Chin. Ann. Math.
29B(5), 2008, 459–486
DOI: 10.1007/s11401-008-0082-1

**Chinese Annals of Mathematics, Series B**


# On Lower Dimensional Invariant Tori in $C^d$ Reversible Systems**

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**Abstract** In this paper, a result on the persistence of lower dimensional invariant tori in $C^d$ reversible systems is obtained under some conditions. The theorem is proved for any $d$ which is larger than some constants.

**Keywords** Reversible systems, Lower dimensional invariant tori, KAM step
**2000 MR Subject Classification** 37C55

## 1 Introduction

We consider the following reversible system with respect to the involution $G : (x, y, u, v) \mapsto (-x, y, -u, v)$

$$\begin{aligned} \dot{x} &= \Lambda_1(y, u, v) + R^1(x, y, u, v), \\ \dot{y} &= \Lambda_2(y, u, v) + R^2(x, y, u, v), \\ \dot{u} &= L_1(y, u, v) + R^3(x, y, u, v), \\ \dot{v} &= L_2(y, u, v) + R^4(x, y, u, v), \end{aligned} \tag{1.1}$$

where $\Lambda_1$, $\Lambda_2$, $L_1$, $L_2$, $R^l$ $(1 \le l \le 4)$ are functions of class $C^d$ defined on a neighborhood of $\mathbb{T}^n \times \mathcal{D} \times \{0\} \times \{0\}$ with an open set $\mathcal{D} \subset \mathbb{R}^n$ and $R^l$'s are small perturbation terms. Let $X$ be the vector field of (1.1), i.e.

$$X = (\Lambda_1 + R^1)\frac{\partial}{\partial x} + (\Lambda_2 + R^2)\frac{\partial}{\partial y} + (L_1 + R^3)\frac{\partial}{\partial u} + (L_2 + R^4)\frac{\partial}{\partial v}.$$

The reversibility of the system (1.1) with respect to the involution $G : (x, y, u, v) \mapsto (-x, y, -u, v)$ means that

$$DG \cdot X = -X \circ G, \quad DG := \Big(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial u}, \frac{\partial G}{\partial v}\Big). \tag{1.2}$$

When the system (1.1) does not have the perturbation terms $R^l$ $(1 \le l \le 4)$, the system

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**Project supported by the National Natural Science Foundation of China (Nos. 10325103, 10531010).

(1.1) becomes

$$\begin{aligned}\dot{x} &= \Lambda_1(y,u,v),\\ \dot{y} &= \Lambda_2(y,u,v),\\ \dot{u} &= L_1(y,u,v),\\ \dot{v} &= L_2(y,u,v).\end{aligned} \tag{1.3}$$

If we further assume

$$\Lambda_2(y,0,0) = 0, \quad L_1(y,0,0) = 0, \quad L_2(y,0,0) = 0,$$

the system (1.3) will have an invariant subspace $\{u = 0, v = 0\}$. This subspace is foliated by a family of invariant tori $\mathbb{T}^n \times \{y_0\} \times \{0\} \times \{0\}$ and the flow of (1.3) restricted on each torus is

$$x(t) = x_0 + \Lambda_1(y_0,0,0)t.$$

We consider the linear approximation of the system (1.3) at the invariant torus $\mathbb{T}^n \times \{y_0\} \times \{0\} \times \{0\}$,

$$\begin{aligned}\dot{x} &= \omega,\\ \dot{y} &= \Lambda_{2y}(y_0,0,0)(y-y_0) + \Lambda_{2u}(y_0,0,0)u + \Lambda_{2v}(y_0,0,0)v,\\ \dot{u} &= L_{1y}(y_0,0,0)(y-y_0) + L_{1u}(y_0,0,0)u + L_{1v}(y_0,0,0)v,\\ \dot{v} &= L_{2y}(y_0,0,0)(y-y_0) + L_{2u}(y_0,0,0)u + L_{2v}(y_0,0,0)v,\end{aligned} \tag{1.4}$$

where $\omega = \Lambda_1(y_0,0,0)$. By (1.2), we have

$$\begin{aligned}\Lambda_1(y,-u,v) &= \Lambda_1(y,u,v),\\ \Lambda_2(y,-u,v) &= -\Lambda_2(y,u,v),\\ L_1(y,-u,v) &= L_1(y,u,v),\\ L_2(y,-u,v) &= -L_2(y,u,v),\end{aligned}$$

which in turn implies that

$$\begin{aligned}&\Lambda_{2y}(y,0,0) \equiv 0, \quad \Lambda_{2v}(y,0,0) \equiv 0,\\ &L_{1u}(y,0,0) \equiv 0,\\ &L_{2y}(y,0,0) \equiv 0, \quad L_{2v}(y,0,0) \equiv 0.\end{aligned}$$

If the matrix $\Lambda_{1y}(y_0,0,0)$ is nonsingular, the function $\phi : y_0 \mapsto \Lambda_1(y_0,0,0)$ is a local diffeomorphism. Therefore $\phi^{-1}$ exists and is also a function of class $C^d$ defined on an open set $\Theta = \phi(\mathcal{D}) \subset \mathbb{R}^n$. Define

$$\begin{aligned}A(\omega) &= L_{1v}(\phi^{-1}(\omega),0,0),\\ B(\omega) &= L_{2u}(\phi^{-1}(\omega),0,0),\\ C(\omega) &= L_{1y}(\phi^{-1}(\omega),0,0),\\ D(\omega) &= \Lambda_{2u}(\phi^{-1}(\omega),0,0)\end{aligned}$$

and

$$\mathcal{A}(\omega) = \begin{pmatrix} 0 & D(\omega) & 0 \\ C(\omega) & 0 & A(\omega) \\ 0 & B(\omega) & 0 \end{pmatrix}.$$

In a small neighborhood of the set $\{y = y_0,\ u = 0,\ v = 0\}$, (1.1) can be viewed as a small perturbation of (1.4), that is, the system (1.1) can be rewritten as

$$\begin{aligned} \dot{x} &= \omega + f^1(x, y, u, v, \omega), \\ \dot{y} &= D(\omega)u + f^2(x, y, u, v, \omega), \\ \dot{u} &= C(\omega)y + A(\omega)v + f^3(x, y, u, v, \omega), \\ \dot{v} &= B(\omega)u + f^4(x, y, u, v, \omega), \end{aligned} \tag{1.5}$$

where $(x, y, u, v) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$, $\omega$ is an independent parameter varying over a positive measure set $\Theta \subset \mathbb{R}^n$, and the variables $(y, u, v)$ vary on a small neighborhood of the origin of the space $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q$. Notice that here our variable $y$ is actually $y - y_0$ in the equation (1.4). The reversibility of the system (1.5) with respect to $G : (x, y, u, v) \mapsto (-x, y, -u, v)$ means that

$$f^1 \circ G = f^1, \quad f^2 \circ G = -f^2, \quad f^3 \circ G = f^3, \quad f^4 \circ G = -f^4. \tag{1.6}$$

If we denote $Z = (x, y, u, v)$,

$$L(Z, \omega) = \begin{pmatrix} \omega \\ D(\omega)u \\ C(\omega)y + A(\omega)v \\ B(\omega)u \end{pmatrix}, \quad \mathcal{F}(Z, \omega) = \begin{pmatrix} f^1(Z, \omega) \\ f^2(Z, \omega) \\ f^3(Z, \omega) \\ f^4(Z, \omega) \end{pmatrix}, \tag{1.7}$$

the system (1.5) can be written as

$$\dot{Z} = L(Z, \omega) + \mathcal{F}(Z, \omega). \tag{1.8}$$

In this article, we state our result for (1.5), or equivalently for (1.8), instead of the original system (1.1).

If the system (1.5) is a Hamiltonian system, the persistence of the lower dimensional tori has been studied extensively. For instance, if all eigenvalues of $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ are not purely imaginary and $C = 0$, $D = 0$, Moser [10], Graff [6] and Zehnder [19] proved that, for any $\omega = (\omega_1, \cdots, \omega_n) \in \Theta$ satisfying the Diophantine condition

$$|\langle k, \omega \rangle| > \gamma |k|^{-\tau}, \quad \tau > n - 1, \quad k \in \mathbb{Z}^n \setminus \{0\},$$

there is an $\omega^*$ close to $\omega$ such that (1.5) at $\omega^*$ has an invariant $n$-torus with prescribed frequencies $\omega$ if the perturbations $f^l$ $(1 \le l \le 4)$ are sufficiently small; if all the eigenvalues of $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ are simple and purely imaginary and $C = 0$, $D = 0$, Melnikov [9] in 1967 annouced that for a positive Lebesgue measure subset $\Theta_\gamma \subset \Theta$, (1.5) possesses a lower dimensional invariant torus. Eliasson [5], Kuksin [7] and Pöschel [11] gave a complete proof. You [18] proved the persistence of $n$-dimensional invariant tori for (1.5) under the condition $\det \left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \ne 0$. In particular, his result can be applied to the case that $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ has multiple eigenvalues. Developing Craig and Wayne's method [4], Bourgain [1] proved the existence of quasi-periodic solutions. Such approach applies to some PDEs with periodic boundary condition (see [2]). Especially, when the small perturbations $f^i$ $(i = 1, 2, 3, 4)$ are functions of class $C^d$ $(d > 6n + 5)$ and normal frequencies $\Omega_i$ are simple, Chierchia and Qian [3] showed the persistence and regularity of the lower $n$-dimensional elliptic tori.

On the other hand, when the system (1.5) is reversible with respect to the involution $G : (x, y, u, v) \mapsto (-x, y, -u, v)$, Sevryuk studied the persistence of $n$-dimensional invariant tori for $\{p = 0\}$, $\{p = q > 0\}$ and $\{x \in \mathbb{T}^n, y \in \mathbb{R}^m\,(m \neq n), p = q > 0\}$, respectively. We refer to [13–16] and references therein. However, in the last two cases, there are two assumptions required in his results:

(i) $C = 0, D = 0$, $\det\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right) \neq 0$, this means $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ has no eigenvalue being zero;

(ii) Any eigenvalue of $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$ is simple.

Liu proved the persistence of $n$-dimensional invariant tori in the reversible system under small perturbations in the case $p \leq q$ in [8]. He did not require the assumption on the simplicity of the eigenvalues of $\left(\begin{smallmatrix} 0 & A \\ B & 0 \end{smallmatrix}\right)$.

In this paper, motivated by the above papers, we are mainly concerned with the persistence of the lower dimensional tori of the reversible system (1.5) under some $C^d$ perturbations $f^i$ $(i = 1, 2, 3, 4)$. We show that the lower dimensional tori are persistent under these $C^d$ perturbations.

To give the main result of our paper, we need the following conditions.

(1) There is a constant $M > 0$, such that for the elements of the matrices $A$, $B$, $C$ and $D$, $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$, the following inequalities hold for all $\omega \in \mathcal{W}_h = \{\omega \in \mathbb{C}^n : |\omega - \Theta| < h\}$,

$$\max_{|l| \leq N^2} \left|\frac{\partial^l a_{ij}}{\partial \omega^l}\right|, \quad \max_{|l| \leq N^2} \left|\frac{\partial^l b_{ij}}{\partial \omega^l}\right|, \quad \max_{|l| \leq N^2} \left|\frac{\partial^l c_{ij}}{\partial \omega^l}\right|, \quad \max_{|l| \leq N^2} \left|\frac{\partial^l d_{ij}}{\partial \omega^l}\right| \leq M,$$

where $N = n + p + q$.

(2) The rank of the matrix $A$ is $p$, which implies that $p \leq q$. Without loss of generality, we assume $\det(a_{ij})_{1 \leq i,j \leq p} \neq 0$ on $\mathcal{W}_h$.

(3) $\operatorname{meas}(\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3) = 0$, where

$$\begin{aligned}
\mathcal{R}_1 &= \{\omega \in \Theta \mid \langle k, \omega \rangle = 0,\ k \in \mathbb{Z}^n \setminus \{0\}\}, \\
\mathcal{R}_2 &= \{\omega \in \Theta \mid \det(\mathrm{i}\langle k, \omega \rangle \mathbb{E}_N - \mathcal{A}) = 0,\ k \in \mathbb{Z}^n \setminus \{0\}\}, \\
\mathcal{R}_3 &= \{\omega \in \Theta \mid \det(\mathrm{i}\langle k, \omega \rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}^T - \mathcal{A} \otimes \mathbb{E}_N) = 0,\ k \in \mathbb{Z}^n \setminus \{0\}\}.
\end{aligned}$$

Here and hereafter, we set $\mathrm{i} = \sqrt{-1}$, $\mathbb{E}_j$ is the $j \times j$ identity matrix, and $\otimes$ is the tensor product of matrices. $\left(\text{The tensor product of matrices } A_{mn}, B_{kl} \text{ is an } mk \times nl \text{ matrix: } A \otimes B = (a_{ij}B) = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.\right)$

In the following, we always assume $1 < \kappa < \frac{3}{2}$, $\tau > N^2 n$,

$$\lambda < \frac{1}{(N^2 + 1)(N^2 + \tau + 1)}$$

and
$$\zeta < \min\Big\{\lambda,\ \frac{3-2\kappa}{N^2+\tau+n+1},\ \frac{2-\kappa}{N^2+n+\tau+1},\ \frac{N^2}{(N^2+1)(N^2+n+\tau+2)}\Big\}.$$
For $s>0$, let $\mathbb{T}^n\times\mathcal{B}_s := \mathbb{T}^n\times\{|y|<s\}\times\{|u|<s\}\times\{|v|<s\}$ be a neighborhood of $\mathbb{T}^n\times\{y=0\}\times\{u=0\}\times\{v=0\}$. Now we can state our main result of this paper.

**Theorem 1.1** *Suppose that the above assumptions* (1)–(3) *hold for the reversible system* (1.5). *And assume that $f^l$ are of class $C^d$ and $\|f^l\|_{C^d}$ $(l=1,2,3,4)$ are bounded in a neighborhood of $\mathbb{T}^n\times\{y=0\}\times\{u=0\}\times\{v=0\}\times\Theta$ with $d>\frac{2}{\zeta}$. Then for any $\gamma>0$ there are a pair of positive constants $\varepsilon_0$ and $s_0$ depending on $n,p,q,\ \tau,M,\gamma$, such that if*
$$\|f^1\|_{C^0},\ \frac{1}{s_0}\|f^2\|_{C^0},\ \frac{1}{s_0}\|f^3\|_{C^0},\ \frac{1}{s_0}\|f^4\|_{C^0} < \frac{\varepsilon_0}{2},$$
*where the norm $\|\cdot\|_{C^0}$ is the maximum norm on the set $\mathbb{T}^n\times\mathcal{B}_{s_0}\times\mathcal{W}_h$, there exists a differentiable map*
$$\Phi:\ \mathbb{T}^n\times\Theta_\gamma\to\mathbb{T}^n\times\mathbb{R}^n\times\mathbb{R}^p\times\mathbb{R}^q,$$
*with $\Theta_\gamma\subset\Theta$ a positive measure set, and a diffeomorphism $\Psi:\Theta_\gamma\to\mathbb{R}^n$ such that $\Phi(\mathbb{T}^n\times\{\omega\})$ is an invariant torus of the system* (1.5) *with frequencies $\Psi(\omega)$ at $\omega$. Moreover,*
$$\mathrm{meas}(\Theta-\Theta_\gamma)\to 0,\quad \text{as } \gamma\to 0.$$

**Remark 1.1** $\omega$ satisfies the diophantine condition. See (2.8) in the next section for more details.

## 2 The KAM Step

The lemma given in [12] is very important to our paper. For easy reference, we list it as follows.

**Lemma 2.1** (see [12]) *Suppose $f\in C^p(\mathbb{R}^k)$ for some $p>0$ with finite $C^p$ norm over $\mathbb{R}^k$. Let $\psi$ be a radial-symmetric, $C^\infty$ function, having as support the closure of the unit ball centered at the origin, where $\psi$ is completely flat and takes value* 1 *and let $K=\widehat{\psi}$ be its Fourier transform. For all $\sigma>0$, define*
$$f_\sigma(x) := K_\sigma * f(x) = \sigma^{-k}\int_{\mathbb{R}^k} K\Big(\frac{x-y}{\sigma}\Big)f(y)\,\mathrm{d}y.$$
*Then there exists a constant $\widetilde{c}\geq 1$ depending only on $p$ and $k$ such that the following holds. For any $\sigma>0$, the function $f_\sigma(x)$ is a real-analytic function on $\mathbb{C}^k$ such that, if $\Delta^k_\sigma$ denotes the $k$-dimensional complex strip of width $\sigma$*
$$\Delta^k_\sigma := \{x\in\mathbb{C}^k : |\mathrm{Im}\ x_j|\leq\sigma,\ \forall\, j\},$$
*then, for all $\alpha\in\mathbb{N}^k$ such that $|\alpha|\leq p$, one has*
$$\sup_{x\in\Delta^k_\sigma}\Big|\partial^\alpha f_\sigma(x) - \sum_{|\beta|\leq p-|\alpha|}\frac{\partial^{\alpha+\beta}f(\mathrm{Re}\,x)}{\beta!}(\mathrm{i}\,\mathrm{Im}\,x)^\beta\Big| \leq \widetilde{c}\ |f|_{\mathbb{C}^p}\ \sigma^{p-|\alpha|}$$

*and, for all* $0 \le s \le \sigma$,

$$\sup_{x\in\Delta_\sigma^k} |\partial^\alpha f_\sigma - \partial^\alpha f_s| \le \widetilde{c}\ |f|_{\mathbb{C}^p}\ \sigma^{p-|\alpha|}. \tag{2.1}$$

*Moreover, the Hölder norms of* $f_\sigma$ *satisfy, for all* $0 \le q \le p \le r$,

$$|f_\sigma - f|_{\mathbb{C}^q} \le \widetilde{c}\ |f|_{\mathbb{C}^p}\ \sigma^{p-q}.$$

*The function* $f_\sigma$ *preserves periodicity (i.e., if* $f$ *is T-periodic in any of its variable* $x_j$*, so is* $f_\sigma$*). Finally, if* $f$ *depends on some parameter* $\xi \in \Pi \subset \mathbb{R}^n$ *and if the Lipschitz semi-norm of* $f$ *and its x-derivatives are uniformly bounded by* $|f|_{\mathbb{C}^l}^{\mathrm{Lip}}$*, then all the above estimates hold with* $|\cdot|$ *replaced by* $|\cdot|^{\mathrm{Lip}}$.

**Remark 2.1** If $f$ is defined on

$$\mathbb{T}^n \times \mathcal{B}_{s_1,s_2,s_3} := \mathbb{T}^n \times \{|y| < s_1\} \times \{|u| < s_2\} \times \{|v| < s_3\}, \quad s_i < 1,$$

then one can easily construct a $C^l$-extension $f_{\mathrm{ext}}$ of $f_{\mathbb{T}^n\times\mathcal{B}_{\frac{s_1}{2},\frac{s_2}{2},\frac{s_3}{2}}}$ to $\mathbb{R}^{2(n+m)}$, such that

$$|f_{\mathrm{ext}}|_{C^l(\mathbb{R}^{2(n+m)})} \le a|f|_{C^l(\mathbb{T}^n\times\mathcal{B}_{s_1,s_2,s_3})},$$

where $a$ is a positive constant depending only on $l$ and $s_i$.

For $\sigma_\nu \to 0$ as $\nu \to \infty$ $(\sigma_\nu > 0)$, $f_{\sigma_\nu} := K_{\sigma_\nu} * f$, then $f_{\sigma_\nu} \overset{\mathbf{C}^q}{\to} f\ (q < p)$ as $\nu \to \infty$.

**Remark 2.2** If $f \circ G = -DG \cdot f$, then

$$f_{\sigma_\nu} \circ G = -DG \cdot f_{\sigma_\nu}.$$

In fact, $\psi$ is a radial-symmetric function, and so is $K = \widehat{\psi}$.

$$\begin{aligned} f_{\sigma_\nu} \circ G &= \frac{1}{\sigma_\nu^{2n+p+q}} \int_{\mathbb{R}^{2n+p+q}} K\Big(\frac{(-x,y,-u,v) - (\widetilde{x},\widetilde{y},\widetilde{u},\widetilde{v})}{\sigma_\nu}\Big) f(\widetilde{x},\widetilde{y},\widetilde{u},\widetilde{v})\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{u}\,\mathrm{d}\widetilde{v} \\ &= \frac{1}{\sigma_\nu^{2n+p+q}} \int_{\mathbb{R}^{2n+p+q}} K\Big(\frac{(x,y,u,v) - (-\widetilde{x},\widetilde{y},-\widetilde{u},\widetilde{v})}{\sigma_\nu}\Big) f(\widetilde{x},\widetilde{y},\widetilde{u},\widetilde{v})\,\mathrm{d}\widetilde{x}\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{u}\,\mathrm{d}\widetilde{v} \\ &\quad (\text{Let } \widetilde{x}_1 = -\widetilde{x},\ \widetilde{u}_1 = -\widetilde{u}.) \\ &= \frac{1}{\sigma_\nu^{2n+p+q}} \int_{\mathbb{R}^{2n+p+q}} K\Big(\frac{(x,y,u,v) - (\widetilde{x}_1,\widetilde{y},\widetilde{u}_1,\widetilde{v})}{\sigma_\nu}\Big) f(-\widetilde{x}_1,\widetilde{y},-\widetilde{u}_1,\widetilde{v})\,\mathrm{d}\widetilde{x}_1\,\mathrm{d}\widetilde{y}\,\mathrm{d}\widetilde{u}_1\mathrm{d}\widetilde{v} \\ &= -DG \cdot f_{\sigma_\nu}. \end{aligned}$$

### 2.1 Main idea of the proof

According to Lemma 2.1, for a sequence of numbers $\sigma_\nu \to 0$ $(\nu \to \infty)$, we can find a sequence of analytic reversible systems

$$\dot{Z} = L(Z,\omega) + \mathcal{F}_{\sigma_\nu}(Z,\omega), \tag{2.2}$$

where $\mathcal{F}_{\sigma_\nu}(Z,\omega) = (f^1_{\sigma_\nu}, f^2_{\sigma_\nu}, f^3_{\sigma_\nu}, f^4_{\sigma_\nu})$ is defined on $\Delta_{\sigma_\nu}$ as in Lemma 2.1, and $\mathcal{F}_{\sigma_\nu} \to \mathcal{F}$ as $\nu \to \infty$.

Now we hope to find a mapping $\Phi_\nu$

$$Z = \Phi_\nu(Z_\nu, \omega_\nu), \tag{2.3}$$

which commutes with $G$, such that it transforms the system

$$\dot{Z}_\nu = L_\nu(Z_\nu, \omega_\nu) + \mathcal{F}_\nu(Z_\nu, \omega_\nu) \tag{2.4}$$

to the system (2.2), where $\mathcal{F}_\nu$ and $\Phi_\nu$ are defined on

$$\mathcal{D}_{r_\nu, s_\nu} \times \mathcal{W}_{h_\nu} = \{|\text{Im } x| < r_\nu, |y| < s_\nu, |u| < s_\nu, |v| < s_\nu\} \times \mathcal{W}_{h_\nu},$$

and $\mathcal{F}_\nu \to 0$ as $\nu \to \infty$. Since $\Phi_\nu$ commutes with $G$ and the system (2.2) is reversible from the remarks of Lemma 2.1, the system (2.4) is also reversible. Taking derivative with respect to $t$ on both side of (2.3), we get

$$\dot{Z} = D\Phi_\nu \cdot \dot{Z}_\nu, \tag{2.5}$$

where $D\Phi_\nu = \frac{\partial \Phi_\nu}{\partial Z_\nu}$. Combining (2.2), (2.4) and (2.5), we get

$$L(Z, \omega) + \mathcal{F}_{\sigma_\nu}(Z, \omega) = D\Phi_\nu \cdot (L_\nu(Z_\nu, \omega_\nu) + \mathcal{F}_\nu(Z_\nu, \omega_\nu)). \tag{2.6}$$

Once we get $\Phi_\nu$, we want to find two other mappings $\phi_\nu(Z, \omega)$ and $\psi_\nu(\omega)$, and therefore we can find $\Phi_{\nu+1} = \Phi_\nu(\phi_\nu, \psi_\nu)$. We can keep doing so and get a series of functions $\Phi_\nu$. If we can prove that $\Phi_\nu$ and $D\Phi_\nu$ converge as $\nu \to \infty$ and $\Phi_\nu$ commutes with $G$, then by taking $\nu \to \infty$ on both sides of (2.6) we get

$$L(Z, \omega) + \mathcal{F}(Z, \omega) = D\Phi_\infty \cdot L_\infty(Z_\infty, \omega_\infty), \tag{2.7}$$

i.e.

$$\begin{pmatrix} \omega + f^1(Z, \omega) \\ D(\omega)u + f^2(Z, \omega) \\ C(\omega)y + A(\omega)v + f^3(Z, \omega) \\ B(\omega)u + f^4(Z, \omega) \end{pmatrix} = D\Phi_\infty \begin{pmatrix} \omega_\infty \\ D_\infty(\omega_\infty)u_\infty \\ C_\infty(\omega_\infty)y_\infty + A_\infty(\omega_\infty)v_\infty \\ B_\infty(\omega_\infty)u_\infty \end{pmatrix}.$$

So $\Phi_\infty(\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \{\omega_\infty\})$ is the invariant torus of the system (1.5). The frequency of the system (1.5) restricted on the torus $\Phi_\infty(\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \{\omega_\infty\})$ is $\omega = \Psi_\infty(\omega_\infty)$.

### 2.2 Construction of the function $\phi_\nu$

For any positive integer $K_\nu$, we denote by $\mathcal{W}_{h_\nu}(K_\nu)$ the complex neighborhood of radius $h_\nu$ of $\Theta_{\gamma_\nu}(K_\nu)$, where

$$\Theta_{\gamma_\nu}(K_\nu) = \{\omega \in \Theta \mid \text{for any } 0 \neq |k| \leq K_\nu \text{ and } \tau > N^2 n,\ \omega \text{ satisfies (2.8) below}\},$$

$$\begin{aligned} &|\langle k, \omega \rangle| > \gamma |k|^{-\tau}, \quad |\det(\mathrm{i}\langle k, \omega \rangle \mathbb{E}_N - \mathcal{A}_\nu)| > \gamma |k|^{-\tau}, \\ &|\det(\mathrm{i}\langle k, \omega \rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}_\nu^T + \mathcal{A}_\nu \otimes \mathbb{E}_N)| > \gamma |k|^{-\tau}. \end{aligned} \tag{2.8}$$

Since $\Phi_{\nu+1} = \Phi_\nu(\phi_\nu, \psi_\nu)$ and $\Psi_{\nu+1} = \Psi_\nu \circ \psi_\nu$ for some $\phi_\nu : \mathcal{D}_{r_{\nu+1}, s_{\nu+1}} \times \mathcal{W}_{h_{\nu+1}}(K_{\nu+1}) \to \mathcal{D}_{r_\nu, s_\nu}$ and $\psi_\nu : \mathcal{W}_{h_{\nu+1}}(K_{\nu+1}) \to \mathcal{W}_{h_\nu}(K_\nu)$, we have

$$D\Phi_{\nu+1} = D\Phi_\nu \cdot D\phi_\nu.$$

Then from (2.6),

$$L(Z,\omega)+\mathcal{F}_{\sigma_{\upsilon+1}}(Z,\omega)=D\Phi_\nu\cdot D\phi_\nu[L_{\nu+1}(Z_{\nu+1},\omega_{\nu+1})+\mathcal{F}_{\nu+1}(Z_{\nu+1},\omega_{\nu+1})]. \tag{2.9}$$

If, furthermore, $\Phi_\nu$ satisfies $(\Phi_\nu(\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)),\ \Psi_\nu(\mathcal{W}_{h_\nu}(K_\nu)))\subset\Delta_{\sigma_\nu}$ and $D\Phi_\nu^{-1}$ exists, we subtract the equation (2.9) from the equation (2.6), apply $D\Phi_\nu^{-1}$ on both sides of the equation and get

$$D\phi_\nu\cdot[L_{\nu+1}(Z_{\nu+1},\omega_{\nu+1})+\mathcal{F}_{\nu+1}(Z_{\nu+1},\omega_{\nu+1})]=L_\nu(Z_\nu,\omega_\nu)+\widetilde{\mathcal{F}}_\nu(Z_\nu,\omega_\nu), \tag{2.10}$$

where

$$\widetilde{\mathcal{F}}_\nu(Z_\nu,\omega_\nu)=\mathcal{F}_\nu(Z_\nu,\omega_\nu)-D\Phi_\nu^{-1}(\mathcal{F}_{\sigma_\nu}(Z,\omega)-\mathcal{F}_{\sigma_{\nu+1}}(Z,\omega)). \tag{2.11}$$

Let $\Phi_0=\phi_0=\mathrm{id}$ and $\mathcal{F}_0=\mathcal{F}_{\sigma_0}$. According to (2.11) and (2.10), we have

$$\widetilde{\mathcal{F}}_0=\mathcal{F}_{\sigma_1}.$$

Suppose that we have finished $\nu$ steps, and the transformed system is of the form

$$\dot{Z}_\nu=L_\nu(Z_\nu,\omega_\nu)+\widetilde{\mathcal{F}}_\nu(Z_\nu,\omega_\nu), \tag{2.12}$$

where the functions $\widetilde{\mathcal{F}}_\nu=(\widetilde{f}_\nu^1,\widetilde{f}_\nu^2,\widetilde{f}_\nu^3,\widetilde{f}_\nu^4)$ and $L_\nu$ satisfy

$$\begin{aligned}
&\|\widetilde{f}_\nu^1\|_{\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}},\quad \frac{1}{s_\nu}\|\widetilde{f}_\nu^j\|_{\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}}<\varepsilon_\nu,\quad 2\le j\le 4,\\
&\max_{|\ell|\le N^2}\Big|\frac{\partial^\ell a_{ij}^\nu}{\partial\omega_\nu^\ell}\Big|,\quad \max_{|\ell|\le N^2}\Big|\frac{\partial^\ell b_{ij}^\nu}{\partial\omega_\nu^\ell}\Big|,\quad \max_{|\ell|\le N^2}\Big|\frac{\partial^\ell c_{ij}^\nu}{\partial\omega_\nu^\ell}\Big|,\quad \max_{|\ell|\le N^2}\Big|\frac{\partial^\ell d_{ij}^\nu}{\partial\omega_\nu^\ell}\Big|\le M_\nu,\\
&\det(a_{ij}^\nu)_{1\le i,j\le p}\ne 0.
\end{aligned} \tag{2.13}$$

In what follows, the notations without subscript mean $\nu$-th step, those with subscript "+" mean $(\nu+1)$-th step, and those with subscript "++" mean $(\nu+2)$-th step. Thus (2.12) becomes

$$\dot{Z}=L(Z,\omega)+\widetilde{\mathcal{F}}(Z,\omega), \tag{2.14}$$

where $L$ and $\widetilde{\mathcal{F}}$ are defined on $\mathcal{D}_{r,s}\times\mathcal{W}_h$.

Since $\mathcal{W}_h(K)\subset\mathcal{W}_h$, the inequalities (2.13) still hold if we replace $\mathcal{W}_h$ by $\mathcal{W}_h(K)$. Assume that the desired change of variables $\phi$ defined in a smaller domain $\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)$ has the form

$$\begin{aligned}
x&=x_++\sum_{|k|\le K}h_k^1\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},\\
y&=y_++\sum_{|k|\le K}(h_k^2+H_k^1y_++H_k^2u_++H_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},\\
u&=u_++\sum_{|k|\le K}(h_k^3+I_k^1y_++I_k^2u_++I_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},\\
v&=v_++\sum_{|k|\le K}(h_k^4+J_k^1y_++J_k^2u_++J_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},
\end{aligned} \tag{2.15}$$

such that $\phi$ transforms the following system to system (2.14)

$$\dot{Z}_+ = L_+(Z_+, \omega_+) + \mathcal{F}_+(Z_+, \omega_+), \tag{2.16}$$

where $(x_+, y_+, u_+, v_+) \in \mathcal{D}_{r_+, s_+}$ and $\omega_+ \in \mathcal{W}_{h_+}$. Then from (2.11) we can calculate $\widetilde{\mathcal{F}}_+$ and repeat the same procedure by using $L_+$ and $\widetilde{\mathcal{F}}_+$ as $L$ and $\widetilde{\mathcal{F}}$. Moreover, when we solve $\phi$ and $\psi$, we hope that the derived new perturbation terms $\mathcal{F}_+$ and $\widetilde{\mathcal{F}}_+$ are much smaller than $\widetilde{\mathcal{F}}$.

Remember that $\kappa$ is a real number between 1 and $\frac{3}{2}$, $\zeta$ is a positive real number and $c$ is a large positive number. Fixing $\varepsilon_0$, $\gamma_0$, $s_0$ and $M_0 = M$, we can define the following numbers by iteration,

$$\begin{aligned}
&\varepsilon_{\nu+1} = c^{\kappa-1}\varepsilon_\nu^\kappa, \quad r_\nu = \varepsilon_\nu^\zeta, \quad \sigma_\nu = 2(N+n)r_\nu, \quad h_\nu = \varepsilon_\nu^{\overline{\frac{1}{N^2+1}}},\\
&\alpha_\nu = \varepsilon_\nu^{\kappa-1}, \quad s_{\nu+1} = \frac{1}{2}\alpha_\nu s_\nu, \quad K_\nu = [\varepsilon_\nu^{-\lambda}],\\
&M_{\nu+1} = M_\nu + c^{(\kappa-1)}\varepsilon_\nu^{\overline{\frac{1}{N^2+1}}}, \quad \gamma_\nu = \gamma_0\Big(\frac{1}{2} + \Big(\frac{1}{2}\Big)^{\nu+1}\Big),
\end{aligned}$$

where $[x]$ equals the integer part of $x$.

**Lemma 2.2** *If $(c\varepsilon_0)^{\zeta(\kappa-1)} < \frac{1}{2}$, $s_0 < \frac{\varepsilon_0^\zeta}{2}$, $\kappa > 1$ and $1 > \zeta > 0$, for any $\nu \in \mathbb{N}$, we have $r_\nu < \frac{r_{\nu-1}}{2}$, $s_\nu < \frac{s_{\nu-1}}{2}$ and $s_\nu < r_\nu$.*

**Proof** We only prove the first result. The other two results can be proved in a similar method. From definition, we have

$$r_1 = \varepsilon_1^\zeta = (c^{\kappa-1}\varepsilon_0^\kappa)^\zeta < \frac{\varepsilon_0^\zeta}{2} = \frac{r_0}{2}.$$

If we have $r_\nu < \frac{r_{\nu-1}}{2}$, again by definition,

$$r_{\nu+1} = \varepsilon_{\nu+1}^\zeta = (c^{\kappa-1}\varepsilon_\nu^\kappa)^\zeta = (c\varepsilon_\nu)^{\zeta(\kappa-1)}\varepsilon_\nu^\zeta \le (c\varepsilon_0)^{\zeta(\kappa-1)}\varepsilon_\nu^\zeta \le \frac{\varepsilon_\nu^\zeta}{2} = \frac{r_\nu}{2}.$$

**Lemma 2.3** *If $\zeta < \lambda$, we have*

$$\int_K^\infty x^n e^{-(r-r_+)x}\,\mathrm{d}x < c_0\varepsilon$$

*for some constant $c_0$.*

**Proof** Directly calculating the integral, we get

$$\begin{aligned}
\int_K^\infty x^n \mathrm{e}^{-(r-r_+)x}\,\mathrm{d}x &= \frac{1}{r-r_+}K^n\mathrm{e}^{-(r-r_+)K} + \frac{n}{(r-r_+)^2}K^{n-1}\mathrm{e}^{-(r-r_+)K}\\
&\quad + \cdots + \frac{n!}{(r-r_+)^{n+1}}\mathrm{e}^{-\frac{rK}{2}}.
\end{aligned}$$

From Lemma 2.2, we know $r - r_+ > \frac{r}{2}$. Hence

$$\begin{aligned}
\int_K^\infty x^n \mathrm{e}^{-(r-r_+)x}\,\mathrm{d}x &\le (n+1)!\max_{0\le j\le n}\Big\{\frac{2^{j+1}K^{n-j}}{r^{j+1}}\mathrm{e}^{-(r-r_+)K}\Big\}\\
&\le (n+1)!\max_{0\le j\le n} 2^{j+1}\big\{\varepsilon^{-\lambda(n-j)-\zeta(j+1)}\mathrm{e}^{-\frac{\varepsilon^{\zeta-\lambda}}{2}}\big\}\\
&\le c_0\varepsilon,
\end{aligned}$$

where $c_0$ is a constant independent of $\nu$ and the last inequality comes from the definitions of $K$ and $r$.

## 2.3 Homological equations

From (2.10), (2.14), (2.15) and (2.16), we have

$$\begin{aligned}
&\omega+\widetilde{f}^1\circ\phi=\omega_++f_+^1+\mathcal{F}_+^1,\\
&Du+\widetilde{f}^2\circ\phi=D_+u_++f_+^2+\mathcal{F}_+^2,\\
&Cy+Av+\widetilde{f}^3\circ\phi=C_+y_++A_+v_++f_+^3+\mathcal{F}_+^3,\\
&Bu+\widetilde{f}^4\circ\phi=B_+u_++f_+^4+\mathcal{F}_+^4,
\end{aligned}\tag{2.17}$$

where

$$\begin{aligned}
\mathcal{F}_+^1&=\sum_{|k|\le K}\mathrm{i}\langle k,\omega_++f_+^1\rangle \mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}h_k^1,\\
\mathcal{F}_+^2&=\sum_{|k|\le K}\mathrm{i}\langle k,\omega_++f_+^1\rangle (h_k^2+H_k^1y_++H_k^2u_++H_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad+\sum_{|k|\le K}H_k^1(D_+u_++f_+^2)e^{\mathrm{i}\langle k,x_+\rangle}+\sum_{|k|\le K}H_k^2(C_+y_++A_+v_++f_+^3)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad+\sum_{|k|\le K}H_k^3(B_+u_++f_+^4)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},\\
\mathcal{F}_+^3&=\sum_{|k|\le K}\mathrm{i}\langle k,\omega_++f_+^1\rangle (h_k^3+I_k^1y_++I_k^2u_++I_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad+\sum_{|k|\le K}I_k^1(D_+u_++f_+^2)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}+\sum_{|k|\le K}I_k^2(C_+y_++A_+v_++f_+^3)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad+\sum_{|k|\le K}I_k^3(B_+u_++f_+^4)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},\\
\mathcal{F}_+^4&=\sum_{|k|\le K}\mathrm{i}\langle k,\omega_++f_+^1\rangle (h_k^4+J_k^1y_++J_k^2u_++J_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad+\sum_{|k|\le K}J_k^1(D_+u_++f_+^2)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}+\sum_{|k|\le K}J_k^2(C_+y_++A_+v_++f_+^3)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad+\sum_{|k|\le K}J_k^3(B_+u_++f_+^4)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}.
\end{aligned}$$

Omitting all terms of order $O(\varepsilon^2)$, we get

$$\begin{aligned}
&\mathrm{i}\langle k,\omega\rangle h_k^1=\widetilde{f}_k^1,\quad 0<|k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle h_k^2=Dh_k^3+\widetilde{f}_k^2,\quad |k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle h_k^3=Ch_k^2+Ah_k^4+\widetilde{f}_k^3,\quad 0<|k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle h_k^4=Bh_k^3+\widetilde{f}_k^4,\quad |k|\le K,\\
&Ch_0^2+Ah_0^4+\widetilde{f}_0^{\,3}=0
\end{aligned}\tag{2.18}$$

and

$$
\begin{aligned}
&\mathrm{i}\langle k,\omega\rangle H_k^1 + H_k^2 C = DI_k^1 + \Big(\frac{\partial \widetilde{f}^2}{\partial y}\Big)_k, \quad |k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle H_k^2 + H_k^1 D + H_k^3 B = DI_k^2 + \Big(\frac{\partial \widetilde{f}^2}{\partial u}\Big)_k, \quad 0<|k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle H_k^3 + H_k^2 A = DI_k^3 + \Big(\frac{\partial \widetilde{f}^2}{\partial v}\Big)_k, \quad |k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle I_k^1 + I_k^2 C = CH_k^1 + AJ_k^1 + \Big(\frac{\partial \widetilde{f}^3}{\partial y}\Big)_k, \quad 0<|k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle I_k^2 + I_k^1 D + I_k^3 B = CH_k^2 + AJ_k^2 + \Big(\frac{\partial \widetilde{f}^3}{\partial u}\Big)_k, \quad |k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle I_k^3 + I_k^2 A = CH_k^3 + AJ_k^3 + \Big(\frac{\partial \widetilde{f}^3}{\partial v}\Big)_k, \quad 0<|k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle J_k^1 + J_k^2 C = BJ_k^1 + \Big(\frac{\partial \widetilde{f}^4}{\partial y}\Big)_k, \quad |k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle J_k^2 + J_k^1 D + J_k^3 B = BJ_k^2 + \Big(\frac{\partial \widetilde{f}^4}{\partial u}\Big)_k, \quad 0<|k|\le K,\\
&\mathrm{i}\langle k,\omega\rangle J_k^3 + J_k^2 A = BJ_k^3 + \Big(\frac{\partial \widetilde{f}^4}{\partial v}\Big)_k, \quad |k|\le K,
\end{aligned}
\tag{2.19}
$$

where

$$
\begin{aligned}
\widetilde{f}_k^l &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \widetilde{f}^l(x,0,0,0,\omega)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\,\mathrm{d}x,\\
\Big(\frac{\partial \widetilde{f}^j}{\partial y}\Big)_k &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^j}{\partial y}(x,0,0,0,\omega)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\,\mathrm{d}x,\\
\Big(\frac{\partial \widetilde{f}^j}{\partial u}\Big)_k &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^j}{\partial u}(x,0,0,0,\omega)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\,\mathrm{d}x,\\
\Big(\frac{\partial \widetilde{f}^j}{\partial v}\Big)_k &= \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^j}{\partial v}(x,0,0,0,\omega)\mathrm{e}^{-\mathrm{i}\langle k,x\rangle}\,\mathrm{d}x
\end{aligned}
$$

for $1\le l\le 4$ and $2\le j\le 4$. The equations (2.18) and (2.19) are usually called homological equations. From (1.6) we have

$$
\begin{aligned}
&\widetilde{f}^1(-x,y,-u,v) = \widetilde{f}^1(x,y,u,v), \quad \widetilde{f}^2(-x,y,-u,v) = -\widetilde{f}^2(x,y,u,v),\\
&\widetilde{f}^3(-x,y,-u,v) = \widetilde{f}^3(x,y,u,v), \quad \widetilde{f}^4(-x,y,-u,v) = -\widetilde{f}^4(x,y,u,v),
\end{aligned}
\tag{2.20}
$$

which implies that

$$
\widetilde{f}_{-k}^1 = \widetilde{f}_k^1, \quad \widetilde{f}_{-k}^2 = -\widetilde{f}_k^2, \quad \widetilde{f}_{-k}^3 = \widetilde{f}_k^3, \quad \widetilde{f}_{-k}^4 = -\widetilde{f}_k^4 \tag{2.21}
$$

and

$$
\begin{aligned}
&\Big(\frac{\partial \widetilde{f}^2}{\partial y}\Big)_{-k} = -\Big(\frac{\partial \widetilde{f}^2}{\partial y}\Big)_k, \quad \Big(\frac{\partial \widetilde{f}^2}{\partial u}\Big)_{-k} = \Big(\frac{\partial \widetilde{f}^2}{\partial u}\Big)_k, \quad \Big(\frac{\partial \widetilde{f}^2}{\partial v}\Big)_{-k} = -\Big(\frac{\partial \widetilde{f}^2}{\partial v}\Big)_k,\\
&\Big(\frac{\partial \widetilde{f}^3}{\partial y}\Big)_{-k} = \Big(\frac{\partial \widetilde{f}^3}{\partial y}\Big)_k, \quad \Big(\frac{\partial \widetilde{f}^3}{\partial u}\Big)_{-k} = -\Big(\frac{\partial \widetilde{f}^3}{\partial u}\Big)_k, \quad \Big(\frac{\partial \widetilde{f}^3}{\partial v}\Big)_{-k} = \Big(\frac{\partial \widetilde{f}^3}{\partial v}\Big)_k,\\
&\Big(\frac{\partial \widetilde{f}^4}{\partial y}\Big)_{-k} = -\Big(\frac{\partial \widetilde{f}^4}{\partial y}\Big)_k, \quad \Big(\frac{\partial \widetilde{f}^4}{\partial u}\Big)_{-k} = \Big(\frac{\partial \widetilde{f}^4}{\partial u}\Big)_k, \quad \Big(\frac{\partial \widetilde{f}^4}{\partial v}\Big)_{-k} = -\Big(\frac{\partial \widetilde{f}^4}{\partial v}\Big)_k.
\end{aligned}
\tag{2.22}
$$

Therefore one has

$$\begin{aligned} &\widetilde{f}_0^2=0, \quad \widetilde{f}_0^4=0, \quad \Big(\frac{\partial \widetilde{f}^2}{\partial y}\Big)_0=0, \quad \Big(\frac{\partial \widetilde{f}^2}{\partial v}\Big)_0=0, \\ &\Big(\frac{\partial \widetilde{f}^3}{\partial u}\Big)_0=0, \quad \Big(\frac{\partial \widetilde{f}^4}{\partial y}\Big)_0=0, \quad \Big(\frac{\partial \widetilde{f}^4}{\partial v}\Big)_0=0. \end{aligned} \tag{2.23}$$

Let

$$\begin{aligned} &\omega_+=\omega+\widetilde{f}_0^1, \quad A_+=A+\Big(\frac{\partial \widetilde{f}^3}{\partial v}\Big)_0, \quad B_+=B+\Big(\frac{\partial \widetilde{f}^4}{\partial u}\Big)_0, \\ &C_+=C+\Big(\frac{\partial \widetilde{f}^3}{\partial y}\Big)_0, \quad D_+=D+\Big(\frac{\partial \widetilde{f}^2}{\partial u}\Big)_0. \end{aligned} \tag{2.24}$$

Now we solve the homological equations (2.18) and (2.19).

For $k=0$, we choose

$$\begin{aligned} &h_0^1=0, \quad h_0^2=0, \quad h_0^3=0, \\ &H_0^1=0, \quad H_0^2=0, \quad H_0^3=0, \\ &I_0^1=0, \quad I_0^2=0, \quad I_0^3=0, \\ &J_0^1=0, \quad J_0^2=0, \quad J_0^3=0. \end{aligned}$$

From condition (2), we know that $\widetilde{A}=(a_{ij})_{1\le i,j\le p}$ has an inverse matrix $\widetilde{A}^{-1}$. We can define

$$h_0^4=\mathrm{col.}\Big(-[(\widetilde{A})^{-1}\widetilde{f}_0^3]^T, \underbrace{0,\cdots,0}_{q-p}\Big).$$

If $\omega\notin\mathcal{R}_1\cup\mathcal{R}_2\cup\mathcal{R}_3$, we have

$$h_k^1=\mathrm{i}\langle k,\omega\rangle^{-1}\widetilde{f}_k^1, \qquad \begin{pmatrix} h_k^2\\ h_k^3\\ h_k^4\end{pmatrix}=(\mathrm{i}\langle k,\omega\rangle\mathbb{E}_N-\mathcal{A})^{-1}\begin{pmatrix}\widetilde{f}_k^2\\ \widetilde{f}_k^3\\ \widetilde{f}_k^4\end{pmatrix} \tag{2.25}$$

and

$$(\mathrm{i}\langle k,\omega\rangle\mathbb{E}_N-\mathcal{A})\widetilde{Z}+\widetilde{Z}\mathcal{A}=\mathcal{F}, \tag{2.26}$$

where

$$\widetilde{Z}=\begin{pmatrix} H_k^1 & H_k^2 & H_k^3\\ I_k^1 & I_k^2 & I_k^3\\ J_k^1 & J_k^2 & J_k^3\end{pmatrix}, \quad \mathcal{F}=\begin{pmatrix} \big(\frac{\partial \widetilde{f}^2}{\partial y}\big)_k & \big(\frac{\partial \widetilde{f}^2}{\partial u}\big)_k & \big(\frac{\partial \widetilde{f}^2}{\partial v}\big)_k\\ \big(\frac{\partial \widetilde{f}^3}{\partial y}\big)_k & \big(\frac{\partial \widetilde{f}^3}{\partial u}\big)_k & \big(\frac{\partial \widetilde{f}^3}{\partial v}\big)_k\\ \big(\frac{\partial \widetilde{f}^4}{\partial y}\big)_k & \big(\frac{\partial \widetilde{f}^4}{\partial u}\big)_k & \big(\frac{\partial \widetilde{f}^4}{\partial v}\big)_k\end{pmatrix}. \tag{2.27}$$

### 2.4 The Commutability of the function $\phi$ with the involution $G$

In this subsection, we prove that $\phi$ commutes with the involution $G$. Such a property guarantees that the transformed system (2.16) is also reversible with respect to the involution $G:(x_+,y_+,u_+,v_+)\mapsto(-x_+,y_+,-u_+,v_+)$. We know that $\phi\circ G=G\circ\phi$ holds if and only if

$$h_{-k}^1=-h_k^1, \quad h_{-k}^2=h_k^2, \quad h_{-k}^3=-h_k^3, \quad h_{-k}^4=h_k^4 \tag{2.28}$$

and

$$\begin{pmatrix} H_k^1 & H_k^2 & H_k^3 \\ I_k^1 & I_k^2 & I_k^3 \\ J_k^1 & J_k^2 & J_k^3 \end{pmatrix} = \begin{pmatrix} H_{-k}^1 & -H_{-k}^2 & H_{-k}^3 \\ -I_{-k}^1 & I_{-k}^2 & -I_{-k}^3 \\ J_{-k}^1 & -J_{-k}^2 & J_{-k}^3 \end{pmatrix}. \tag{2.29}$$

By (2.21) and (2.18), we obtain

$$\begin{aligned} &\mathrm{i}\langle -k,\omega\rangle h_{-k}^1 = \widetilde{f}_{-k}^1 = \widetilde{f}_k^1, \\ &\mathrm{i}\langle -k,\omega\rangle h_{-k}^2 = Dh_{-k}^3 + \widetilde{f}_{-k}^2 = Dh_{-k}^3 - \widetilde{f}_k^2, \\ &\mathrm{i}\langle -k,\omega\rangle h_{-k}^3 = Ch_{-k}^2 + Ah_{-k}^4 + \widetilde{f}_{-k}^3 = Ch_{-k}^2 + Ah_{-k}^4 + \widetilde{f}_k^3, \\ &\mathrm{i}\langle -k,\omega\rangle h_{-k}^4 = Bh_{-k}^3 + \widetilde{f}_{-k}^4 = Bh_{-k}^3 - \widetilde{f}_k^4. \end{aligned}$$

The last three equalities imply that

$$(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A}) \begin{pmatrix} h_{-k}^2 \\ -h_{-k}^3 \\ h_{-k}^4 \end{pmatrix} = \begin{pmatrix} \widetilde{f}_k^2 \\ \widetilde{f}_k^3 \\ \widetilde{f}_k^4 \end{pmatrix}.$$

Hence if $\omega \notin \mathcal{R}_1 \cup \mathcal{R}_2$, $(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A})^{-1}$ exits and (2.28) holds.

Similarly, from (2.22) and the homological equation (2.19), it follows that

$$(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A})Z + Z\mathcal{A} = \mathcal{F},$$

where

$$Z = \begin{pmatrix} H_{-k}^1 & -H_{-k}^2 & H_{-k}^3 \\ -I_{-k}^1 & I_{-k}^2 & -I_{-k}^3 \\ J_{-k}^1 & -J_{-k}^2 & J_{-k}^3 \end{pmatrix}.$$

If the matrix equation (2.26) has a unique solution, then $Z = \widetilde{Z}$, which is equivalent to the equation (2.29).

The following lemma gives the uniqueness of the solution to the matrix equation (2.26).

**Lemma 2.4** *The matrix equation* (2.26) *has a unique solution $Z$ if and only if the matrix $\mathbb{E}_N \otimes (\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A})^T + \mathcal{A} \otimes \mathbb{E}_N$ is nonsingular. Moreover, in this case one has*

$$\|Z\| \le \|(\mathbb{E}_N \otimes (\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A})^T + \mathcal{A} \otimes \mathbb{E}_N)^{-1}\| \cdot \|\mathcal{F}\|,$$

*where $\|\cdot\|$ is an operator-norm of matrices.*

The proof of this lemma can be found in [18]. Therefore, if $\omega \notin \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, we have $\phi \circ G = G \circ \phi$ and $\Phi \circ G = G \circ \Phi$.

## 2.5 Estimates $\phi$ and $\phi^{-1}$ and their derivatives

We will show that $\phi$ is well-defined and close to the identity on the set $\mathcal{D}_{r_+,s_+} \times \mathcal{W}_{h_+}(K_+)$.

In the sequel, '$\prec$' and '$\ll$' respectively stand for '$< c$' and '$< c^{-1}$' with a large positive constant $c$ independent of iteration steps. An $N \times M$ matrix $C = (c_{ij})$ can be viewed as a linear operator from $\mathbb{R}^M$ to $\mathbb{R}^N$. We denote $\|C\| = \max\limits_{i,j} |c_{ij}|$.

In order to obtain the estimates of the solutions to the homological equations (2.18) and (2.19), we need the following two lemmas. The proofs of these two lemmas can be found in [18].

**Lemma 2.5** *If $\omega \in \Theta_\gamma(K)$, then for any $k$ with $0 < |k| \le K$, one has*

$$\|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A}(\omega))^{-1}\|, \quad \|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}^T(\omega) + \mathcal{A}(\omega)\otimes \mathbb{E}_N)^{-1}\| \prec \gamma^{-1}|k|^{N^2+\tau}.$$

**Lemma 2.6** *Let $\mathcal{W}_h(K)$ be an open complex neighborhood of radius $h$ of $\Theta_\gamma(K)$ with respect to the sup-norm in $\mathbb{C}^n$. Then for any $\omega \in \mathcal{W}_h(K)$ we have*

$$\begin{aligned}&|(\mathrm{i}\langle k,\omega\rangle)^{-1}|, \quad \|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A}(\omega))^{-1}\|,\\ &\|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}^T(\omega) + \mathcal{A}(\omega)\otimes \mathbb{E}_N)^{-1}\| \prec \gamma^{-1}|k|^{N^2+\tau},\end{aligned} \tag{2.30}$$

*for $0 < |k| \le K$ provided that*

$$hK^{N^2+\tau}\max\{K, M+1\} \ll \gamma.$$

**Remark 2.3** If

$$\lambda < \frac{1}{(N^2+1)(N^2+\tau+1)}, \quad \varepsilon_0^{\frac{1}{N^2+1}-\lambda(N^2+\tau+1)} \ll \frac{\gamma_0}{M_0}$$

and

$$(c\varepsilon_0)^{(\kappa-1)(\frac{1}{N^2+1}-\lambda(N^2+\tau+1))} < \frac{1}{3},$$

we have $hK^{N^2+\tau}\max\{K, M+1\} \ll \gamma$.

Now we estimate the solutions to the equations (2.18) and (2.19). Since $\widetilde{f}^l$ are analytic and $\|\widetilde{f}^l\|_{C^0}$ are small, we have

$$\Big\|\frac{\partial \widetilde{f}^j}{\partial y}(x_+,0,0,0)\Big\| \le \varepsilon, \quad \Big\|\frac{\partial \widetilde{f}^j}{\partial u}(x_+,0,0,0)\Big\| \le \varepsilon, \quad \Big\|\frac{\partial \widetilde{f}^j}{\partial v}(x_+,0,0,0)\Big\| \le \varepsilon, \quad 2 \le j \le 4$$

from Cauchy's estimate. Hence we have

$$|\widetilde{f}^1_k| \le \varepsilon \mathrm{e}^{-|k|r}, \quad |\widetilde{f}^j_k| \le \varepsilon s \mathrm{e}^{-|k|r}, \quad 2 \le j \le 4$$

and

$$\Big|\Big(\frac{\partial \widetilde{f}^j}{\partial y}\Big)_k\Big| \le \varepsilon \mathrm{e}^{-|k|r}, \quad \Big|\Big(\frac{\partial \widetilde{f}^j}{\partial u}\Big)_k\Big| \le \varepsilon \mathrm{e}^{-|k|r}, \quad \Big|\Big(\frac{\partial \widetilde{f}^j}{\partial v}\Big)_k\Big| \le \varepsilon \mathrm{e}^{-|k|r}, \quad 2 \le j \le 4$$

for $(x,y,u,v,\omega) \in \mathcal{D}_{r,s}$. By (2.25), (2.27) and (2.30), we obtain, for $\omega \in \mathcal{W}_h(K)$ and $0 < |k| \le K$,

$$\begin{aligned}&|h^1_k| < |\mathrm{i}\langle k,\omega\rangle|^{-1}\cdot|\widetilde{f}^1_k|, \quad |h^j_k| < \|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A})^{-1}\|\cdot|\widetilde{f}^j_k|, \quad 2 \le j \le 4,\\ &\|H^j_k\| < \|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathcal{A}\otimes\mathbb{E}_N + \mathbb{E}_N\otimes\mathcal{A})^{-1}\|\cdot\Big|\Big(\frac{\partial \widetilde{f}^{j+1}}{\partial y}\Big)_k\Big|, \quad 1 \le j \le 3,\\ &\|I^j_k\| < \|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathcal{A}\otimes\mathbb{E}_N + \mathbb{E}_N\otimes\mathcal{A})^{-1}\|\cdot\Big|\Big(\frac{\partial \widetilde{f}^{j+1}}{\partial u}\Big)_k\Big|, \quad 1 \le j \le 3,\\ &\|J^j_k\| < \|(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathcal{A}\otimes\mathbb{E}_N + \mathbb{E}_N\otimes\mathcal{A})^{-1}\|\cdot\Big|\Big(\frac{\partial \widetilde{f}^{j+1}}{\partial v}\Big)_k\Big|, \quad 1 \le j \le 3.\end{aligned} \tag{2.31}$$

For $k=0$, we have

$$\begin{aligned}&|h_0^1|=|h_0^2|=|h_0^3|=0,\\&|h_0^4|\prec\varepsilon s,\\&\|H_0^j\|=\|I_0^j\|=\|J_0^j\|=0,\quad 1\le j\le 3.\end{aligned}\tag{2.32}$$

Denote $\mathcal{D}_j=\mathcal{D}_{r_++\frac{1}{2}j\rho,\frac{1}{4}(j+1)s}$, $0\le j\le 3$, where $\rho=\frac{r-r_+}{2}>0$. Then by (2.31) and (2.32), we have

$$\begin{aligned}&\Big\|\sum_{|k|\le K}h_k^4\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}<|h_0^4|+\sum_{0<|k|\le K}\|(\mathrm{i}\langle k,\omega\rangle\mathbb{E}_N-\mathcal{A})^{-1}\|\cdot\|\widetilde{f}^4\|\cdot\mathrm{e}^{|k|\frac{(r_+-r)}{2}},\\&\Big\|\sum_{|k|\le K}h_k^j\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}<\sum_{0<|k|\le K}\|(\mathrm{i}\langle k,\omega\rangle\mathbb{E}_N-\mathcal{A})^{-1}\|\cdot\|\widetilde{f}^j\|\cdot e^{|k|\frac{(r_+-r)}{2}},\quad j\ne 4,\\&\|(\mathbb{E}_N-\mathcal{A})^{-1}\|\prec\gamma^{-1}|k|^{N^2+\tau},\quad\forall\,0<|k|<K.\end{aligned}\tag{2.33}$$

By (2.33) and Lemma 2.6, we get

$$\begin{aligned}&\Big\|\sum_{|k|\le K}h_k^1\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\varepsilon\sum_{j=1}^{K}\gamma^{-1}|j|^{N^2+\tau}|j|^{n-1}(\mathrm{e}^{|j|\frac{(r_+-r)}{2}})\prec\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)},\\&\Big\|\sum_{|k|\le K}h_k^j\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\varepsilon s\sum_{j=1}^{K}\gamma^{-1}|j|^{N^2+\tau}|j|^{n-1}(\mathrm{e}^{-|j|\frac{(r_+-r)}{2}})\prec\gamma^{-1}s\varepsilon^{1-\zeta(N^2+\tau+n)}.\end{aligned}$$

Similarly, one can get

$$\begin{aligned}&\Big\|\sum_{|k|\le K}(H_k^1y_++H_k^2u_++H_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\gamma^{-1}s\varepsilon^{1-\zeta(N^2+\tau+n)},\\&\Big\|\sum_{|k|\le K}(I_k^1y_++I_k^2u_++I_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\gamma^{-1}s\varepsilon^{1-\zeta(N^2+\tau+n)},\\&\Big\|\sum_{|k|\le K}(J_k^1y_++J_k^2u_++J_k^3v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\Big\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\gamma^{-1}s\varepsilon^{1-\zeta(N^2+\tau+n)}.\end{aligned}$$

In conclusion, we get the following results.

**Lemma 2.7** *Denote $\Xi=\mathrm{Diag}(\mathbb{E}_n,s^{-1}\mathbb{E}_n,s^{-1}\mathbb{E}_p,s^{-1}\mathbb{E}_q)$. Then we have*

$$\begin{aligned}&\|\Xi(\phi-\mathrm{id})\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)},\\&\|\Xi(D\phi-\mathbb{E}_{n+N})\Xi^{-1}\|_{\mathcal{D}_2\times\mathcal{W}_h(K)}\prec\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n+1)}.\end{aligned}$$

*Moreover, if $\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)}<\min\{\frac{1}{2}\rho,\frac{1}{4}\}$ (This can be satisfied when $\varepsilon_0$ is small enough and $\zeta<\frac{1}{N^2+\tau+n+1}$), then*

$$\phi(\mathcal{D}_2\times\mathcal{W}_h(K))\subset\mathcal{D}_3.$$

By (2.24) we know that

$$\begin{aligned}\omega_+ &= \omega + \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \widetilde{f}^1(x,0,0,0)\,\mathrm{d}x,\\ A_+ &= A + \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^3}{\partial v}(x,0,0,0)\,\mathrm{d}x,\\ B_+ &= B + \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^4}{\partial u}(x,0,0,0)\,\mathrm{d}x,\\ C_+ &= C + \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^3}{\partial y}(x,0,0,0)\,\mathrm{d}x,\\ D_+ &= D + \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} \frac{\partial \widetilde{f}^2}{\partial u}(x,0,0,0)\,\mathrm{d}x.\end{aligned}$$

Therefore

$$|\omega_+ - \omega|,\ |a_{ij}^+ - a_{ij}|,\ |b_{ij}^+ - b_{ij}|,\ |c_{ij}^+ - c_{ij}|,\ |d_{ij}^+ - d_{ij}| \le \varepsilon.$$

From Cauchy's estimates it follows that for $\omega \in \mathcal{W}_{\frac{h}{4}}$ and $|l| \le N^2$,

$$\begin{aligned}&h^{|l|}\Big|\frac{\partial^l(\omega_+ - \omega)}{\partial \omega^l}\Big| \prec \varepsilon,\\ &h^{|l|}\Big|\frac{\partial^l(a_{ij}^+ - a_{ij})}{\partial \omega^l}\Big|,\ h^{|l|}\Big|\frac{\partial^l(b_{ij}^+ - b_{ij})}{\partial \omega^l}\Big| \prec \varepsilon\\ &h^{|l|}\Big|\frac{\partial^l(c_{ij}^+ - c_{ij})}{\partial \omega^l}\Big|,\ h^{|l|}\Big|\frac{\partial^l(d_{ij}^+ - d_{ij})}{\partial \omega^l}\Big| \prec \varepsilon.\end{aligned}$$

From these inequalities and (2.36) below, it follows that

$$\begin{aligned}&\Big|\frac{\partial^l(\omega_+ - \omega)}{\partial \omega_+^l}\Big| \prec \varepsilon h^{-|l|},\\ &\Big|\frac{\partial^l a_{ij}^+}{\partial \omega_+^l} - M\Big|,\ \Big|\frac{\partial^l b_{ij}^+}{\partial \omega_+^l} - M\Big| \prec \varepsilon h^{-|l|},\\ &\Big|\frac{\partial^l c_{ij}^+}{\partial \omega_+^l} - M\Big|,\ \Big|\frac{\partial^l d_{ij}^+}{\partial \omega_+^l} - m\Big| \prec \varepsilon h^{-|l|}.\end{aligned}$$

Let $\widetilde{A}_+ = (a_{ij}^+)_{1\le i,j\le p}$ and

$$\mathcal{A}_+ = \begin{pmatrix} 0 & D_+ & 0\\ C_+ & 0 & A_+\\ 0 & B_+ & 0\end{pmatrix}.$$

Then

$$|\det \widetilde{A}_+| \ge |\det \widetilde{A}| - p!(M+\varepsilon)^{p-1}\cdot 2^p\varepsilon > \frac{1}{2}|\det \widetilde{A}| > 0$$

provided $\varepsilon < \frac{1}{2}(p!2^p(M+1)^p)^{-1}|\det \widetilde{A}|$.

Define $C_\Theta = \sup\limits_{\omega\in\Theta}|\omega| + 1$. If $K^{N+\tau}\varepsilon < (\gamma-\gamma_+)(N!(2C_\Theta(M+1))^N)^{-1}$, for $0<|k|\le K$ the following holds

$$|\det(\mathrm{i}\langle k,\omega_+\rangle \mathbb{E}_N - \mathcal{A}_+)| > |\det(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A})| - N!2^N(|k|+1)(|k|C_\Theta(M+1))^{N-1}\varepsilon > \gamma_+|k|^{-\tau}.$$

Similarly, if

$$K^{N^2+\tau}\varepsilon < (\gamma-\gamma_+)(N^2!(2C_\Theta(M+1))^{N^2})^{-1},$$

then

$$|\det(\mathrm{i}\langle k,\omega_+\rangle\mathbb{E}_{N^2}-\mathbb{E}_N\otimes\mathcal{A}_+^T+\mathcal{A}_+\otimes\mathbb{E}_N)|>\gamma_+|k|^{-\tau}$$

and

$$|\langle k,\omega_+\rangle|>\gamma_+|k|^{-\tau}$$

for $0<|k|\le K$. Therefore we have proved the following lemma.

**Lemma 2.8** *Suppose*

$$K^{N^2+\tau}\varepsilon < (N^2!(2C_\Theta(M+1))^{N^2})^{-1}\min\{(\gamma-\gamma_+),|\det\widetilde{A}|\}.$$

*Then for* $0<|k|\le K$,

$$\begin{aligned}
&\det\widetilde{A}_+\ne 0,\\
&|\langle k,\omega_+\rangle|>\gamma_+|k|^{-\tau},\\
&|\det(\mathrm{i}\langle k,\omega_+\rangle\mathbb{E}_N-\mathcal{A}_+)|>\gamma_+|k|^{-\tau},\\
&|\det(\mathrm{i}\langle k,\omega_+\rangle\mathbb{E}_{N^2}-\mathbb{E}_N\otimes\mathcal{A}_+^T+\mathcal{A}_+\otimes\mathbb{E}_N)|>\gamma_+|k|^{-\tau}.
\end{aligned}$$

**Remark 2.4** If

$$\begin{aligned}
&\varepsilon_0^{1-\lambda(N^2+\tau)}\ll\frac{\gamma_0}{4},\quad \lambda<\frac{1}{N^2+\tau},\\
&(c\varepsilon_0)^{(\kappa-1)(1-\lambda(N^2+\tau))}<\frac{1}{2},
\end{aligned}$$

then for any $\nu\in\mathbb{N}$, we have

$$K^{N^2+\tau}\varepsilon < (N^2!(2C_\Theta(M+1))^{N^2})^{-1}\min\{(\gamma-\gamma_+),|\det\widetilde{A}|\}.$$

Since $f^l$ $(1\le l\le 4)$ in (1.5) are defined in the domain $\mathcal{D}_{r,s}\times\mathcal{W}_h(K)$, we have to show that when $\varepsilon$ is sufficiently small,

$$\phi:\ \mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)\to\mathcal{D}_{r,s},\quad (x_+,y_+,u_+,v_+)\mapsto(x,y,u,v), \tag{2.34}$$

$$\psi:\ \mathcal{W}_{h_+}(K_+)\to\mathcal{W}_h(K_+),\quad \omega_+\mapsto\omega, \tag{2.35}$$

where the mappings $\phi$ and $\psi$ are defined by (2.15) and the first equality in (2.24) respectively. Lemma 2.7 leads to (2.34), and (2.35) is a direct consequence of the following lemma.

**Lemma 2.9** *Suppose that $f$ is a real analytic mapping from $\mathcal{W}_h(K)$ into $\mathbb{C}^n$. If*

$$|f-\mathrm{id}|_{\mathcal{W}_h(K)}\le\delta<\frac{h}{4},$$

*then $f$ has a real analytic inverse $\psi$ on $\mathcal{W}_{\frac{h}{4}}(K)$. Moreover,*

$$|\psi-\mathrm{id}|_{\mathcal{W}_{\frac{h}{4}}(K)},\quad \frac{h}{4}\|D\psi-\mathbb{E}_n\|_{\mathcal{W}_{\frac{h}{4}}(K)}\le\delta.$$

The proof of this lemma can be found in [11]. From this lemma it follows that if

$$\varepsilon < \frac{h}{4}, \quad h_+ < \frac{h}{4},$$

then $\psi : \omega_+ \mapsto \omega$ is well-defined on $\mathcal{W}_{h_+}(K_+)$ and $\psi(\mathcal{W}_{h_+}(K_+)) \subset \mathcal{W}_h(K)$. Moreover, let

$$\psi(\omega_+) = \omega_+ + \eta(\omega_+).$$

Then

$$\|\eta(\omega_+)\|_{\mathcal{W}_{\frac{h}{4}}(K)} < \varepsilon.$$

By Cauchy's estimates, we have

$$\Big\|\frac{\partial^l \eta}{\partial \omega_+^l}\Big\|_{\mathcal{W}_{\frac{h}{8}}(K)} \prec \varepsilon h^{-|l|}, \quad |l| \le N^2. \tag{2.36}$$

To finish one step of KAM iteration, we have to estimate the new perturbation terms $f_+^l$, $\widetilde{f}_+^l$ $(1 \le l \le 4)$ in (2.10) and (2.11).

## 2.6 Estimates of new perturbation terms $f_+^l$ and $\widetilde{f}_+^l$ $(1 \le l \le 4)$

(a) Estimates of $f_+^1$ and $\widetilde{f}_+^1$

Using the first equation in (2.17) and (2.11), we have

$$f_+^1(x_+, y_+, u_+, v_+) = \widetilde{f}^1 \circ \phi + \omega - \omega_+ - \sum_{|k| \le K} \mathrm{i}\langle k, \omega_+ + f_+^1\rangle \mathrm{e}^{\mathrm{i}\langle k, x_+\rangle} h_k^1.$$
$$\widetilde{f}_+^l = f_+^l - (D\Phi_+)^{-1}(f_{\sigma_+}^l - f_{\sigma_{++}}^l) \circ \Phi_+.$$

From (2.25) and $h_0^1 = 0$, we have

$$\begin{aligned} f_+^1 &= \widetilde{f}^1 \circ \phi - \widetilde{f}_0^1 - \sum_{0<|k|\le K} \widetilde{f}_k^1 e^{\mathrm{i}\langle k, x_+\rangle} - \sum_{|k|\le K} \mathrm{i}\langle k, \widetilde{f}_0^1 + f_+^1\rangle \mathrm{e}^{\mathrm{i}\langle k, x_+\rangle} h_k^1 \\ &= \widetilde{f}^1 \circ \phi - T_K \widetilde{f}^1(x_+, 0, 0, 0) - \sum_{|k|\le K} \mathrm{i}\langle k, \widetilde{f}_0^1 + f_+^1\rangle \mathrm{e}^{\mathrm{i}\langle k, x_+\rangle} h_k^1, \end{aligned}$$

where $T_K \widetilde{f}^1(x_+, 0, 0, 0) = \sum_{|k|\le K} \widetilde{f}_k^1 \mathrm{e}^{\mathrm{i}\langle k, x\rangle}$. Therefore

$$\begin{aligned} \|f_+^1\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)} &\le \|\widetilde{f}^1 \circ \phi - T_K \widetilde{f}^1(x_+, 0, 0, 0)\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)} \\ &\quad + (|\widetilde{f}_0^1| + \|f_+^1\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)}) \sum_{|k|\le K} |k||h_k^1| e^{|k| r_+} \\ &\prec \|\widetilde{f}^1 \circ \phi - T_K \widetilde{f}^1(x_+, 0, 0, 0)\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)} \\ &\quad + (\varepsilon + \|f_+^1\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)}) \gamma^{-1} \varepsilon^{1 - \zeta(N^2 + \tau + n + 1)}. \end{aligned}$$

If $\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n+1)} \ll 1$, then

$$\|f_+^1\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)} \prec \|\widetilde{f}^1 \circ \phi - T_K \widetilde{f}^1(x_+, 0, 0, 0)\|_{\mathcal{D}_{r_+, s_+} \times \mathcal{W}_{h_+}(K_+)} + \gamma^{-1} \varepsilon^{2 - \zeta(N^2 + \tau + n + 1)}.$$

Now we give an estimate of $\|\widetilde{f}^1\circ\phi - T_K\widetilde{f}^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}$. Set

$$\begin{aligned}F_1^1 &= \widetilde{f}^1\circ\phi - \widetilde{f}^1(x_+,y_+,u_+,v_+),\\ F_2^1 &= \widetilde{f}^1(x_+,y_+,u_+,v_+) - \widetilde{f}^1(x_+,0,0,0),\\ F_3^1 &= \widetilde{f}^1(x_+,0,0,0) - T_K\widetilde{f}^1(x_+,0,0,0),\end{aligned}$$

and we have

$$\widetilde{f}^1\circ\phi - T_K\widetilde{f}^1(x_+,0,0,0) = F_1^1 + F_2^1 + F_3^1.$$

If $(c\varepsilon_0)^{(\kappa-1)(2-\kappa-\zeta(N^2+n+\tau+1))} < \frac{1}{2}$, $\varepsilon_0^{2-\kappa-\zeta(N^2+n+\tau+1)} \prec \gamma_0^{-1}$ and $\zeta < \frac{2-\kappa}{N^2+\tau+n+1}$, we have, by Lemma 2.7 and the Cauchy's estimates,

$$\begin{aligned}\|F_1^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}} &\prec \max\Big\{\frac{\|f^1\|_{\mathcal{D}_{r,s}\times\mathcal{W}_h(K)}}{r-r_+}\|\phi-\mathrm{id}\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)},\\ &\qquad\qquad \frac{\|f^1\|_{\mathcal{D}_{r,s}\times\mathcal{W}_h(K)}}{s-s_+}\|\phi-\mathrm{id}\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\Big\}\\ &\prec \max\Big\{\frac{\varepsilon^{2-\zeta(N^2+\tau+n)}}{r-r_+}\gamma^{-1}, \frac{\varepsilon^{2-\zeta(N^2+\tau+n)}s_+}{s-s_+}\gamma^{-1}\Big\}\\ &\prec \max\Big\{\varepsilon^{2-\zeta(N^2+\tau+n+1)}\gamma^{-1}, \varepsilon^{2-\zeta(N^2+\tau+n)+(\kappa-1)}\gamma^{-1}\Big\}\\ &\prec \varepsilon^{\kappa},\\ \|F_2^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)} &\le \frac{1}{s-s_+}\varepsilon s_+ < \varepsilon\alpha \le \varepsilon^{\kappa},\\ \|F_3^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)} &\le \sum_{|k|\ge K}\|\widetilde{f}_k^1 \mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\\ &\le \sum_{|k|\ge K}|\widetilde{f}_k^1|\mathrm{e}^{|k|r_+} \le \varepsilon\sum_{|k|\ge K}\mathrm{e}^{-|k|(r-r_+)}\\ &\prec \varepsilon\sum_{j\ge K} j^{n-1}\mathrm{e}^{-j(r-r_+)} \le \varepsilon\int_K^\infty x^{n-1}\mathrm{e}^{-(r-r_+)x}\,\mathrm{d}x\\ &\prec \varepsilon^2.\end{aligned}$$

Now we get

$$\|\widetilde{f}^1\circ\phi - T_K\widetilde{f}^1\| \prec \varepsilon^{\kappa},$$

that is,

$$\|f_+^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}} \prec \varepsilon^{\kappa}.$$

Before we prove $\|\widetilde{f}_+^1\| < \varepsilon_+$, we need to prove $\|D\Phi_\nu^{-1}\| < 2$ and $\|D\Psi_\nu^{-1}\| < 2$ by induction. Suppose

$$\varepsilon_0^{1-\zeta(N^2+\tau+n+1)}\gamma_0^{-1} \ll \frac{1}{16(N+n)} \quad \text{and} \quad \Phi_0 = \mathrm{id}.$$

When $\nu = 0$, $\|D\Phi_0\| < 2$. If $\|D\Phi_j\| < 2$ for all $j \le \nu - 1$, we have

$$\begin{aligned}\|D\Phi_\nu - \mathbb{E}_{N+n}\| &\le \sum_{j=1}^{\nu-1}\|D\Phi_j\cdot D\phi_j - D\Phi_j\|\\ &\prec 2(N+n)\sum_{j=1}^{\nu-1}\varepsilon_j^{1-\zeta(N^2+\tau+n+1)}\gamma_j^{-1}\end{aligned}$$

$$\prec 2(N+n)\sum_{j=1}^{\nu-1}\varepsilon_0^{1-\zeta(N^2+\tau+n+1)\kappa^j}\gamma_j^{-1}$$
$$\prec 8(N+n)\varepsilon_0^{\kappa(1-\zeta(N^2+\tau+n+1))}\gamma_0^{-1}<\frac{1}{2}.$$

Therefore, $\|D\Phi_\nu\|<2$ for any $\nu$. If $\gamma_0^{-1}\varepsilon_0^{\frac{N^2}{N^2+1}-\zeta(N^2+n+\tau+2)}\ll\frac{1}{8(N+n)}$, we get

$$\begin{aligned}\Big\|\frac{\partial\Phi_\nu}{\partial\omega_\nu}\Big\|_{\mathcal{W}_{h_{\nu+1}}}&\le(N+n)\|D\Phi_{\nu-1}\|\cdot\Big\|\frac{\partial\phi_{\nu-1}}{\partial\omega_\nu}\Big\|_{\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)}\\&\quad+(N+n)\Big\|\frac{\partial\Phi_{\nu-1}}{\partial\omega_{\nu-1}}\Big\|\cdot\Big\|\frac{\partial\psi_{\nu-1}}{\partial\omega_\nu}\Big\|_{\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)}\\&\prec 2(N+n)\big(\gamma_{\nu-1}^{-1}\varepsilon_{\nu-1}^{\frac{N^2}{N^2+1}-\zeta(N^2+n+\tau+1)}+\varepsilon_{\nu-1}^{\frac{N^2}{N^2+1}}\big)\\&\le 2(N+n)\big(\varepsilon_{\nu-1}^{\zeta}+2\varepsilon_{\nu-1}^{\frac{N^2}{N^2+1}}\big)\le\frac{1}{2}.\end{aligned}$$

So $\|\frac{\partial\Phi_\nu}{\partial\omega_\nu}\|<2$ is proved for any $\nu$. We can also prove $\|\frac{\partial\Psi_\nu}{\partial\omega_\nu}\|<2$ by the same method.

We know that (2.10) is well-defined, if $(\Phi_\nu(\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)),\Psi_\nu(\mathcal{W}_{h_\nu}(K_\nu)))\subset\Delta_{\sigma_\nu}$. Since $r_\nu>s_\nu$, $2(N+n)r_\nu=\sigma_\nu$ and $\zeta<\frac{1}{N^2+1}$, we have

$$\begin{aligned}\|\mathrm{Im}\Phi_\nu(Z,\omega)\|_{\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)}&=\|\mathrm{Im}\Phi_\nu(Z,\omega)-\mathrm{Im}\Phi_\nu(\mathrm{Re}(Z,\omega))\|_{\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)}\\&\le(N+n)\max\Big\{\|D\Phi_\nu\|\cdot\|\mathrm{Im}Z\|,\Big\|\frac{\partial\Phi_\nu}{\partial\omega}\Big\|\cdot\|\mathrm{Im}\omega\|\Big\}\\&\le\max\{2(N+n)r_\nu,2(N+n)s_\nu,2(N+n)h_\nu\}\le\sigma_\nu,\\\|\mathrm{Im}\Psi_\nu(\omega)\|_{\mathcal{W}_{h_\nu}}&=\|\mathrm{Im}\Psi_\nu(\omega)-\mathrm{Im}\Psi_\nu(\mathrm{Re}(\omega))\|_{\mathcal{W}_{h_\nu}}\\&\le(N+n)\|D\Psi_\nu\|\cdot\|\mathrm{Im}(\omega)\|\\&\le 2(N+n)h_\nu\le\sigma_\nu.\end{aligned}$$

Therefore $(\Phi_\nu(\mathcal{D}_{r_\nu,s_\nu}\times\mathcal{W}_{h_\nu}(K_\nu)),\Psi_\nu(\mathcal{W}_{h_\nu}(K_\nu)))\subset\Delta_{\sigma_\nu}$ and (2.10) is well-defined.

From

$$\|D\Phi_\nu^{-1}-\mathbb{E}_{n+N}\|\le\sum_{j=1}^{\infty}\|D\Phi_\nu-\mathbb{E}_{n+N}\|^j\le 2\|D\Phi_\nu-\mathbb{E}_{n+N}\|<1,$$

we get $\|D\Phi_\nu^{-1}\|<2$.

Now we estimate $\widetilde{f}_+^1$. If $\zeta d>2$, $\varepsilon_0^{d\zeta-1}\le[2\widetilde{c}(2(N+n))^{d+1}\|f^1\|_{C^d}]^{-1}$ and $c^{d\zeta-1}\varepsilon_0^{d\zeta-2}<(\frac{1}{2})^{\frac{1}{\kappa-1}}$, then by Lemma 2.1

$$\begin{aligned}\|D\Phi_+^{-1}(f_{\sigma_+}^1-f_{\sigma_{++}}^1)\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}&<2(N+n)\|f_{\sigma_+}^1-f_{\sigma_{++}}^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\\&<2(N+n)\widetilde{c}\|f^1\|_{C^d}\sigma_+^d\\&=2^{d+1}(N+n)^{d+1}\widetilde{c}\|f^1\|_{C^d}\varepsilon_+^{d\zeta}<\frac{\varepsilon_+}{2}.\end{aligned}\tag{2.37}$$

At last we get $\|\widetilde{f}_+^1\|_{\mathcal{D}_+\times\mathcal{W}_{h_+}(K_+)}<\varepsilon_+$.

(b) Estimate of $f_+^i$ and $\widetilde{f}_+^i$ $(i=2,3,4)$

From the last three equalities of (2.17) and the homological equation (2.19), it follows that

$$\begin{aligned}
f_+^2 &= \widetilde{f}^2\circ\phi + (D-D_+)u_+ + \sum_{|k|\le K}(Dh_k^3 + DI_k^1 y_+ + DI_k^2 u_+ + DI_k^3 v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad - \sum_{|k|\le K}\mathrm{i}\langle k,\omega\rangle(h_k^2 + H_k^1 y_+ + H_k^2 u_+ + H_k^3 v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad - \sum_{|k|\le K}\mathrm{i}\langle k,\widetilde{f}_0^1 + f_+^1\rangle(h_k^2 + H_k^1 y_+ + H_k^2 u_+ + H_k^3 v_+)\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&\quad - \sum_{|k|\le K}[H_k^1(D_+u_+ + f_+^2) + H_k^2(C_+y_+ + A_+v_+ + f_+^3) + H_k^3(B_+u_+ + f_+^4)]\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle}\\
&= \widetilde{f}^2\circ\phi - T_K\widetilde{f}^2 - T_K\Big(\frac{\partial\widetilde{f}^2}{\partial y}\Big)y_+ - T_K\Big(\frac{\partial\widetilde{f}^2}{\partial u}\Big)u_+ - T_K\Big(\frac{\partial\widetilde{f}^2}{\partial v}\Big)v_+\\
&\quad - \sum_{|k|\le K}\Big[\mathrm{i}\langle k,\widetilde{f}_0^1 + f_+^1\rangle(h_k^2 + H_k^1 y_+ + H_k^2 u_+ + H_k^3 v_+) + H_k^1 f_+^2 + H_k^2 f_+^3 + H_k^3 f_+^4\\
&\quad + H_k^1\Big(\frac{\partial\widetilde{f}^2}{\partial u}\Big)_0 u_+ + H_k^2\Big(\frac{\partial\widetilde{f}^2}{\partial y}\Big)_0 y_+ + H_k^3\Big(\frac{\partial\widetilde{f}^2}{\partial u}\Big)_0 u_+ + H_k^2\Big(\frac{\partial\widetilde{f}^2}{\partial v}\Big)_0 v_+\Big]\mathrm{e}^{\mathrm{i}\langle k,x_+\rangle},
\end{aligned}$$

which implies that, for $(x_+,y_+,u_+,v_+)\in\mathcal{D}_{r_+,s_+}$ and $\omega_+\in\mathcal{W}_{h_+}$,

$$\begin{aligned}
\|f_+^2\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)} &\prec \Big\|\widetilde{f}^2\circ\phi - T_K\widetilde{f}^2 - T_K\Big(\frac{\partial\widetilde{f}^2}{\partial y}\Big)y_+ - T_K\Big(\frac{\partial\widetilde{f}^2}{\partial u}\Big)u_+ - T_K\Big(\frac{\partial\widetilde{f}^2}{\partial v}\Big)v_+\Big\|\\
&\quad + (|\widetilde{f}_0^1| + \|f_+^1\|)\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n+1)}s_+ + \varepsilon^{2-\zeta(N^2+n+\tau)}\\
&\quad + \gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)}(\|f_+^2\| + \|f_+^3\| + \|f_+^4\|).
\end{aligned}$$

We first estimate the first term in the above inequality. Similarly to the discussions in part (a), we have

$$\widetilde{f}^2\circ\phi - T_K\Big(\widetilde{f}^2 + \frac{\partial\widetilde{f}^2}{\partial y}y_+ + \frac{\partial\widetilde{f}^2}{\partial u}u_+ + \frac{\partial\widetilde{f}^2}{\partial v}v_+\Big) = F_1^2 + F_2^2 + F_3^2,$$

where

$$\begin{aligned}
F_1^2 &= \widetilde{f}^2\circ\phi - \widetilde{f}^2(x_+,y_+,u_+,v_+),\\
F_2^2 &= \widetilde{f}^2(x_+,y_+,u_+,v_+) - \widetilde{f}^2(x_+,0,0,0) - \frac{\partial\widetilde{f}^2}{\partial y}y_+ - \frac{\partial\widetilde{f}^2}{\partial u}u_+ - \frac{\partial\widetilde{f}^2}{\partial v}v_+,\\
F_3^2 &= (\mathrm{id} - T_K)\Big(\widetilde{f}^2(x_+,0,0,0) + \frac{\partial\widetilde{f}^2}{\partial y}y_+ + \frac{\partial\widetilde{f}^2}{\partial u}u_+ + \frac{\partial\widetilde{f}^2}{\partial v}v_+\Big).
\end{aligned}$$

If $\zeta < \frac{3-2\kappa}{N^2+\tau+n+1}$, $\gamma_0^{-1}\varepsilon_0^{3-2\kappa-\zeta(N^2+n+\tau+1)} \ll 1$, $(c\varepsilon_0)^{(\kappa-1)(3-2\kappa-\zeta(N^2+n+\tau+1))} < \frac{1}{2}$ and $\kappa < \frac{3}{2}$, we have the following result from Lemma 2.7 and Cauchy's estimates

$$\|F_1^2\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)} \prec \max\Big\{\frac{\varepsilon^{2-\zeta(N^2+\tau+n)}s}{\gamma(r-r_+)}, \frac{\varepsilon^{2-\zeta(N^2+\tau+n)}s^2}{\gamma(s-s_+)}\Big\} \prec s_+\varepsilon^{\kappa}.$$

By the mean value theorem and Cauchy's estimates, we get

$$\|F_2^2\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\prec\frac{\varepsilon s}{s^2}\cdot(\alpha s)^2=\varepsilon^\kappa s_+.$$

Since $\zeta<\lambda$ and $\kappa<\frac{3}{2}$, the following holds by Lemma 2.2

$$\|F_3^2\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\le\varepsilon s\sum_{|k|\ge K}\mathrm{e}^{-|k|(r-r_+)}\le\varepsilon s\int_K^\infty x^n\mathrm{e}^{-x(r-r_+)}\,\mathrm{d}x\prec\varepsilon^2 s\le\varepsilon^\kappa s_+.$$

Hence

$$\|f_+^2\|\prec\varepsilon^\kappa s_++\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)}(\|f_+^2\|+\|f_+^3\|+\|f_+^4\|)$$

on $\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)$.

Similarly, one can obtain

$$\|f_+^3\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\prec\varepsilon^\kappa s_++\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)}(\|f_+^2\|+\|f_+^3\|+\|f_+^4\|),$$
$$\|f_+^4\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\prec\varepsilon^\kappa s_++\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n)}(\|f_+^2\|+\|f_+^3\|+\|f_+^4\|).$$

If $\gamma^{-1}\varepsilon^{1-\zeta(N^2+\tau+n+1)}\ll 1$, $(c\varepsilon_0)^{(\kappa-1)(1-\zeta(N^2+n+\tau+1))}<\frac{1}{2}$, and $\kappa<\frac{3}{2}$, then

$$\frac{1}{s_+}(\|f_+^2\|+\|f_+^3\|+\|f_+^4\|)\prec\varepsilon^\kappa<\frac{1}{2}\varepsilon_+.$$

Now we estimate $\widetilde{f}_+^2$. Since

$$\widetilde{f}_+^2=f_+^2+(D\Phi_+)^{-1}(f_{\sigma_+}^2-f_{\sigma_{++}}^2)\circ\Phi_+,$$

we only need to estimate $(D\Phi_+)^{-1}(f_{\sigma_+}^2-f_{\sigma_{++}}^2)\circ\Phi_+$.

If $d\zeta>2$, $2(2(N+n))^{d+1}\widetilde{c}\,\|f^i\|_{C^d}\varepsilon_0^{\zeta d-1}\le s_0$ $(i=2,3,4)$ and $c^{\zeta d-1}\varepsilon_0^{\zeta d-2}<\left(\frac{1}{2}\right)^{\frac{1}{\kappa-1}}$, similarly to (2.37) we get

$$\|(D\Phi_+)^{-1}(f_{\sigma_+}^i-f_{\sigma_{++}}^i)\circ\Phi_+\|<2(N+n)\widetilde{c}\,\|f^i\|_{C^d}\cdot\sigma_+^d<\frac{1}{2}\varepsilon_+s_+.$$

To summarize this section, we have the following proposition.

**Proposition 2.1** *There is a large constant $c>0$ such that if*

$$\gamma^{-1}\varepsilon^{\frac{1}{N^2+1}-\lambda(N^2+\tau+1)}<c^{-1},$$
$$\gamma^{-1}hK^{N^2+\tau}\max\{K,M+1\}<c^{-1},$$
$$K^{N^2+\tau}\varepsilon<c^{-1}\min\{\gamma-\gamma_{+,}\ |\det(a_{ij})_{1\le i,j\le p}|\},$$

*where $\varepsilon$ is small enough and $\zeta<\min\left\{\frac{2-\kappa}{N^2+\tau+n+1},\frac{3-2\kappa}{N^2+\tau+n+1},\frac{1}{N^2+1}\right\}$, then there exist a G-commute transformation $\phi$ and a mapping $\psi:\omega_+\mapsto\omega$, such that under this $\phi$, the system* (2.16) *is changed into* (2.14)*. Moreover, the following conclusions hold:*

(1)
$$|\langle k,\omega_+\rangle|>\gamma_+|k|^{-\tau},$$
$$|\det(\mathrm{i}\langle k,\omega_+\rangle\mathbb{E}_N-\mathcal{A}_+)|>\gamma_+|k|^{-\tau},$$
$$|\det(\mathrm{i}\langle k,\omega_+\rangle\mathbb{E}_{N^2}-\mathbb{E}_N\otimes\mathcal{A}_+^T+\mathcal{A}_+^T\otimes\mathbb{E}_N)|>\gamma_+|k|^{-\tau},$$
$$|\det(a_{ij}^+)_{1\le i,j\le p}|>0,$$

*on an open set $\mathcal{W}_h(K)$ of $\mathbb{R}^n$ for $|k|\le K$;*

(2) *$\omega_+(\omega)$, $a_{ij}^+(\omega)$, $c_{ij}^+(\omega)$ and $d_{ij}^+(\omega)$ are real analytic in $\omega$ and satisfy, for $|l|\le N^2$,*

$$\Big|\frac{\partial^l(\omega_+-\omega)}{\partial\omega^l}\Big|\le c\varepsilon^{\frac{1}{N^2+1}},$$
$$\Big|\frac{\partial^l a_{ij}^+}{\partial\omega^l}\Big|,\ \Big|\frac{\partial^l b_{ij}^+}{\partial\omega^l}\Big|,\ \Big|\frac{\partial^l c_{ij}^+}{\partial\omega^l}\Big|,\ \Big|\frac{\partial^l d_{ij}^+}{\partial\omega^l}\Big|\le M+c\varepsilon^{\frac{1}{N^2+1}}=M_+,$$

*on the complex $h$-neighborhood of $\Theta$, and*

$$\Big|\frac{\partial^l a_{ij}^+}{\partial\omega_+^l}\Big|,\ \Big|\frac{\partial^l b_{ij}^+}{\partial\omega_+^l}\Big|,\ \Big|\frac{\partial^l c_{ij}^+}{\partial\omega_+^l}\Big|,\ \Big|\frac{\partial^l d_{ij}^+}{\partial\omega_+^l}\Big|\le M+c\varepsilon^{\frac{1}{N^2+1}}=M_+,$$

*on the complex $h_+$-neighborhood of $\Theta$;*

(3) *The new perturbation terms $\widetilde{f}_+^l$ $(1\le l\le 4)$ satisfy*

$$\|\widetilde{f}_+^1\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)},\quad \frac{1}{s_+}\|\widetilde{f}_+^j\|_{\mathcal{D}_{r_+,s_+}\times\mathcal{W}_{h_+}(K_+)}\le c\varepsilon^{\kappa}=\varepsilon_+,\quad 2\le j\le 4.$$

## 3 Proof of Theorem 1.1

Without loss of generality, we assume that $r<1$, $s<1$. For any $\nu\ge 0$, we define

$$\mathcal{D}_\nu=\mathcal{D}_{r_\nu,s_\nu},\quad \mathcal{W}_\nu=\mathcal{W}_{h_\nu}(K_\nu).$$

**Step 1** The choice of $\varepsilon_0$

In order to apply Proposition 2.1 for any $\nu\ge 0$, we choose $\varepsilon_0$ sufficiently small and $s_0$ such that

$$\varepsilon_0^{1-\zeta(N^2+\tau+n+3)}\gamma_0^{-1}\ll\frac{1}{16(N+n)},\quad \varepsilon_0^{\frac{N^2}{N^2+1}-\zeta(N^2+\tau+n+2)}\gamma_0^{-1}\ll\frac{1}{8(N+n)},$$
$$\varepsilon_0^{\frac{1}{N^2+1}-\lambda(N^2+\tau+n)}\gamma_0^{-1}M_0\ll 1,\quad \varepsilon_0^{3-2\kappa-\zeta(N^2+\tau+n+1)}\gamma_0^{-1}\ll 1,$$
$$\varepsilon_0^{d\zeta-1}<\min\{(2(2(N+n))^{d+1}\widetilde{c}\|f^1\|_{C^d})^{-1},\quad \min_{2\le j\le 4}s_0(2(2(N+n))^{d+1}\widetilde{c}\|f^j\|_{C^d})^{-1}\},$$
$$c\varepsilon_0<\min\Big\{\Big(\frac13\Big)^{\frac{1}{(\kappa-1)\left(\frac{1}{N^2+1}-\lambda(N^2+\tau+n+1)\right)}},\ \Big(\frac12\Big)^{\frac{1}{(\kappa-1)(2-\kappa-\zeta(N^2+n+\tau+1))}}\Big\},$$
$$(c\varepsilon_0)^{\zeta d-1}\varepsilon_0^{-1}<\Big(\frac12\Big)^{\frac{1}{\kappa-1}},\quad \varepsilon_0^{1-\lambda(N^2+\tau)}\ll\frac{\gamma_0}{4},\quad s_0<\frac{\varepsilon_0^{\zeta}}{2}.$$

We take $\Phi_0=\phi_0=\mathrm{id}$, $f_{\sigma_0}^i=f_0^i$, $i=1,2,3,4$.

**Step 2** The definition of $\Theta_\gamma$

Set $\Theta_{-1}=\Theta$, $\mathcal{A}_0=\mathcal{A}$ and $K_{-1}=0$. From the discussions in Section 2, it follows that for each $\nu\ge 0$,

$$\Theta_\nu=\bigcap_{K_{\nu-1}<|k|\le K_\nu}\left\{\omega\in\Theta_{\nu-1}\;\middle|\;\begin{array}{l}|\langle k,\omega\rangle|\ge\gamma_\nu|k|^{-\tau}\text{ and}\\ |\det(\mathrm{i}\langle k,\omega\rangle\mathbb{E}_N-\mathcal{A}_\nu)|\ge\gamma_\nu|k|^{-\tau}\text{ and}\\ |\det(\mathrm{i}\langle k,\omega\rangle\mathbb{E}_{N^2}-\mathbb{E}_N\otimes\mathcal{A}_\nu^T+\mathcal{A}_\nu\otimes\mathbb{E}_N)|\ge\gamma_\nu|k|^{-\tau}\end{array}\right\}.$$

Now we define

$$\Theta_\gamma = \bigcap_{\nu=0}^{\infty} \Theta_\nu.$$

**Step 3** The measure of $\Theta_\gamma$

By the definition of $\Theta_\gamma$, we have

$$\Theta - \Theta_\gamma \subset \bigcup_{\nu=0}^{\infty} \mathcal{R}^\nu(\gamma_\nu),$$

where

$$\mathcal{R}^\nu(\gamma_\nu) = \mathcal{R}_1^\nu(\gamma_\nu) \cup \mathcal{R}_2^\nu(\gamma_\nu) \cup \mathcal{R}_3^\nu(\gamma_\nu)$$

with

$$\mathcal{R}_1^\nu(\gamma_\nu) = \bigcap_{K_{\nu-1}<|k|\le K_\nu} \{\omega \in \Theta_{\nu-1} \mid |\langle k,\omega\rangle| \le \gamma_\nu |k|^{-\tau}\},$$
$$\mathcal{R}_2^\nu(\gamma_\nu) = \bigcap_{K_{\nu-1}<|k|\le K_\nu} \{\omega \in \Theta_{\nu-1} \mid |\det(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A}_\nu)| \le \gamma_\nu |k|^{-\tau}\},$$
$$\mathcal{R}_3^\nu(\gamma_\nu) = \bigcap_{K_{\nu-1}<|k|\le K_\nu} \{\omega \in \Theta_{\nu-1} \mid |\det(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}_\nu^T + \mathcal{A}_\nu \otimes \mathbb{E}_N)| \le \gamma_\nu |k|^{-\tau}\}.$$

Before we estimate the measure of the set $\mathcal{R}^\nu(\gamma_\nu)$, we need the following lemma.

**Lemma 3.1** *Let $g(t) = g_1(t) + \mathrm{i}g_2(t) : I \subset \mathbb{R} \to \mathbb{C}$ be of class $\mathcal{C}^m$, $\forall\, m \in \mathbb{N}$. Define $I_\beta = \{t \in I \mid |g(t)| \le \beta\}$ with $\beta > 0$. If there is a constant $d > 0$ such that one of the following conditions holds:*

(1) $|g_1^{(m)}(t)| > d > 0, \quad \forall\, t \in I,$

(2) $|g_2^{(m)}(t)| > d > 0, \quad \forall\, t \in I,$

*then*

$$\operatorname{meas} I_\beta \le c_0 \beta^{\frac{1}{m}},$$

*where $c_0 = 2(2 + 3 + \cdots + m + d^{-1})$.*

The proof of this lemma can be found in [8, 17].

For each $k$ satisfying $K_{\nu-1} < |k| \le K_\nu$, let

$$\mathcal{R}_{k1}^\nu(\gamma_\nu) = \{\omega \in \Theta_{\nu-1} \mid |\langle k,\omega\rangle| \le \gamma_\nu |k|^{-\tau}\},$$
$$\mathcal{R}_{k2}^\nu(\gamma_\nu) = \{\omega \in \Theta_{\nu-1} \mid |\det(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_N - \mathcal{A}_\nu)| \le \gamma_\nu |k|^{-\tau}\},$$
$$\mathcal{R}_{k3}^\nu(\gamma_\nu) = \{\omega \in \Theta_{\nu-1} \mid |\det(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}_\nu^T + \mathcal{A}_\nu \otimes \mathbb{E}_N)| \le \gamma_\nu |k|^{-\tau}\}.$$

Now we estimate the measure of the set $\mathcal{R}_{k3}^\nu(\gamma_\nu)$. Let

$$g(\omega) = \det(\mathrm{i}\langle k,\omega\rangle \mathbb{E}_{N^2} - \mathbb{E}_N \otimes \mathcal{A}_\nu^T + \mathcal{A}_\nu \otimes \mathbb{E}_N).$$

We get

$$g(\omega) = \mathrm{i}^{N^2} \langle k,\omega\rangle^{N^2} + \sum_{l \le N^2-1} p_l(\omega) \langle k,\omega\rangle^l,$$

where the coefficients $p_l(\omega)$ depend on $\mathcal{A}_\nu$, but not on $k$. Suppose that the real and imaginary part of $g$ are $g_1$ and $g_2$, respectively, that is,

$$g(\omega) = g_1(\omega) + \mathrm{i}g_2(\omega).$$

If $N \equiv 0 \pmod 2$, then

$$g_1 = \pm\langle k,\omega\rangle^{N^2} + \sum_{l\le N^2-1} \operatorname{Re} p(\omega)\langle k,\omega\rangle^l.$$

By (2.36), the choice of $\varepsilon_\nu$ and Proposition 2.1, it follows that

$$\Big|\frac{\partial^l(\omega_\nu(\omega)-\omega)}{\partial\omega^l}\Big| \le \frac{1}{2}, \quad \Big|\frac{\partial^l a_{ij}^\nu(\omega)}{\partial\omega^l}\Big| \le M_\nu \le M + c',$$

where $c' = 2c^{\kappa-1}\varepsilon_0^{\frac{1}{N^2+1}}$. Without loss of generality, we assume that

$$|k_1| = \max_{1\le j\le n} |k_j|.$$

Then

$$|k_1| \ge \frac{K_{\nu-1}}{n}, \quad \text{if } |k| \ge K_{\nu-1}.$$

Hence we have

$$\Big|\frac{\partial^{N^2}}{\partial\omega_1^{N^2}} g_1(\omega)\Big| \ge (N^2)!|k_1|^{N^2}(1-O(|k_1|^{-1})) \ge 1,$$

if $K_{\nu-1} > K_*$ with a sufficiently large positive integer $K_*$. By Lemma 3.1 we have

$$\operatorname{meas}\mathcal{R}_{k3}^\nu(\gamma_\nu) \le \widetilde{c}_0\Big(\frac{\gamma_\nu}{|k|^\tau}\Big)^{\frac{1}{N^2}} \le \widetilde{c}_0\gamma^{\frac{1}{N^2}}|k|^{-\frac{\tau}{N^2}},$$

where $\widetilde{c}_0 = N(N+1)\cdot(\text{diameter of }\Theta)^{n-1}$. Similarly, one can get the same estimate when $N \equiv 1 \pmod 2$. The estimates for the other two sets $\mathcal{R}_{k1}^\nu(\gamma_\nu)$ and $\mathcal{R}_{k2}^\nu(\gamma_\nu)$ are analogous. Therefore we have

$$\begin{aligned}\operatorname{meas}\Big(\bigcup_{\nu=1}^\infty \mathcal{R}^\nu\Big) &\le \sum_{\nu=1}^\infty \operatorname{meas}\mathcal{R}^\nu(\gamma_\nu) \le 3\widetilde{c}_0\gamma^{\frac{1}{N^2}} \sum_{\nu=1}^\infty \sum_{|k|=K_{\nu-1}}^{K_\nu} |k|^{-\frac{\tau}{N^2}} \\ &\le 3\widetilde{c}_0\gamma^{\frac{1}{N^2}} \sum_{|k|\ge K_*} |k|^{-\frac{\tau}{N^2}} = O(\gamma^{\frac{1}{N^2}})\end{aligned}$$

because $\tau > N^2 n$.

On the other hand, if $\varepsilon_0$ is very small, then $K_0 \ge K_*$. By the condition (3) in Theorem 1.1, it follows that for any fixed $K_0$,

$$\operatorname{meas}\mathcal{R}^0(\gamma) \to 0, \quad \text{as } \gamma \to 0.$$

Hence

$$\operatorname{meas}(\Theta - \Theta_\gamma) \to 0, \quad \text{as } \gamma \to 0.$$

**Step 4** The convergences of $\Phi_\nu$, $D\Phi_\nu$ and $\Psi_\nu$

For any $\nu \geq 0$, by the discussions in Section 2, we can find a sequence of $G$-commute transformations $\phi_0$, $\phi_1, \cdots, \phi_\nu$. Let

$$\Phi_\nu = \phi_0 \circ \phi_1 \circ \cdots \circ \phi_{\nu-1} : \mathcal{D} \times \mathcal{W} \to \mathcal{D}_0.$$

Then under $\Phi_\nu$, the system (2.4) is changed into (2.2).

By the choice of $\varepsilon_0$ and the definition of $\varepsilon_\nu$, we can conclude that $\varepsilon_\nu \to 0$ as $\nu \to \infty$. If $\gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+\tau+n+2)} \ll \frac{1}{16(N+n)}$, $\forall\, \nu, \varrho \in \mathbf{N},\ \nu > \varrho$, we can prove the convergence of $\Phi_\nu$:

$$\begin{aligned}
\|\Phi_\nu - \Phi_\varrho\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} &\leq \sum_{j=\varrho}^{\nu-1} \|(\Phi_{j+1} - \Phi_j)\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \\
&\leq \sum_{j=\varrho}^{\nu-1} \|D\Phi_j(\phi_j - \mathrm{id})\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \\
&\leq (N+n) \sum_{j=\varrho}^{\nu-1} \|D\Phi_j\| \|\phi_j - \mathrm{id}\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \\
&\prec 2(N+n) \sum_{j=\varrho}^{\nu-1} \max\{\gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+n+\tau)},\ \gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+n+\tau)} s_j\} \\
&\prec 2(N+n) \sum_{j=\varrho}^{\nu-1} \gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+\tau+n)} \\
&\prec 4(N+n)\gamma_\varrho^{-1}\varepsilon_\varrho^{1-\zeta(N^2+n+\tau)} \leq \frac{\varepsilon_\varrho^\zeta}{4} \to 0, \quad \varrho \to +\infty. \qquad (3.1)
\end{aligned}$$

Then $\Phi_\nu$ is a Cauchy sequence, and has a continuous function $\Phi_\infty$ as its limit.

Now we prove that $D\Phi_\nu$ is a Cauchy sequence.

$$\begin{aligned}
\|D\Phi_\nu - D\Phi_\varrho\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} &\leq \sum_{j=\varrho}^{\nu-1} \|(D\Phi_{j+1} - D\Phi_j)\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \\
&\leq \sum_{j=\varrho}^{\nu-1} \|(D\Phi_j D\phi_j - D\Phi_j)\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \\
&\leq (N+n) \sum_{j=\varrho}^{\nu-1} \|D\Phi_j\| \|D\phi_j - \mathbb{E}_{2n+p+q})\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \\
&\prec 2(N+n) \sum_{j=\varrho}^{\nu-1} \max\{\gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+n+\tau+1)},\ \gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+n+\tau+1)} s_j\} \\
&\prec 2(N+n) \sum_{j=\varrho}^{\nu-1} \gamma_j^{-1}\varepsilon_j^{1-\zeta(N^2+\tau+n+1)} \\
&\prec 4(N+n)\gamma_\varrho^{-1}\varepsilon_\varrho^{1-\zeta(N^2+n+\tau+1)} \leq \frac{\varepsilon_\varrho^\zeta}{4} \to 0, \quad \varrho \to +\infty. \qquad (3.2)
\end{aligned}$$

Therefore $D\Phi_\nu$ is a Cauchy sequence, and hence $D\Phi_\nu$ converges to $D\Phi_\infty$. Especially when $\varrho = 0$, the inequality (3.2) becomes

$$\|D\Phi_\nu - \mathbb{E}_{2n+p+q}\|_{\mathcal{D}_\nu \times \mathcal{W}_\nu} \leq \frac{\varepsilon_0^\zeta}{4}.$$

Let $\nu$ tend to infinity. We get

$$\|D\Phi_\infty - \mathbb{E}_{2n+p+q}\|_{\mathcal{D}_\infty \times \Theta_\gamma} \leq \frac{\varepsilon_0^\zeta}{4}.$$

Similarly, from (3.1) we get

$$\|\Phi_\infty - \mathrm{id}\|_{\mathcal{D}_\infty \times \Theta_\gamma} \leq \frac{\varepsilon_0^\zeta}{4}.$$

Hence if $\varepsilon_0$ is small enough, $\Phi_\infty$ must be a diffeomorphism. By the same method we can prove the convergence of $\Psi_\nu$. The convergences of $\Phi_\nu$ and $D\Phi_\nu$ yield that

$$\varphi^t = \Phi_\infty \circ \varphi_\infty^t,$$

on $\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \Theta_\gamma$, where $\varphi^t$ is the flow of system (1.5) and $\varphi_\infty^t$ the flow of the system

$$\begin{aligned} \dot{x} &= \omega_\infty, \\ \dot{y} &= D_\infty(\omega_\infty)u, \\ \dot{u} &= C_\infty(\omega_\infty)y + A_\infty(\omega_\infty)v, \\ \dot{v} &= B_\infty(\omega_\infty)u. \end{aligned} \tag{3.3}$$

Hence, for each $\omega \in \Theta_\gamma$, the embedding torus $\Phi_\infty(\mathbb{T}^n \times \{0\} \times \{0\} \times \{0\} \times \{\omega_\infty\})$ is invariant under $\varphi^t$. Moreover, on this torus, we have

$$x = x_0 + \omega_\infty t,$$

where $\omega = \Psi_\infty(\omega_\infty)$. This completes the proof.

**Acknowledgement** The author sincerely thanks Professor B. Liu, who kindly gave some useful advice and suggestions.

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