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Coincidence Properties for Maps from the Torus to the Klein Bottle

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Abstract The authors study the coincidence theory for pairs of maps from the Torus to the Klein bottle. Reidemeister classes and the Nielsen number are computed, and it is shown that any given pair of maps satisfies the Wecken property. The 1-parameter Wecken property is studied and a partial negative answer is derived. That is for all pairs of coincidence free maps a countable family of pairs of maps in the homotopy class is constructed such that no two members may be joined by a coincidence free homotopy.

Keywords Coincidence point, Nielsen number, Wecken property
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1 Introduction

In this paper, we study various coincidence point properties for pairs of maps from the 2-dimensional torus into the Klein bottle. Much of this work is an extension of the ideas developed in [8, 9] for self-coincidences of maps on the torus and on the Klein bottle. As was done in [9], we try to understand how Nielsen theoretic properties for these maps compare with those of maps from the torus into itself. One objective of this work is to present a computation of the Nielsen theoretic data for pairs of maps from the torus, denoted by T in this paper, to the Klein bottle, denoted by K. We compute the Reidemeister classes, characterize defective classes and give a formula for the Nielsen coincidence number. This formula appears as Theorem 3.1 in Section 3 of the paper.

Given the formula for the Nielsen coincidence number we then verify that the Wecken property holds. That is, for each homotopy class of pairs of maps we produce a representative that realizes the Nielsen coincidence number. The result is stated as Theorem 4.1 in Section 4. This result has been obtained independently in [2], using different methods, where they study completely maps from torus to an arbitrary closed surface. Let us also point out that every map $f : T \to K$ has the Wecken property with respect to roots. This follows from a result of Kneser [12], that the geometric degree is equal to the absolute degree, and the absolute degree is the root Nielsen number for maps from the torus to the Klein bottle [3, Theorem 2].

The second direction of this paper is a study of the 1-parameter Wecken problem for coincidences from the torus to the Klein bottle. This problem, following the work in [9], can

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be described as follows: Can one find a pair H, G of homotopies from f_1 to f_2 and g_1 to g_2 respectively, such that $\#\mathrm{Coin}(H(\cdot\,,t), G(\cdot\,,t)) = \mathrm{MC}[f_1, g_1]$ for all $t \in [0,1]$? We refer to this as the 1-parameter minimal coincidence problem or simply the minimal coincidence problem when the context is clear. We note that this problem, when specialized to the case where the domain space and target are the same, both g_1 and g_2 are the identity and the homotopy G remains constant is known as the 1-parameter fixed point problem and has been considered in [14, 11, 4, 7, 6]. Results in the last citation are partially generalized to the coincidence setting in [10].

In [7–9] the authors study the 1-parameter minimal coincidence problem, and also the related restricted minimal coincidence problem in the setting of surface mappings. That is, when we have $g_1 = g_2$ and all homotopies under consideration keep the second leg fixed at the map g_1. In [9] we see that this problem is much more difficult for self-coincidences on the Klein bottle than it was for the torus. The fact that the torus has a multiplicative structure was crucial in resolving the minimal coincidence problem, and also implies that the restricted problem is equivalent. For the Klein bottle we needed also to take into account liftings of maps to the torus and were able to solve the restricted minimal coincidence problem. This difficultly will persist when we consider coincidences from T to K in this paper.

The results on the 1-parameter problem presented in this work is a solution to the restricted minimal coincidence problem for pairs of maps from T to K under the assumption that each pair is coincidence free. The result is stated in Theorem 5.1 of Section 5.

A brief summary of the structure of the four sections in this paper is as follows. In Section 2 we classify homotopy classes of pairs of maps (base point homotopies and free homotopy, respectively) from the torus T to the Klein bottle K. In Section 3 we compute the Reidemeister classes and the Nielsen coincidence number for all homotopy classes. The formula given for the Nielsen number is for a class of maps which are defined in Section 2. The proof of Lemma 2.2 shows how to transform an arbitrarily given map into this class to obtain the value of the Nielsen number in general. In Section 4 we establish the Wecken property for all coincidences from T to K. In Section 5 we consider the 1-parameter Wecken problem and show that the restricted problem has a negative solution in the case when the Nielsen coincidence number is equal to zero.

2 A Classification of Maps, and Pairs of Maps, from the Torus to the Klein Bottle

The purpose of this section is to give a characterization for the induced map on fundamental group for all maps from the 2-dimensional torus, denoted by T, into the Klein bottle K. Let $\pi_1(K, y_0)$ be the group having the presentation $\pi_1(K, y_0) = \langle \alpha, \beta \mid R = \alpha\beta\alpha\beta^{-1} \rangle$ and let a, b denote generators for the free Abelian group $\pi_1(T, x_0)$.

Lemma 2.1 *The homomorphism induced in the fundamental group by a map $f : T \to K$ is of the form $f_\#(a) = \alpha^{r_1}\beta^{s_1}$, $f_\#(b) = \alpha^{r_2}\beta^{s_2}$, where one of the conditions below holds:*

(I) *s_1 and s_2 are both even,*

(II) (a) *s_1 odd, s_2 even and $r_2 = 0$, or* (b) *s_1 even, s_2 odd and $r_1 = 0$, or* (c) *s_1, s_2 odd* *and* $r_1 = r_2$.

Proof Since a commutes with b we must have that $f_\#(a)$ commutes with $f_\#(b)$. Using the relation R we obtain that

$$f_\#(a)f_\#(b) = \alpha^{r_1}\beta^{s_1}\alpha^{r_2}\beta^{s_2} = \alpha^{r_1+(-1)^{s_1}r_2}\beta^{s_1+s_2},$$
$$f_\#(b)f_\#(a) = \alpha^{r_2}\beta^{s_2}\alpha^{r_1}\beta^{s_1} = \alpha^{r_2+(-1)^{s_2}r_1}\beta^{s_1+s_2}.$$

Therefore we must have

$$r_1 + (-1)^{s_1}r_2 = r_2 + (-1)^{s_2}r_1$$

or

$$(1-(-1)^{s_2})r_1 = (1-(-1)^{s_1})r_2.$$

The result follows by a routine analysis of this last equation.

With $f_\#$ as in Lemma 2.1 we say that the map f is of type I if $f_\#$ satisfies case (I) and is of type II(a), type II(b), type II(c) when $f_\#$ satisfies, respectively, case (II)a, (II)b, (II)c. By type II we mean one of the last three types.

Lemma 2.1 gives an algebraic description of the set of based point preserving homotopy classes of maps from T to K. As a consequence we can also deduce the following corollary for the set of free homotopy classes. This will be useful to establish the Wecken property for maps from T to K.

Corollary 2.1 *The set of base point preserving homotopy classes of maps from T to K is in one-to-one correspondence with the homomorphism given in Lemma* 2.1. *The set of free homotopy classes of maps are classified as follows:*

(a) *Maps of type* I *are in one-to-one correspondence with homomorphisms where* $r_1 \geq 0$.

(b) *Maps of type* II(a) *are in one-to-one correspondence with homomorphisms where* r_1 *is equal to* 0 *or* 1.

(c) *Maps of type* II(b) *are in one-to-one correspondence with homomorphisms where* r_2 *is equal to* 0 *or* 1.

(d) *Maps of type* II(c) *are in one-to-one correspondence with homomorphisms where* $r = r_1 = r_2$ *is equal to* 0 *or* 1.

Proof The correspondence with the set of base point homotopy class of maps is clear, since the spaces involved are Eilenberg-MacLane spaces $K(\pi,1)'s$. The set of free homotopy classes of maps are in one-to-one correspondence with the set of conjugacy classes of the homomorphisms. For case (a) we use conjugation by β to obtain the result. The other three cases are similar. Using the relation $\alpha\beta\alpha\beta^{-1} = 1$ on $\pi_1(K)$ we obtain $\alpha^{-1}\beta^{2s+1} = \beta^{2s+1}\alpha$ and hence $\alpha^{-1}\alpha^r\beta^{2s+1}\alpha = \alpha^{r-2}\beta^{2s+1}$. This implies that after we conjugate the homomorphism by a suitable power of α, the exponent of α becomes either zero or one, and the result follows.

Now we will consider pair of maps $f, g : T \to K$. Our goal is to study the coincidence theory of the homotopy class of the pairs of maps. To reduce the amount of work in some arguments it will be useful to divide the set of all pairs (f, g) into as few cases as possible. By Lemma 2.1 each map is of type I, II(a), II(b) or II(c), and so we have 16 possible cases. By observing that $\mathrm{coin}(f,g) = \mathrm{coin}(g,f)$, for the Wecken problem it is clear that the order of the functions is not relevant. So we can reduce the number of cases to 10. Still a further reduction can be done. The Nielsen theory (meaning the Reidemeister classes, Nielsen number, Wecken property) is completely determined for $(h_2 \circ f \circ h_1, h_2 \circ g \circ h_1)$ if we know the answer for (f,g), where $h_1 \in \mathrm{Homeo}(T)$ and $h_2 \in \mathrm{Homeo}(K)$. The following lemma is to be used later to reduce the number of cases which we need to consider.

Lemma 2.2 *In order to decide if the Wecken property holds for maps from T to K it suffices to consider pairs of maps (f,g) in one of the following four cases:*

(1) *Both maps are of type* I*,*

(2) *One map is of type* I *and the other is of type* II(a)*,*

(3) *Both maps are of type* II(a)*,*

(4) *One map is of type* II(a) *and the other is of type* II(b)*.*

Furthermore, we can assume that a map f of type II(a) *satisfies $r_1 = 0$, and that in the cases* (3) *and* (4) *that the other map g satisfies $r_i \in \{0,1\}$ as in Corollary* 2.1*.*

Proof If we compose the homeomorphism of the torus given by the matrix

$$\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

with a map of type II(a) we obtain a map of type II(c). Similarly, compose with a map of type II(c) to obtain a map of type II(b), and with a map of type II(b) to obtain a map of type II(a). Type I remains unchanged by composition with such a homeomorphism, so we can reduce the 10 cases to the four cases claimed. To prove the furthermore part we first observe that composition by the above torus homeomorphism preserves the property given in Corollary 2.1 that, for maps of type II, the value of r_i is equal to 0 or 1. Now let f be of type II(a) with $f_\#(a) = \alpha^{r_1}\beta^{2s_1+1}$, $f_\#(b) = \beta^{2s_2}$. If we denote $w = \alpha^{r_1}\beta$ then using the relations for the Klein bottle we see that $f_\#(a) = (w)^{2s_1+1}$, $f_\#(b) = (w)^{2s_2}$. Let $h : K \to K$ be the homeomorphism of the Klein bottle which induces the homomorphism $h_\#(\alpha) = \alpha$, $h_\#(\beta) = \alpha^{-r_1}\beta$. Then the composition $h \circ f$ induces in the fundamental group the homomorphism $(h \circ f)_\#(a) = \beta^{2s_1+1}$, $(h \circ f)_\#(b) = \beta^{2s_2}$. Thus, if all maps of type II(a) have $r_1 = 1$, then apply h to convert to 0. Finally, note that a map of type II(b) may now have $r_2 = -1$, but conjugation by α corrects to 1. So the result follows.

Remark 2.1 For the 1-parameter problems (both the minimal coincidence problem and the restricted minimal coincidence problem) it is not so clear how to reduce the question in terms of homotopy classes of the pair. This analysis will be done later in Section 5.

Our strategy will be to study maps from $T \to K$ in terms of certain maps from $T \to T$. Let $p : T \to K$ be an orientable covering of K, which is a 2-fold covering. Denote by θ the deck transformation which corresponds to the element $\beta \in \pi_1(K, y_0)$. The maps of type I are also known as orientation true maps (see [13]). They are the ones which lift to a map $\widetilde{f} : T \to T$ with respect to the orientable covering $p : T \to K$. The maps of type II do not lift, but for any such map, denoted by f, there is a two-fold covering $\phi : T \to T$ such that $f \circ \phi$ admits a lift. Each two-fold covering corresponds to a subgroup of index two of $\pi_1(T)$, and there are exactly three possibilities for such coverings. They correspond to the subgroups of $\pi_1(T)$ given by $G_1 = \langle a^2, b\rangle$, $G_2 = \langle a, b^2\rangle$ and $G_3 = \langle a^2, ab\rangle$. These subgroups are the kernel of the three possible non trivial homomorphisms from $\pi_1(T) \to Z_2$. We use the coverings corresponding to these subgroups G_1, G_2 and G_3 to lift the composite of the covering with the map f in the three cases; s_1 odd and s_2 even, s_1 even and s_2 odd, and s_1, s_2 odd, respectively. We denote by $p_1 : T \to T$ the 2-fold covering which corresponds to the subgroup G_1. Finally we will make use of another finite covering. Consider the 4-fold covering $p_2 : T \to T$ which corresponds to the subgroup $\langle a^2, b^2\rangle \subset Z \oplus Z$. Denote by γ_a, γ_b the deck transformations which correspond to the elements $a, b \in \pi_1(T, x_0)$, respectively.

Remark 2.2 There are homeomorphisms $h_1, h_2 : T \to T$ such that $h_{i\#}(G_1) = G_{i+1}$, $i = 1, 2$. Namely let $h_1(a) = b$, $h_1(b) = a$ and $h_2(a) = a$, $h_2(b) = ab$. Therefore there are homeomorphisms $h : T \to T$ which takes G_i to G_j for any $i, j \in \{1, 2, 3\}$.

3 Coincidence Reidemeister Classes and the Nielsen Number of a Pair of Maps

In this section, we compute the coincidence Reidemeister classes, the coincidence Reidemeister number and the coincidence Nielsen number $N(f, g)$ of a pair of maps $f, g : T \to K$. Then some Nielsen theoretic properties can be compared with those of maps from the torus into itself. We will establish that when $R(f, g)$ is finite, then the Nielsen number is equal to the Reidemeister number and all Nielsen coincidence classes have index either $+1$ or -1. The converse is not true, namely we can have Reidemeister number infinite but the Nielsen number different from zero. This is something that does not happen for self coincidences of the torus. For simplicity of presentation we only present the calculations for pairs of maps as given by Lemma 2.2. There is no theoretical difficulty to write the formulas given below for any of the 16 cases for pairs of maps.

In order to study Reidemeister classes we need to introduce some new terminology for Reidemeister classes for coincidences which appears for maps between manifolds where at least one of them is not orientable. For fixed points this notion was considered in [5].

Definition 3.1 *A Reidemeister class* $[\alpha]$ *is said to be defective, if there exists a loop* γ *in the domain such that* $\alpha = g_\#(\gamma)\,\alpha\,(f_\#(\gamma)^{-1})$ *and for some* $\alpha \in [\alpha]$, $\mathrm{sign}(\gamma) \cdot \mathrm{sign}(f_\#(\gamma)) = -1$.

Remark 3.1 Since T is orientable $\mathrm{sign}(\gamma) = 1$, so we reduce to consideration of loops whose image under f is orientation reversing. On the other hand from the equality $\alpha = g_\#(\gamma)\alpha(f_\#(\gamma)^{-1})$ it follows that $\mathrm{sign}(g_\#(\gamma)) = \mathrm{sign}((f_\#(\gamma)))$. If one of the maps is of type I, the image of γ by the other map has also sign 1, therefore α cannot be defective. Thus, for a pair of maps of type I, or one of type I and the other of type II(a), there are no defective Reidemeister classes. These are cases (1) and (2) given in Lemma 2.2. In the remaining two cases in Lemma 2.2 there is the possibility of defective Reidemeister classes. The details will be addressed below.

In order to compute the Reidemeister classes in the four cases given in Lemma 2.2 we introduce the following notation:

Case (1) $f_\#(a) = \alpha^{r_1}\beta^{2s_1}$, $f_\#(b) = \alpha^{r_2}\beta^{2s_2}$ and $g_\#(a) = \alpha^{t_1}\beta^{2v_1}$, $g_\#(b) = \alpha^{t_2}\beta^{2v_2}$.
Case (2) $f_\#(a) = \beta^{2s_1+1}$, $f_\#(b) = \beta^{2s_2}$ and $g_\#(a) = \alpha^{t_1}\beta^{2v_1}$, $g_\#(b) = \alpha^{t_2}\beta^{2v_2}$.
Case (3) $f_\#(a) = \beta^{2s_1+1}$, $f_\#(b) = \beta^{2s_2}$ and $g_\#(a) = \alpha^{t_1}\beta^{2v_1+1}$, $g_\#(b) = \beta^{2v_2}$.
Case (4) $f_\#(a) = \beta^{2s_1+1}$, $f_\#(b) = \beta^{2s_2}$ and $g_\#(a) = \beta^{2v_1}$, $g_\#(b) = \alpha^{t_2}\beta^{2v_2+1}$.

Remark 3.2 These four cases will also be used in the proof of the Wecken property given in Section 4. But, as a result of Lemma 2.2, we will have the further restriction that t_i is either 0 or 1 in Cases (3) and (4).

We now proceed with the characterization of the Reidemeister classes in Cases (1) and (2). We define two subsets $H, H_1 \subset \pi_1(K)$, and as a notation vH (resp. vH_1) refers to a set of the form vw where $w \in H$ (resp. H_1).

For Case (1), let

$$H = \{\alpha^{(t_1-r_1)k+(t_2-r_2)l}\beta^{(2v_1-2s_1)k+(2v_2-2s_2)l} \mid k, l \in Z\},$$
$$H_1 = \{\alpha^{(-t_1-r_1)k+(-t_2-r_2)l}\beta^{(2v_1-2s_1)k+(2v_2-2s_2)l} \mid k, l \in Z\}.$$

It is not difficult to see that H and H_1 are subgroups. For case (2), we similarly define subsets $H, H_1 \subset \pi_1(K)$. Let

$$H = \{\alpha^{t_1k+t_2l+\epsilon_k r_1}\beta^{(2v_1-2s_1-1)k+(2v_2-2s_2)l} \mid k, l \in Z\},$$
$$H_1 = \{\alpha^{-t_1k-t_2l+\epsilon_k r_1}\beta^{(2v_1-2s_1-1)k+(2v_2-2s_2)l} \mid k, l \in Z\},$$

where ϵ_k is zero if k is even and 1 if k is odd.

Proposition 3.1 *For a pair of maps $f, g : T \to K$ the Reidemeister class of a generic element $\alpha^m\beta^n$ is given as follows:*
Case (1) *$\alpha^m\beta^n H$ if n is even, and $\alpha^m\beta^n H_1$ if n is odd.*
Case (2) *$\alpha^m\beta^n H$ if n is even, and $\alpha^m\beta^n H_1$ if n is odd.*

Proof In Case (1), by definition, a generic element of the Reidemeister class has the form

$$(\alpha^{t_1}\beta^{2v_1})^k(\alpha^{t_2}\beta^{2v_2})^l\alpha^m\beta^n(\alpha^{r_2}\beta^{2s_2})^{-l}(\alpha^{r_1}\beta^{2s_1})^{-k},$$

which by straightforward calculation, using the relation $\alpha\beta = \beta\alpha^{-1}$, is identified with $\alpha^m\beta^n H$ when n is even, and with $\alpha^m\beta^n H_1$ when n is odd.

In Case (2), a generic element of the Reidemeister class is of the form

$$(\alpha^{t_1}\beta^{2v_1})^k(\alpha^{t_2}\beta^{2v_2})^l\alpha^m\beta^n(\beta^{2s_2})^{-l}(\beta^{2s_1+1})^{-k}$$

and the same calculation reduces to $\alpha^m\beta^n H$ when n is even, and $\alpha^m\beta^n H_1$ when n is odd.

Remark 3.3 In Case (1) we can have $H \subset Z \oplus Z$ a subgroup of finite index but not H_1. This will imply that the number of Reidemeister classes is infinite but the Nielsen number is not zero. This does not happen for self-maps of the torus.

Now let us consider a pair of maps in the two Cases (3) and (4). For a given integer n, let $|n|_2$ denote the highest integer ℓ such that 2^ℓ divides $|n|$.

Proposition 3.2 *For a pair of maps $f, g : T \to K$ the Reidemeister class of a generic element $\alpha^m\beta^n$ and the defective classes are given as follows:*

Case (3) *$\alpha^{m'}\beta^{n'}$ is in the same class as the element $\alpha^m\beta^n$ if and only if either $m' = m$ and $n' - n = 2p(v_1 - s_1) + 2q(v_2 - s_2)$, where p is even and q is an arbitrary integer, or $m' = t_1 - m$ and $n' - n = 2p(v_1 - s_1) + 2q(v_2 - s_2)$, where p is odd and q arbitrary. Defective classes exist if and only if t_1 is even and $|v_1 - s_1|_2 \geq |v_2 - s_2|_2$, and the defective classes are those given by the elements $\alpha^{\frac{t_1}{2}}\beta^n$).*

Case (4) *$\alpha^{m'}\beta^{n'}$ is in the same class as the element $\alpha^m\beta^n$ if and only if either $m' = m$ and $n' - n = p(2v_1 - 2s_1 - 1) + q(2v_2 - 2s_2 + 1)$, where p is arbitrary and q is even, or $m' = t_2 - m$ and $n' - n = p(2v_1 - 2s_1 - 1) + q(2v_2 - 2s_2 + 1)$, where p is arbitrary and q is odd. In this case defective classes exist if and only if t_2 is even. The defective Reidemeister classes are those given by the elements $\alpha^{\frac{t_1}{2}}\beta^n$. Further, two such elements (given by n and n') are in the same defective class if and only if n' is congruent to n mod the* g.c.d. *of $2v_1 - 2s_1 - 1$ and $2v_2 - 2s_2 + 1$.*

Proof The two cases are similar. We give the argument for Case (3) here and leave the other to the reader.

Using the relation $\alpha\beta = \beta\alpha^{-1}$ the image of an arbitrary element a^pb^q can be represented by $f_\#(a^pb^q) = \beta^{(p(2s_1+1)+2qs_2)}$ and $g_\#(a^pb^q) = \beta^{(p(2v_1+1)+2qv_2)}$ if p is even and $g_\#(a^pb^q) = \alpha^{t_1}\beta^{(p(2v_1+1)+2qv_2)}$ if p is odd. But in $\pi_1(K)$ the element

$$g_\#(a^pb^q)\ \alpha^m\beta^n\ (f_\#(a^pb^q))^{-1}$$

is equal to $\alpha^m\beta^{(n+2p(v_1-s_1)+2q(v_2-s_2))}$ or $\alpha^{(-m+t_1)}\beta^{(n+2p(v_1-s_1)+2q(v_2-s_2))}$ for p even or odd, respectively. This establishes the first part. In order to find the defective classes we must have that $m = m'$ and that $f_\#(a^pb^q)$ is an orientation reversing element. As a result, p must be odd, and so $m = t_1 - m$. Thus, $m = \frac{t_1}{2}$. In addition, we must find p and q such that $2p(s_1 - v_1) + 2q(s_2 - v_2) = 0$. This is possible with p odd if and only if $|s_1 - v_1|_2 \geq |v_2 - s_2|_2$ and the result follows.

Based on the proposition given above we will describe all the Reidemeister classes for Cases (3) and (4). For Case (3), let $k = \text{g.c.d.}(s_1 - v_1, s_2 - v_2)$ and $(s_1 - v_1, s_2 - v_2) = k(w, z)$. The set of integers $\{2p(s_1 - v_1) + 2q(s_2 - v_2)\}$ can be described as follows:

The set $\{2p(s_1 - v_1) + 2q(s_2 - v_2)\}$ is the same as the set $\{2k(pw + qz)\}$. Since (w, z) are relatively primes, there exist x, y such that $xw + yz = 1$. If z is even then x is necessarily odd. If z is odd then we claim that there is a solution with x even and another solution with x odd. To see this observe that the equation $x_0w + y_0z = 0$ has the solution $(z, -w)$. Hence if (x, y) is a solution of the equation $xw + yz = 1$, then $(x + z, y - w)$ is also a solution and one of the two solutions satisfies our hypothesis. So we have the cases:

(a) If p is even and $\frac{s_2 - v_2}{k}$ is even, we get $\{2p(s_1 - v_1) + 2q(s_2 - v_2)\}$ is equal to $4kZ$;

(b) If p is even and $s_2 - v_2$ is odd, we get $\{2p(s_1 - v_1) + 2q(s_2 - v_2)\}$ is equal to $2kZ$;

(c) If p is odd, we get $\{2p(s_1 - v_1) + 2q(s_2 - v_2)\}$ is equal to $2kZ$, independent of the parity of $s_2 - v_2$.

Similarly, for Case (4), in order to describe the set $\{p(2v_1 - 2s_1 - 1) + q(2v_2 - 2s_2 + 1)\}$ let $k = \text{g.c.d.}(2v_1 - 2s_1 - 1, 2v_2 - 2s_2 + 1)$ and $(2v_1 - 2s_1 - 1, 2v_2 - 2s_2 + 1) = k(w, z)$. Because z

is odd, from the above analysis we conclude that the set $\{p(2v_1 - 2s_1 - 1) + q(2v_2 - 2s_2 + 1)\}$ is $2kZ$ for p odd or even. Consequently, we may summarize as follows:

Corollary 3.1 *The elements $\alpha^{m'}\beta^{n'}$ in the Reidemeister class of a given element $\alpha^m\beta^n$ are those values m', n' which satisfy one of the conditions below:*

For Case (3) :

(I) $m' = m$ *and* $n' = n + 4kZ$ *if* $\frac{s_2 - v_2}{k}$ *is even, or*

(II) $m' = m$ *and* $n' = n + 2kZ$ *if* $s_2 - v_2$ *is odd, or*

(III) $m' = -m + t_1$ *and* $n' = n + 2kZ$*;*

For Case (4) :

(I) $m' = m$ *and* $n' = n + 2kZ$*, or*

(II) $m' = -m + t_1$ *and* $n' = n + 2kZ$*.*

Remark 3.4 Although there are defective Reidemeister classes, we will see later that all defective Nielsen classes for maps from T to K are inessential.

Now we will compute the Nielsen coincidence number for a pair of maps. Since $N(f,g) = N(g,f)$ we have 10 possibilities to consider for the pairs, as $[f]$, $[g]$ run through the four cases given by Lemma 2.1. The Nielsen number is presented here for all these cases, instead of just the four cases given by Lemma 2.2, so as to have explicit formulae for the Nielsen number for any given pair of maps. On the other hand, and in order to save on notation, we will borrow the notation set up at the beginning of this section based on Lemma 2.2. For maps of types II(b),(c) we make the obvious substitution in the notation.

Theorem 3.1 *The value of the Nielsen coincidence number $N(f,g)$ is as follows:*

(1) *Both maps are of type* I*,*

$$|(r_1 - t_1)(s_2 - v_2) - (r_2 - t_2)(s_1 - v_1)| + |(r_1 + t_1)(s_2 - v_2) - (r_2 + t_2)(s_1 - v_1)|,$$

(2i) f *is of type* II(a) *and* g *is of type* I

$$|2t_1(v_2 - s_2) - t_2(2v_1 - 2s_1 - 1)|,$$

(2ii) f *is of type* II(b) *and* g *is of type* I

$$|t_1(2v_2 - 2s_2 - 1) - 2t_2(v_1 - s_1)|,$$

(2iii) f *is of type* II(c) *and* g *is of type* I

$$|2t_1(2v_2 - 2s_2 - 1) - 2t_2(2v_1 - 2s_1 - 1)|.$$

In the remaining cases, that is when both maps are of type II*, $N(f,g) = 0$.*

Proof (1) First consider the case when both f and g are of type I. Recall the orientable double covering $p : T \to K$ and the corresponding deck transformation θ given in the previous section. Consider $\widetilde{f} : T \to T$ a lift of f and $\widetilde{g}_1$, $\widetilde{g}_2$ the two lifts of g.

By a routine argument one can show that $\text{coin}(f,g)$ is the disjoint union of $\text{coin}(\widetilde{f}, \widetilde{g}_1)$ and $\text{coin}(\widetilde{f}, \widetilde{g}_2)$, and that the intersection of $\text{coin}(\widetilde{f}, \widetilde{g}_1)$ with $\text{coin}(\widetilde{f}, \widetilde{g}_2)$ is empty since the two lifts

g_1, g_2 have no coincidence. Further, a Nielsen class of (f, g) is entirely contained in one of the two sets $\mathrm{coin}(\widetilde{f}, \widetilde{g}_1)$, $\mathrm{coin}(\widetilde{f}, \widetilde{g}_2)$. Let C be a Nielsen class of (f, g) which, without loss of generality, we assume that it is also a Nielsen class of $\mathrm{coin}(\widetilde{f}, \widetilde{g}_1)$. The index of C as a Nielsen class of (f, g) is the same as the index as a Nielsen class of $(\widetilde{f}, \widetilde{g}_1)$ since these maps are orentation true (see [5] for more details). Hence $N(f, g) = N(\widetilde{f}, \widetilde{g}_1) + N(\widetilde{f}, \widetilde{g}_2)$, and the result follows directly from the formulas for the Nielsen number for torus maps (see [1]).

(2i) Let f be of type II(a) and let g be of type I. Consider the double covering of T which corresponds to the subgroup $G_1 = \langle a^2, b \rangle$, and denote by $p_1 : T \to T$ the projection. Let $\widetilde{f}$ be a lift of $f \circ p_1$ and let $\widetilde{g}_1, \widetilde{g}_2 : T \to T$ be the two distinct lifts of $g \circ p_1$. These two lifts $\widetilde{g}_1, \widetilde{g}_2 : T \to T$ are the composite of p_1 with the two lifts of $g : T \to K$. Certainly $x \in \mathrm{coin}(f, g)$ if and only if $p_1^{-1}(x)$ contains a point in $\mathrm{coin}(\widetilde{f}, \widetilde{g}_1) \cup \mathrm{coin}(\widetilde{f}, \widetilde{g}_2)$. Observe that $\widetilde{g}_i$ sends the points of a fibre $p_1^{-1}(x)$ into a single point and $\widetilde{f}$ does not, and so the Nielsen number $N(f, g)$ is equal to $\frac{1}{2}(N(\widetilde{f}, \widetilde{g}_1) + N(\widetilde{f}, \widetilde{g}_2))$. As $N(\widetilde{f}, \widetilde{g}_1) = N(\widetilde{f}, \widetilde{g}_2)$, a straightforward calculation leads to the desired result.

The proofs of cases (2ii) and (2iii) in the theorem are similar and left to the reader. We now proceed to the last part, illustrating with one representative case. Suppose one map is of type II(a) and the other is of type II(b). From Lemma 2.2 we will assume that f is given by $f_\#(a) = \beta^{2s_1+1}$, $f_\#(b) = \beta^{2s_2}$ and $g_\#(a) = \beta^{2v_1}$, $g_\#(b) = \alpha^{t_2}\beta^{2v_2+1}$. We take the 4-fold covering $p_2 : T \to T$ which corresponds to the subgroup $\langle a^2, b^2 \rangle$ and the 2-fold covering of K which corresponds to the subgroup α, β^2. As in the previous cases the Nielsen number of the pair (f, g) is a linear combination of the Nielsen number of the various lifts to the torus. A routine check on the matrices of the lifts of (f, g) shows that the first horizontal line is zero for each such lift. So each entry is zero and the result follows.

Remark 3.5 The conclusion $N(f, g) = 0$ for the cases stated in the theorem above, also will follow from the work in the next section where it will be proved that for the said pairs the pair can be deformed to be coincidence free.

4 Wecken Property

The purpose of this section is to give a proof of the following Wecken theorem for pairs of maps from the torus to the Klein bottle.

Theorem 4.1 *Any pair $(f, g) : T \to K$ satisfies $MC[f, g] = N(f, g)$, i.e. the Nielsen number is equal to the minimum number of coincidence points in the homotopy class of the pair.*

In order to prove the Wecken property for all pairs of maps, we will use Corollary 2.1 and Lemma 2.2 of Section 2 to reduce the problem to the 4 cases given in Section 3. Recall that for Cases (3) and (4) we can assume that t_i is either 0 or 1.

As in [9] the proofs divide into two types. When $N(f, g) > 0$ we show that for most maps the linear model carried by lifts to the torus will have $N(f, g)$ coincidence points. The one exception to this appears in the first proposition below. In case of $N(f, g) = 0$ we convert to a root problem to show that the pair can be made coincidence free. Also, we will use the same notational convention as used in [9] for the covering of K and the deck transformation θ.

We begin by considering the case where both maps are of type I (This is Case (1) in Lemma 2.2).

Proposition 4.1 *Let $f, g : T \to K$ be a pair of maps of type* I. *Then $MC[f, g] = N(f, g)$.*

Proof Let us consider $\widetilde{f}, \widetilde{g} : T \to T$ the lifts of f, g, respectively. Consider the linear transformations $T_f, T_g : R^2 \to R^2$ which covers up to homotopy the maps $\widetilde{f}, \widetilde{g}$, respectively.

First suppose that the Lefschetz numbers $L(\widetilde{f}, \widetilde{g})$ and $L(\widetilde{f}, \theta\widetilde{g})$ are both non-zero, where θ is the deck transformation which corresponds to the orientable covering of K. The number of coincidence points of the induced maps $\overline{T}_f, \overline{T}_g : T \to T$ by the linear transformations $T_f, T_g : R^2 \to R^2$, which have matrices M_1, M_2, is the number of solution of the system $M_1(x_1, x_2) \equiv M_2(x_1, x_2) \bmod Z \times Z$, which is also the modulus of the Lefschetz number $L(\widetilde{f}, \widetilde{g})$. Similarly the number of coincidence points of the pair $\overline{T}_f, \overline{\theta T}_g : T \to T$ reduces to solving the system $M_1(x_1, x_2) \equiv \widetilde{M_2}(x_1, x_2) + (0, \frac{1}{2}) \bmod Z \times Z$, where $\widetilde{M_2}$ is the matrix obtained from the matrix M_2 by changing the sign of the first column. But the number of solutions of the latter system is given by the modulus of the Lefschetz number $L(\widetilde{f}, \theta\widetilde{g})$. Since $N(f, g) = |L(\widetilde{f}, \widetilde{g})| + |L(\widetilde{f}, \theta\widetilde{g})|$ the result follows.

Now suppose that at least one of the Lefschetz numbers $L(\widetilde{f}, \widetilde{g})$ or $L(\widetilde{f}, \theta\widetilde{g})$ is zero. Without loss of generality let us assume that $L(\widetilde{f}, \widetilde{g}) = 0$. Consider the deviation map defined by

$$h(z) = \widetilde{f}(z) \cdot (\widetilde{g}(z))^{-1},$$

where $\cdot$ represents a multiplication defined on the torus. By [15, Theorem 1] the degree of h is zero, so it induces in the fundamental group a homomorphism whose image has rank at most one. With $h_\theta = \widetilde{f} \cdot (\theta\widetilde{g})^{-1}$ the theorem just cited also yields $|L(\widetilde{f}, \theta\widetilde{g})| = |\deg(h_\theta)|$.

Now we deform h so that its entire image lies on a geodesic simple closed curve which represents the image of the fundamental group. Choose a nowhere zero direction field on T which is transverse to this curve, and also the similarly obtained curve in the case that $\deg(h_\theta) = 0$. Now consider the two maps $\overline{T}_f$ and $\epsilon \circ \overline{T}_g$, where ϵ is a small deformation of the identity on R^2 which is determined by the direction field. Now it is easy to see that the pair $\epsilon \circ \overline{T}_g, \overline{T}_f$ has no coincidence points. Generic properties of matrices ensure that $\epsilon \circ \overline{T}_g, \theta\overline{T}_f$ have exactly $|\deg(h_\theta)|$ coincidence points. This completes Case (1).

Now we divide the remaining pairs into two families. The pairs where the $N(f, g) \neq 0$ and those where $N(f, g) = 0$. According to Theorem 3.1 we have that the former corresponds to those in Case (2) of Lemma 2.2 with Nielsen number different from zero, while the latter corresponds to Case (2) with Nielsen number zero together with Cases (3) and (4).

Proposition 4.2 *Let $f, g : T \to K$ be a pair of maps corresponding to Case* (2). *Then $MC[f, g] = N(f, g)$.*

Proof From Lemma 2.2 the map f induces in the fundamental group a homomorphism of the form $f_\#(a) = \beta^{2s_1+1}$, $f_\#(b) = \beta^{2s_2}$.

Let $(\widetilde{f}, \widetilde{g}) : T \to T$ be a lift of the pair (f, g) as given in the proof of Theorem 3.1(2i). Also, as seen in that proof we do not need to consider the other lift of g corresponding to the deck transformation θ. The lifts $\widetilde{f}$ and $\widetilde{g}$, respectively, have the following models given by linear transformations of R^2 with respective matrices

$$\begin{bmatrix} 0 & 0 \\ 2s_1+1 & s_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2t_1 & t_2 \\ 2v_1 & v_2 \end{bmatrix}.$$

When $N(f,g) \neq 0$, then $N(\widetilde{f},\widetilde{g}) = N(f,g)$ and by a routine calculation, which is left to the reader to check, one gets that the above models have the same number of coincidence points as given by the Nielsen number in Theorem 3.1.

For the remainder of the proof we assume that $N(f,g) = 0$. This implies that $L(\widetilde{f},\widetilde{g}) = 0$. As in the previous proposition we consider the deviation map $h(z) = \widetilde{f}(z) \cdot (\widetilde{g}(z))^{-1}$. Let γ_a denote the deck transformation corresponding to p_1. Since g is type I, $\widetilde{g}$ sends both points in $p_1^{-1}(z)$ to a single point and so we have $\widetilde{g}(\gamma_a(z)) = \widetilde{g}(z)$. On the other hand f is of type II and so the lift $\widetilde{f}$ satisfies $\widetilde{f}(\gamma_a(z)) = \theta(\widetilde{f}(z))$. Now, since the image of $\widetilde{f}$ is along the curve b this reduces to $\widetilde{f}(\gamma_a(z)) = \widetilde{f}(z) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1)$, as in coordinates the action of θ is given by $\theta(x,y) = (x + \frac{1}{2}, 1-y)$. Together these imply the deck transformation equation

$$h(\gamma_a(z)) = h(z) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1).$$

Let a_1, b_1 be generators for the fundamental group of the torus which are lifts of a, b respectively. Consider a fundamental domain X for the action of γ_a on T, bordered by b_1 and $\gamma_a(b_1)$. In coordinates the action of γ_a is by $\gamma_a(x,y) = (x, y + \frac{1}{2})$. Let σ_a denote the unique proper arc in X which is also contained in the loop a_1. Considering the matrices for $\widetilde{f}$ and $\widetilde{g}$ one sees that, under the assumption that $N(f,g) = 0$, the images $h(a_1)$ and $h(b_1)$ must commute. Hence, they are both multiples of some primitive word u. As a result we get, up to homotopy, that the image of σ_a, and hence the halfspace X, is into a 1-dimensional subspace determined by u. So deform h on this halfspace so that the image does not contain the points $(1,1)$ and $(\mathrm{e}^{\pi \mathrm{i}}, 1)$ and extend to the entire torus using the equation $h(\gamma_a(z)) = h(z) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1)$. Let h' denote the resulting map, and keeping $\widetilde{f}$ fixed define $\widetilde{g}' = h' \cdot \widetilde{f}$. Since h' misses $(1,1)$ and $(\mathrm{e}^{\pi \mathrm{i}}, 1)$ neither of the pairs $(\widetilde{f}, \widetilde{g}')$ nor $(\widetilde{f}, \theta\widetilde{g}')$ contain a coincidence point. Hence, (f, g') is coincidence free, so the result follows.

The following proposition covers Cases (3) and (4), where $N(f,g) = 0$ for all pairs considered in these two cases.

Proposition 4.3 *Suppose that f is of type* II(a) *and g is either type* II(a) *or* II(b). *Then we can deform the pair (f,g) to be coincidence free.*

Proof We first consider Case (3) where, by Lemma 2.2, we have two subcases to consider. Namely: (i) $t_1 = 0$ so $g_{\#}(a) = \beta^{2v_1+1}$, $g_{\#}(b) = \beta^{2v_2}$ and (ii) $t_1 = 1$ so $g_{\#}(a) = \alpha^1\beta^{2v_1+1}$, $g_{\#}(b) = \beta^{2v_2}$.

Let $p_1 : T \to T$ be the double cover as in the previous proposition, and let γ_a denote the corresponding deck transformation. Lifting f and g to the torus we again consider the deviation $h(z) = \widetilde{f}(z) \cdot (\widetilde{g}(z))^{-1}$. Since both maps are type II(a) we have that $\widetilde{f}(\gamma_a(z)) = \theta(\widetilde{f}(z))$ and $\widetilde{g}(\gamma_a(z)) = \theta(\widetilde{g}(z))$. So following Case I in [9] we obtain the deck transformation equation

$$h(\gamma_a(z)) = \theta\widetilde{f}(z) \cdot (\theta\widetilde{g}(z))^{-1} = \theta(h(z)) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1).$$

For Case (3i) let b_1 be a simple closed curve in the lifted domain torus such that $p_{1\#}(b_1) = b$. Consider a fundamental domain for the action of γ_a which is bounded by the loops b_1 and $\gamma_a(b_1)$. As both f and g map the generator b into some power of β, it follows from the deck transformation equation that $h(\gamma_a(z)) = h(z)$. Deform h on this halfspace so that the image does not contain the points $(1, 1)$ and $(e^{\pi i}, 1)$ and extend to the entire torus using the deck transformation equation for this case. The result for Case (3i) now follows.

Case (3ii) is much more direct. We claim that the map g in question is freely homotopic to a map g' such that the image of g' lies in the loop $\{(t, \frac{1}{2}), 0 \le t \le 1\}$ in K. Denote by β' the homotopy class of this loop. To see the claim observe that $\alpha\beta$ is free homotopic to β' and β^2 is free homtopic to β'^2. The isotopy θ_s which is the rotation of the circle $t \times S^1$ of the angle $s\pi(1 - 2t)$ provides the homotopy above and the claim. As a result the images of f and g' are disjoint and the result for Case (3ii) follows.

Finally, in Case (4) we have the two similar subcases: (i) $t_2 = 0$ and (ii) $t_2 = 1$. Let $p_2 : T \to T$ denote the 4-fold cover corresponding to the subgroup $\langle a^2, b^2 \rangle$ and let γ_a and γ_b denote the deck transformations corresponding to a^2 and b^2, respectively. Lifting f and g to the torus we consider the deviation $h(z) = \widetilde{f}(z) \cdot (\widetilde{g}(z))^{-1}$ this time satisfying the deck transformation equations

$$h(\gamma_a(z)) = \theta \widetilde{f}(z) \cdot (\widetilde{g}(z))^{-1} = h(z) \cdot (e^{\pi i}, 1),$$

since the image of f is in β, and

$$h(\gamma_b(z)) = \widetilde{f}(z) \cdot (\theta \widetilde{g}(z))^{-1},$$

which in Case (4i) reduces to $h(z) \cdot (e^{\pi i}, 1)$ as well.

In Case (4i) the image of h lies on the curve b and can be deformed equivariantly to miss $(1, 1)$, $(1, e^{\pi i})$, $(e^{\pi i}, 1)$ and $(e^{\pi i}, e^{\pi i})$ and the result follows. Case (4ii) is proved in the exact same manner as case (3ii).

5 1-Parameter Problem for Coincidence

In this section, we will study the 1-parameter problem for coincidence of pair of maps from T to K. Here we will only consider the case where $N(f, g) = 0$. The case where $N(f, g) \neq 0$ is more subtle and presents technical difficulties that we are not able to address at the present time. For pairs (f, g) such that $N(f, g) = 0$ we show that in any such homotopy class there exist pairs of maps which can not be joined by Wecken homotopies. In fact, we produce an infinite family of pairs in a given homotopy class. To do so we start with some generalities about this problem in terms of the homotopy classes of the maps. The following proposition is proved in [9].

Proposition 5.1 *Let (f, g) be a pair of maps which satisfies $\mathrm{coin}(f, g) = N(f, g)$ and g_1 a map homotopic to g. Then*

(a) *the restricted minimal coincidence problem has a positive solution for (f, g) if and only if it has a solution for a pair (f', g), where f' is homotopic to f.*

(b) *there exists* f_1 *homotopic to* f *such that* $\mathrm{coin}(f_1, g_1) = N(f, g)$ *and the restricted minimal coincidence problem has a positive solution for* (f, g) *if and only if it has a solution for the pair* (f_1, g_1).

Remark 5.1 (1) For two minimal and homotopic pairs (f, g) and (f', g'), the minimal coincidence problem, has a positive solution for (f, g) if and only if it has a solution for a pair (f', g'). This follows straight from the statement of the minimal coincidence problem.

(2) As with the classical Nielsen theory, the 1-parameter problem in coincidence point theory (this means either the restricted minimal coincidence problem or the minimal coincidence problem), is completely determined for $(f \circ h, g \circ h)$ and for $(h \circ f, h \circ g)$ if we know the answer for (f, g), where $h \in \mathrm{Homeo}(T)$, resp. $\mathrm{Homeo}(K)$.

The restricted minimal coincidence problem is not symmetric in the variables. So in principle we will have 6 cases to analyze. Namely:

(a) The two maps are of type I,

(b1) The two maps are of type I and II(a) for the first and second coordinate, respectively;

(b2) the two maps are of type II(a) and I for the first and second coordinate, respectively,

(c) The two maps are of type II(a),

(d1) The two maps are of type II(a) and II(b) for the first and second coordinate, respectively;

(d2) the two maps are of type II(b) and II(a) for the first and second coordinate, respectively.

Our results will show at the end, at least for the case of $N(f, g) = 0$, that the restricted minimal coincidence problem for a pair (f, g) has a positive solution if and only if it has a positive solution for the pair (g, f).

Theorem 5.1 *Let* $f, g : T \to K$ *be a pair of homotopy class of maps* $([f], [g])$ *such that* $N(f, g) = 0$ *(this includes Cases* (3), (4) *and part of Cases* (1) *and* (2)*). Then given a map* $g \in [g]$ *there is a countable family of maps* f_n*, where every* $f_n \in [f]$ *and* $\mathrm{coin}(f_n, g) = \emptyset$*, such that for any two pairs* (f_m, g)*,* (f_n, g) *with* $m \neq n$ *there is no a homotopy* H *between* f_m, f_n *with the property that* $(H(\ , t), g)$ *is coincidence free for all* $t \in [0, 1]$.

Proof As a result of Proposition 5.1, it suffices to show the result for one particular map g which belongs to the free homotopy class $[g]$. The proof is then reduced to the six cases mentioned before the theorem.

Case (a) Fix $g : T \to K$ in its homotopy class and let $f \in [f]$ be any map that satisfies the conclusion of Theorem 4.1. That is, $\mathrm{coin}(f, g) = \emptyset$. Let $\widetilde{f}, \widetilde{g}$ be lifts to the two-fold covering $T \to K$, with corresponding deck transformation θ. Consider the deviation map $h : T \to T$ defined by $h(x) = \widetilde{f}(x) \cdot (\widetilde{g}(x))^{-1}$ using the multiplication on the torus. Since $N(f, g) = 0$, it follows that the degree of h is zero, and since the pair is coincidence free the points $(1, 1)$ and $\theta(1, 1)$ do not belong to the image of h. As a result there exists a word v such that $h(a)_\# = v^r$, $h(b)_\# = v^s$. Now proceeding as in [7], for each integer n we define a family of maps $h_n : T \to T$ such that $h_{n\#}(a) = (vB^{2n})^r$, $h_{n\#}(b) = (vB^{2n})^s$ where B is the commutator $[c, d]$ where c, d are generators of $\pi_1(T)$ which project to α, β^2 respectively. These generators are chosen so that they miss both $(1, 1)$ and $\theta(1, 1)$. Setting, $\widetilde{f}_n(x) = h_n(x) \cdot \widetilde{g}(x)$ (with $\widetilde{f} = \widetilde{f}_0$), one gets the family f_n such that each pair (f_n, g) is coincidence free. It is known from [7, Proposition 1.2] that for $m \neq n$ the two maps h_m, h_n can not be joined by a root free homotopy. Consequently, the family of pairs of maps (f_n, g) gives the desired result.

Remark 5.2 Alternately, we could have used $h_{n\#}(a) = v^r$, $h_{n\#}(b) = (vB^{2n})^s$ for the family. The proof of [7, Proposition 1.2] is followed in the same manner. This will be used in the next case.

Cases (b1) and (b2) Let (f, g) be a pair of maps where f is of type II(a) and g is type I, which is Case (b2). Just as in the proof of Proposition 4.2, we consider the two-fold covering $p_1 : T \to T$ which corresponds to the subgroup $\langle a^2, b\rangle$, and let a_1, b_1 be generators for the fundamental group of the covering torus with $p_{1\#}(a_1) = a^2$ and $p_{1\#}(b_1) = b$. Then the composites $f \circ p_1$, $g \circ p_1 : T \to K$ admits liftings to the covering $p : T \to K$. As in the proof of Proposition 4.2 we construct the deviation map $h : T \to T$ which is an equivariant map with respect to the action γ_a on the domain and θ on the contradomain. With respect to the deck transformation γ_a the deviation map satisfies the equation

$$h(\gamma_a(z)) = h(z) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1).$$

Just as in Case (a) above, the hypothesis that $N(f, g) = 0$ implies that the degree of h is zero and, as a result, there exists a word v such that $h(a_1)_\# = v^r$, $h(b_1)_\# = v^s$. Let X be a fundamental domain for the action of γ_a and let σ_a be as in the proof of Proposition 4.2. We define a family h_n so that $h_n = h$ on the arc σ_a and $h_{n\#}(b_1) = (vB^{2n})^s$. Extend equivariantly using the deck transformation equation to get $h_{n\#}(a_1) = v^r$. Now arguing just as in Case (a) together with Remark 2.2 we obtain the desired pairs of maps (f_n, g).

The proof of Case (b1) is the same and is left to the reader.

Case (c) The two maps are of type II(a). We consider the two subcases: (3i) $t_1 = 0$ and (3ii) $t_1 = 1$. Consider the lifts of f, g as in the proof of Case (3) of Proposition 4.3 and let a_1, b_1 be lifts of a, b respectively.

For Case (3i) the deviation map h sends a_1 to an even power $2k$ of d and b_1 to some power l of d, where d is as in Proposition 4.3. As both f and g map b into some power of β, it follows from the deck transformation equation that $h(\gamma_a(z)) = h(z)$.

Define a family of maps so that $h_{n\#}(a_1) = (d)^{2k}$ and $h_{n\#}(b_1) = (dB^{2n})^l$, which are seen to be equivariant maps as a_1 sent to an even power of d is compatible with $h(\gamma_a(z)) = h(z)$. So they define a family of maps from T to K. The result follows just as in the previous Case (b2).

In Case (3ii) the deviation map h in this case sends b_1 to d^l as above, and leads to the transformation equation

$$h(\gamma_a(z)) = \theta\widetilde{f}(z) \cdot (\theta\widetilde{g}(z))^{-1} = \theta(h(z)) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1).$$

Fix a fundamental region X for the action of γ_a and as before, let $\sigma_a = X \cap a_1$. Define a family h_n by $h_n(\sigma_a) = h(\sigma_a)$ and $h_n(b_1) = (dB^{2n})^l$ and extend to the domain torus using the deck transformation equation. The result follows just as in the previous case.

Cases (d1) and (d2) We focus on Case (d1) where the map f is of type II(a) and g of type II(b). Consider the 4-fold covering which corresponds to the subgroup $\langle a^2, b^2\rangle$. Generators are denoted by γ_a and γ_b are the deck transformations which correspond to a, b respectively. As before, let a_1, b_1 be lifts of a, b respectively.

We can reduce consideration to two cases where for f we can assume $r_1 = 0$. For g we have

(4i) $g_\#(a) = \beta^{2s}$, $g_\#(b) = \beta^{2r+1}$,

or

(4ii) $g_\#(a) = \beta^{2s}$, $g_\#(b) = \alpha\beta^{2r+1}$.

We now look at the deviation map $h = \widetilde{f} \cdot (\widetilde{g})^{-1}$. As in Case (4) of Proposition 4.3 we have

$$h(\gamma_a(z)) = \theta\widetilde{f}(z) \cdot (\widetilde{g}(z))^{-1} = h(z) \cdot (\mathrm{e}^{\pi \mathrm{i}}, 1),$$

and

$$h(\gamma_b(z)) = \widetilde{f}(z) \cdot (\theta\widetilde{g}(z))^{-1},$$

which in Case (4i) reduces to $h(z) \cdot (\mathrm{e}^{\pi i}, 1)$ as before.

Fix a fundamental region X for the action of the group generated by γ_a and γ_b and as before, let $\sigma_a = X \cap a_1$. Also let $\sigma_b = X \cap b_1$, and let c, d be natural generators for the target torus.

Case (4i) Write $d = d_1 d_2$ where $\theta(d_1) = d_2$ and of course $\theta(d_2) = d_1$. In this case the deviation map h takes the form $\sigma_a \mapsto d^p d_1$ and $\sigma_b \mapsto d^q d_1$, where p, q are arbitrary integers that depend on the given data for the maps f, g. We note that if either, say p, is negative, then this reduces to $d^{-p'} d_2^{-1}$.

Define a family h_n by $h_n(\sigma_a) = h(\sigma_a)$ and $h_n(\sigma_b) = B^{2n} d^q d_1$ and extend to the domain torus using the deck transformation equation.

Setting $\widetilde{f}_n(x) = h_n(x) \cdot \widetilde{g}(x)$ (with $\widetilde{f} = \widetilde{f}_0$), one gets the family f_n such that each pair (f_n, g) is coincidence free.

Suppose that for two given integers n, m, (f_n, g) and (f_m, g) can be joined by a coincidence free homotopy. We show that $n = m$ so we have the desired family for the result. By assumption, h_n and h_m can be joined by a root free homotopy at 1. The root free homotopy implies that

$$h_n(\sigma_b\ \gamma_b\sigma_b) = \phi\ h_m(\sigma_b\ \gamma_b\sigma_b)\ \phi^{-1}$$

for some word ϕ in the free group generated by c, d.

Using the action of h, and hence any h_k, we compute

$$h_k(\gamma_b\sigma_b) = d_1^{-1} B^{2k} d^q d_1 d_2,$$

which yields the equation

$$B^{2n} d^q B^{2n} d^{q+1} = \phi\ B^{2m} d^q B^{2m} d^{q+1}\ \phi^{-1}.$$

We now compute an invariant of conjugacy classes for the words of the form given in this equation. This invariant was given in the proof of Theorem 3.10 of [9]. Given a word W in the free group with generators x, y let $t'(W)$ denote the total number of transitions between an $x^{\pm}$ and a $y^{\pm}$ that occur in W. Let $t(W)$ be the minimum of $t'(W')$, where W' is an arbitrary conjugate of W. A direct calculation shows that $t(B^{2k} d^q B^{2k} d^{q+1}) = 16k - l$, where $l = 1$ or $l = 5$ depending on the value of k and q. Thus, when $n \neq m$ we conclude that no such ϕ exists.

Remark 5.3 We could have also defined a family by choosing h_n so that $h_n(\sigma_b) = h(\sigma_b)$ and $h_n(\sigma_a) = B^{2n}d^p d_1$, or by using B^{2n} in both coordinates. Due to the symmetry of this case the argument is exactly the same.

Case (4ii) This case is not symmetric due to the fact that the image of $h(a_1)$ is in d, while $h(b_1)$ includes c. But, if we restrict the family to the type $h_n(\sigma_b) = h(\sigma_b)$ and $h_n(\sigma_a) = B^{2n}d^p d_1$ we get the conclusion by a proof identical to the one used in Case (4i).

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