq C_{11}(2\,C_1C_5 t)^k \frac{T^{\frac{n}{2}}}{t^n} \exp\Big(-\frac{C_6 T}{4}\Big), \tag{9.20}$$

from which one sees that there exist $0 < c_7 \leq \min\{C_0,\ c_1\}$, $C_{12} > 0$, $C_{13} > 0$ such that for any $0 < t \leq c_7$ and $T \geq 1$,

$$\Big|\sum_{k=n}^{+\infty}\int_{\Delta_k} \mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}]\mathrm{d}t_1 \cdots \mathrm{d}t_k\Big| \leq C_{12}\exp(-C_{13}T). \tag{9.21}$$

On the other hand, for any $1 \leq k < n$, proceeding as in (8.8), one sees that for any $0 < t \leq c_7$, $T \geq 1$,

$$\Big|\int_{\Delta_k} \mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}]\mathrm{d}t_1 \cdots \mathrm{d}t_k\Big|$$
$$\leq C_{14} t^{k-n} \|\psi \mathrm{e}^{-\frac{1}{2(k+1)} A^2_{g,t,\frac{T}{t}}}\| \tag{9.22}$$

for some constant $C_{14} > 0$.

Now since $\mathrm{Supp}(\psi) \subset M \setminus \bigcup_{x\in B} B_x(\delta)$, by [4, Proposition 15.1], one sees that there exist $C_{15},\ C_{16} > 0$ such that for any $0 < t \leq c_7$, $T \geq 1$,

$$\int_M \mathrm{Tr}[\psi(x) S_{\frac{1}{\sqrt{k+1}}t,\frac{1}{\sqrt{k+1}}\frac{T}{t}}(x,x)\psi(x)]\mathrm{d}\,\mathrm{vol}_x \leq C_{15}\exp\Big(-\frac{C_{16}T}{t^2}\Big). \tag{9.23}$$

From (8.9), (9.22) and (9.23), one sees immediately that there exist $C_{17} > 0$, $C_{18} > 0$ such that for any $1 \leq k \leq n-1$, $0 < t \leq c_7$ and $T \geq 1$,

$$\Big|\int_{\Delta_k} \mathrm{Tr}_s[N\mathrm{e}^{-t_1 A^2_{g,t,\frac{T}{t}}} C_{t,\frac{T}{t}} \mathrm{e}^{-t_2 A^2_{g,t,\frac{T}{t}}} \cdots C_{t,\frac{T}{t}} \mathrm{e}^{-t_{k+1} A^2_{g,t,\frac{T}{t}}}]\mathrm{d}t_1 \cdots \mathrm{d}t_k\Big| \leq C_{17}\mathrm{e}^{-C_{18}T}. \tag{9.24}$$

From (9.3), (9.21) and (9.24), one gets (9.1).

The proof of Theorem 3.8 is completed.

10 Euler Structure and the Burghelea-Haller Conjecture

In this section, we recall several symmetric bilinear torsions introduced by Burghelea-Haller [10, 11] which are defined by using the Euler structure introduced by Turaev [34]. We then apply our main result, Theorem 3.1, to prove a conjecture due to Burghelea and Haller [11, Conjecture 5.1].

Some applications on comparisons of various torsions are also included.

10.1 Euler and coEuler structures

Let M be a closed oriented smooth manifold, with $\dim M = n$. We assume the vanishing of the Euler-Poincaré characteristics of M, that is, $\chi(M) = 0$. The set of Euler structures with integral coefficients, $\mathrm{Eul}(M;\mathbf{Z})$, introduced by Turaev [34], is an affine version of $H_1(M;\mathbf{Z})$.

Let $X \in \Gamma(TM)$ be a non-degenerate vector field on M which means $X : M \to TM$ is transversal to the zero section. Denote its set of zeros by $\mathrm{zero}(X)$. For every $x \in \mathrm{zero}(X)$, there is a well-defined Hopf index $\mathrm{IND}_X(x) \in \{\pm 1\}$.

Any Euler structure can be represented by a pair (X, c) where $c \in C_1^{\mathrm{sing}}(M; \mathbf{Z})$ is a singular 1-chain satisfying

$$\partial c = \mathrm{e}(X) := \sum_{x \in \mathrm{zero}(X)} \mathrm{IND}_X(x) x. \tag{10.1}$$

Since $\chi(M) = 0$, the existence of c is clear.

Lemma 10.1 (cf. [11, Lemma 2.1]) *Let M be a closed smooth manifold with $\chi(M) = 0$, let $\mathfrak{e} \in \mathrm{Eul}(M; \mathbf{Z})$ be an Euler structure, and let $x_0 \in M$ be a base point. Suppose X is a non-degenerate vector field on M with $\mathrm{zero}(X) \neq \emptyset$. Then there exists a collection of paths σ_x, $\sigma_x(0) = x_0$, $\sigma_x(1) = x$, $x \in \mathrm{zero}(X)$, such that*

$$\mathfrak{e} = \Big[X, \sum_{x \in \mathrm{zero}(X)} \mathrm{IND}_X(x) \sigma_x\Big]. \tag{10.2}$$

The set of coEuler structures $\mathrm{Eul}^*(M; \mathbf{C})$ is an affine version of $H^{n-1}(M; \mathbf{C})$.

Let g^{TM} as before be a Riemannian metric on M with the associated Levi-Civita connection denoted by ∇^{TM}.

Any coEuler structure can be represented by (g^{TM}, α) for some $\alpha \in \Omega^{n-1}(M)$ such that

$$\mathrm{d}\alpha = \mathrm{e}(TM, \nabla^{TM}), \tag{10.3}$$

where $\mathrm{e}(TM, \nabla^{TM})$ is the Euler form defined in (2.27). Since $\chi(M) = 0$, the existence of α is clear.

If $[X, c] \in \mathrm{Eul}(M; \mathbf{Z})$ and $[g^{TM}, \alpha] \in \mathrm{Eul}^*(M; \mathbf{C})$, we say that $[g^{TM}, \alpha]$ is dual to $[X, c]$ if for any closed one form $\omega \in \Omega^1(M)$ which vanishes in a neighborhood of $\mathrm{zero}(X)$,

$$\int_M \omega \wedge (X^* \psi(TM, \nabla^{TM}) - \alpha) = \int_c \omega, \tag{10.4}$$

where $\psi(TM, \nabla^{TM})$ is the Mathai-Quillen current (cf. [22]) associated with g^{TM} defined in [4, Definition 3.6].

For any $[X, c] \in \mathrm{Eul}(M; \mathbf{Z})$ and g^{TM}, the existence of α is proved in [10, 11].

10.2 A proof of the Burghelea-Haller conjecture

We make the same geometric assumptions as in Section 3. We also assume $\chi(M) = 0$ as in the previous subsection.

Recall that we have the Thom-Smale cochain complex $(C^*(W^u, F), \partial)$ associated to a Morse function f and a Riemannian metric g^{TM} verifying conditions in Section 3.1.

Let $x_0 \in M$ be a fixed base point.

Let $\mathfrak{e}$ be an Euler structure.

For every critical point $x \in B$ of f choose a path σ_x with $\sigma_x(0) = x_0$ and $\sigma_x(1) = x$ so that $\Big[\nabla f, \sum_{x \in B} (-1)^{\mathrm{ind}(x)} \sigma_x\Big]$ is a representative of $\mathfrak{e}$ (cf. Lemma 10.1).

Let b_{x_0} be a nondegenerate symmetric bilinear form on the fiber F_{x_0} over x_0. For $x \in B$ define a nondegenerate symmetric bilinear form b_x on F_x by parallel transport of b_{x_0} along σ_x with respect to ∇^F. The collection of symmetric bilinear forms $\{b_x\}_{x\in B}$ defines a nondegenerate symmetric bilinear form on the Thom-Smale cochain complex $(C^*(W^u, F), \partial)$, which in turn defines an induced symmetric bilinear form on $\det H^*(C^*(W^u, F), \partial)$.

Since $\chi(M) = 0$, one sees easily that the above induced symmetric bilinear form on $\det H^*(C^*(W^u, F), \partial)$ does not depend on the choices of $\{\sigma_x\}_{x\in B}$, x_0 and b_{x_0}. It depends only on F, $\mathfrak{e}$ and ∇f. We call it the Milnor-Turaev symmetric bilinear torsion and denote it by $\tau^{\nabla f}_{F,\mathfrak{e}}$.

On the other hand, let b^F be a nondegenerate symmetric bilinear form on the flat vector bundle F.

For any $\alpha \in \Omega^{n-1}(M)$ such that $\mathrm{d}\alpha = \mathrm{e}(TM, \nabla^{TM})$, following Burghelea and Haller [10, 11], we define

$$\tau^{\mathrm{an}}_{F,g^{TM},b^F,\alpha} = b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)} \cdot \exp\Big(\int_M \theta(F, b^F) \wedge \alpha\Big) \tag{10.5}$$

and call it the Burghelea-Haller symmetric bilinear torsion.

By [11, Theorem 4.2], we know that $\tau^{\mathrm{an}}_{F,g^{TM},b^F,\alpha}$ does not depend on the choice of g^{TM} and the smooth deformations of b^F. Thus we now denote it by $\tau^{\mathrm{an}}_{F,b^F,\alpha}$.

We can now state the following equivalent version of the Burghelea-Haller conjecture (cf. [11, Conjecture 5.1]).

Theorem 10.1 *If* $\mathfrak{e} = \Big[\nabla f, \sum\limits_{x\in B} (-1)^{\mathrm{ind}(x)} \sigma_x\Big]$ *and* (g^{TM}, α) *are dual in the sense of* (10.4), *then we have*

$$P^{\det H}_{\infty}(\tau^{\mathrm{an}}_{F,b^F,\alpha}) = \tau^{\nabla f}_{F,\mathfrak{e}}. \tag{10.6}$$

Proof By [11, Theorem 4.2], we may well assume that b^F is flat near B. Then $\theta(F, b^F) = 0$ near B.

By (10.4), one has

$$\int_M \theta(F, b^F)(X^*\psi(TM, \nabla^{TM}) - \alpha) = \int_c \theta(F, b^F), \tag{10.7}$$

where $c = \sum\limits_{x\in B} (-1)^{\mathrm{ind}(x)} \sigma_x$.

From Theorem 3.1 and (10.7), noting $X = \nabla f$, we have

$$\begin{aligned} P^{\det H}_{\infty}(\tau^{\mathrm{an}}_{F,b^F,\alpha}) &= P^{\det H}_{\infty}(b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}) \cdot \exp\Big(\int_M \theta(F, b^F) \wedge \alpha\Big) \\ &= b^{\mathcal{M}}_{(M,F,b^F,-X)} \cdot \exp\Big(-\int_M \theta(F, b^F) X^*\psi(TM, \nabla^{TM})\Big) \cdot \exp\Big(\int_M \theta(F, b^F) \wedge \alpha\Big) \\ &= b^{\mathcal{M}}_{(M,F,b^F,-X)} \cdot \exp\Big(\int_M \theta(F, b^F) \wedge (\alpha - X^*\psi(TM, \nabla^{TM}))\Big) \\ &= b^{\mathcal{M}}_{(M,F,b^F,-X)} \cdot \exp\Big(-\int_c \theta(F, b^F)\Big). \end{aligned} \tag{10.8}$$

By [11, (46)], we have

$$\tau^{X}_{F,\mathfrak{e}} = b^{\mathcal{M}}_{(M,F,b^F,-X)} \cdot \exp\Big(-\int_c \theta(F, b^F)\Big). \tag{10.9}$$

By (10.8) and (10.9), we get (10.6).

The proof of Theorem 10.1 is completed.

Remark 10.1 Some non-trivial special cases of Theorem 10.1 has already been proved in [11]. Moreover Braverman-Kappeler proved in [9] that when dim M is odd, (10.6) holds up to a numerical factor of absolute value one. This later result was generalized to the case of even dimensional manifolds in the second version of [11].

As was pointed out in [11], the following result is a direct consequence of Theorem 10.1.

Corollary 10.1 *The Burghelea-Haller torsion* $\tau^{\mathrm{an}}_{F,b^F,\alpha}$ *does not depend on* b^F *and the representative* α.

10.3 Comparison of $b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}$ with the usual Ray-Singer torsion

We still assume $\chi(M)=0$.

Let g^F be a Hermitian metric on F. Then one can construct the Ray-Singer analytic torsion as an inner product on $\det H^*(M,F)$, or equivalently as a metric on the determinant line (cf. [4]). We denote the resulting inner product by $b^{\mathrm{RS}}_{(M,F,g^{TM},g^F)}$.

In this section, we prove the following comparison result between $b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}$ and $b^{\mathrm{RS}}_{(M,F,g^{TM},g^F)}$, which is also a consequence of [8, (5.13)] and [9, Theorem 1.4].

It is clear that the absolute value of the ratio of the symmetric bilinear form and the inner product is well-defined.

Proposition 10.1 *If* $\dim M$ *is odd, then the following identity holds:*

$$\Big|\frac{b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}}{b^{\mathrm{RS}}_{(M,F,g^{TM},g^F)}}\Big| = 1. \tag{10.10}$$

Proof Let $\mathfrak{e}$ be an Euler class associated to ∇f in the sense of Lemma 10.1. Let $T^{\nabla f}_{F,\mathfrak{e}}$ be the Redemeister inner product induced from the Euler structure $\mathfrak{e}$. Then one verifies easily that

$$\Big|\frac{\tau^{\nabla f}_{F,\mathfrak{e}}}{T^{\nabla f}_{F,\mathfrak{e}}}\Big| = 1. \tag{10.11}$$

Let $[g^{TM},\alpha]$, $\alpha\in\Omega^{n-1}(M)$, be dual to the Euler structure $\mathfrak{e}$ in the sense of (10.4).

From (10.5), (10.6), (10.11) and [4, Theorem 0.2], one deduces that

$$\Big|\frac{b^{\mathrm{RS}}_{(M,F,g^{TM},b^F)}}{b^{\mathrm{RS}}_{(M,F,g^{TM},g^F)}}\Big| = \Big|\exp\Big(\int_M (\theta(F,g^F)-\theta(F,b^F))\wedge\alpha\Big)\Big|. \tag{10.12}$$

Note that the left-hand side of (10.12) does not depend on the Euler structure $\mathfrak{e}$.

By choosing different Euler structures, one sees that for any real closed form $\gamma\in\Omega^{n-1}(M)$ whose image in $H^*(M,\mathbf{R})$ lies in $H^*(M,\mathbf{Z})$,

$$\mathrm{Re}\Big(\int_M (\theta(F,g^F)-\theta(F,b^F))\wedge\gamma\Big) = 0. \tag{10.13}$$

Then it is easy to see that (10.13) should also hold for any real closed form $\gamma\in\Omega^{n-1}(M)$. As a consequence, we get the following equality in $H^1(M,\mathbf{R})$

$$\mathrm{Re}[\theta(F,b^F)] = [\theta(F,g^F)]. \tag{10.14}$$

Since that $\dim M$ is odd implies $\mathrm{e}(TM, \nabla^{TM}) = 0$, by (10.3), (10.12) and (10.14), we get (10.10).

The proof of Proposition 10.1 is completed.

Remark 10.2 In the general case that $\dim M$ need not be odd, by the consideration in the proof of [11, Theorem 5.9], one sees that there exists an anti-linear involution $J^F : F \to F$ such that

$$(J^F)^2 = \mathrm{Id}_F, \quad b^F(J^F u, v) = \overline{b^F(u, J^F v)}, \quad b^F(u, J^F u) \geq 0, \quad u, v \in F. \tag{10.15}$$

Then

$$g^F(u, v) := b^F(u, J^F v), \quad u, v \in F, \tag{10.16}$$

defines a Hermitian metric on F. From (10.16), we get

$$(g^F)^{-1}\nabla^F g^F = (J^F)^{-1}((b^F)^{-1}\nabla^F b^F)J^F + (J^F)^{-1}\nabla^F J^F. \tag{10.17}$$

From (10.17), one gets

$$\theta(F, b^F) = \theta(F, g^F) - \mathrm{Tr}[(J^F)^{-1}\nabla^F J^F], \tag{10.18}$$

from which we get

$$\mathrm{Re}(\theta(F, b^F)) = \theta(F, g^F), \tag{10.19}$$

which provides a direct proof of (10.14).

References

[1] Atiyah, M. F., Bott, R. and Patodi, V. K., On the heat equation and the index theorem, *Invent. Math.*, **19**, 1973, 279–330.

[2] Bismut, J.-M., Gillet, H. and Soulé, C., Analytic torsions and holomorphic determinant line bundles I, *Commun. Math. Phys.*, **115**, 1988, 49–78.

[3] Bismut, J.-M. and Lebeau, G., Complex immersions and Quillen metrics, *Publ. Math. IHES.*, **74**, 1991, 1–297.

[4] Bismut, J.-M. and Zhang, W., An Extension of a Theorem by Cheeger and Müller, Astérisque, **205**, Soc. Math. France, Paris, 1992.

[5] Bismut, J.-M. and Zhang, W., Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle, *Geom. Funct. Anal.*, **4**, 1994, 136–212.

[6] Braverman, M. and Kappeler, T., A refinement of the Ray-Singer torsion, *C. R. Acad. Sci. Paris*, **341**, 2005, 497–502.

[7] Braverman, M. and Kappeler, T., Refined analytic torsion as an element of the determinant line, *Geom. and Topol.*, **11**, 2007, 139–213.

[8] Braverman, M. and Kappeler, T., Ray-Singer type theorem for the refined analytic torsion, *J. Funct. Anal.*, **243**, 2007, 232–256.

[9] Braverman, M. and Kappeler, T., Comparison of the refined analytic and the Burghelea-Haller torsions, *Ann. Inst. Fourier.*, **57**, 2007, 2361–2387.

[10] Burghelea, D. and Haller, S., Torsion, as function on the space of representations, C^*-algebras and Elliptic Theory II (Trends in Mathematics), D. Burghelea, R. Melrose, A. S. Mishchenko, et al (eds.), Birkhäuser, Basel, 2008, 41–66.

[11] Burghelea, D. and Haller, S., Complex valued Ray-Singer torsion, *J. Funct. Anal.*, **248**, 2007, 27–78.

[12] Burghelea, D. and Haller, S., Complex valued Ray-Singer torsion II, preprint. math.DG/0610875

[13] Cheeger, J., Analytic torsion and the heat equation, *Ann. of Math.*, **109**, 1979, 259–332.

[14] Chern, S. and Hu, X., Equivariant Chern character for the invariant Dirac operator, *Michigan Math. J.*, **44**, 1997, 451–473.

[15] Farber, M. and Turaev, V., Poincaré-Reidemeister metric, Euler structures and torsion, *J. Reine Angew. Math.*, **520**, 2000, 195–225.

[16] Feng, H., A remark on the noncommutative Chern character (in Chinese), *Acta Math. Sinica*, **46**, 2003, 57–64.

[17] Getzler, E., Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem, *Commun. Math. Phys.*, **92**, 1983, 163–178.

[18] Getzler, E., A short proof the local Atiyah-Singer index theorem, *Topology*, **25**, 1986, 111–117.

[19] Helffer, B. and Sjöstrand, J., Puis multiples en mécanique semi-classique IV: Etude du complexe de Witten, *Comm. PDE*, **10**, 1985, 245–340.

[20] Knudson, F. F. and Mumford, D., The projectivity of the moduli space of stable curves I: Preliminaries on "det" and "div", *Math. Scand.*, **39**, 1976, 19–55.

[21] Laudenbach, F., On the Thom-Smale complex, Appendix in "An Extension of a Theorem by Cheeger and Müller", J.-M. Bismut and W. Zhang (eds.), Astérisque, **205**, Paris, 1992.

[22] Mathai, V. and Quillen, D., Superconnections, Thom classes, and equivariant differential forms, *Topology*, **25**, 1986, 85–110.

[23] Milnor, J., Whitehead torsion, *Bull. Amer. Math. Soc.*, **72**, 1966, 358–426.

[24] Müller, W., Analytic torsion and the R-torsion of Riemannian manifolds, *Adv. in Math.*, **28**, 1978, 233–305.

[25] Müller, W., Analytic torsion and the R-torsion for unimodular representations, *J. Amer. Math. Soc.*, **6**, 1993, 721–753.

[26] Quillen, D., Determinants of Cauchy-Riemann operators over a Riemann surface, *Funct. Anal. Appl.*, **14**, 1985, 31–34.

[27] Quillen, D., Superconnections and the Chern character, *Topology*, **24**, 1985, 89–95.

[28] Ray, D. B. and Singer, I. M., R-torsion and the Laplacian on Riemannian manifolds, *Adv. in Math.*, **7**, 1971, 145–210.

[29] Shubin, M. A., Semiclassical asymptotics on covering manifolds and Morse inequalities, *Geom. Funct. Anals.*, **6**, 1996, 370–409.

[30] Shubin, M. A., Pseudodifferential Operators and Spectral Theory, Springer-Verlag, Berlin, 2001.

[31] Simon, B., Trace Ideals and Their Applications, Lecture Notes Series, **35**, London Math. Soc., Cambridge University Press, Cambridge, 1979.

[32] Smale, S., On gradient dynamical systems, *Ann. of Math.*, **74**, 1961, 199–206.

[33] Su, G. and Zhang, W., A Cheeger-Müller theorem for symmetric bilinear torsions, preprint. math. DG/0610577

[34] Turaev, V., Euler structures, nonsingular vector fields, and Reidemeister-type torsion, *Math. USSR-Izv.*, **34**, 1990, 627–662.

[35] Witten, E., Supersymmetry and Morse theory, *J. Diff. Geom.*, **17**, 1982, 661–692.

[36] Zhang, W., Lectures on Chern-Weil Theory and Witten Deformations, Nankai Tracts in Mathematics, Vol. 4, World Scientific, Singapore, 2001.