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Lie Bialgebras of a Family of Lie Algebras of Block Type***

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Abstract Lie bialgebra structures on a family of Lie algebras of Block type are shown to be triangular coboundary.

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1 Introduction

Since the notion of Lie bialgebras was introduced by Drinfeld in 1983 (cf. [1, 2]), there have appeared several papers on Lie coalgebras or Lie bialgebras (cf. [3–10]). Lie bialgebras of Witt and Virasoro type were presented in [9]. These types of Lie bialgebras were further classified in [6]. The authors in [8] studied Lie bialgebra structures on Lie algebras of generalized Witt type, which were proved to be coboundary triangular. Lie bialgebra structures on Lie algebras of generalized Virasoro-like type were considered in [10]. Partially due to the fact that constructing quantization of Lie bialgebras is an important tool to produce new quantum groups (e.g., [11, 12]), the study of Lie bialgebra structures becomes more and more important.

In this paper, we study Lie bialgebra structures on a family of Lie algebras of Block type. Lie algebras of this type attract our attention not only because they are closely related to the Virasoro algebra or the Virasoro-like algebra but also because they are special cases of Lie algebras of Cartan type S and Cartan type H (cf. [13–15]).

First, let us recall the definition of Lie bialgebras. Let L be a vector space over a field $\mathbb{F}$ of characteristic zero. Denote by ξ the cyclic map of $L\otimes L\otimes L$ cyclically permuting the coordinates, namely, $\xi(x_1\otimes x_2\otimes x_3)=x_2\otimes x_3\otimes x_1$ for $x_1,x_2,x_3\in L$, and by τ the twist map of $L\otimes L$, i.e., $\tau(x\otimes y)=y\otimes x$ for $x,y\in L$.

To introduce the notion of Lie bialgebras, we first reformulate the definition of a Lie algebra as follows: A Lie algebra is a pair (L,δ) of a vector space L and a linear map $\delta: L\otimes L\to L$

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(the bracket of L) satisfying the conditions:

$$\mathrm{Ker}(1-\tau)\subset \mathrm{Ker}\,\delta, \tag{1.1}$$

$$\delta\cdot(1\otimes\delta)\cdot(1+\xi+\xi^2)=0:\ L\otimes L\otimes L\to L, \tag{1.2}$$

which are called skew-symmetry and Jacobi identity respectively. Dually, one has the notion of Lie coalgebras: A Lie coalgebra is a pair (L,Δ) of a vector space L and a linear map $\Delta: L\to L\otimes L$ (the cobracket of L) satisfying the conditions:

$$\mathrm{Im}\,\Delta\subset \mathrm{Im}(1-\tau), \tag{1.3}$$

$$(1+\xi+\xi^2)\cdot(1\otimes\Delta)\cdot\Delta=0:\ L\to L\otimes L\otimes L, \tag{1.4}$$

which are called anti-commutativity and Jacobi identity respectively. For a Lie algebra L, we always use $[x,y]=\delta(x,y)$ to denote its Lie bracket and use the symbol "$\cdot$" to stand for the diagonal adjoint action

$$x\cdot\Big(\sum_i a_i\otimes b_i\Big)=\sum_i([x,a_i]\otimes b_i+a_i\otimes[x,b_i])\quad\text{for } x,a_i,b_i\in L. \tag{1.5}$$

Definition 1.1 *A Lie bialgebra is a triple (L,δ,Δ) satisfying the conditions:*

$$(L,\delta)\ \textit{is a Lie algebra},\quad (L,\Delta)\ \textit{is a Lie coalgebra}, \tag{1.6}$$

$$\Delta\delta(x,y)=x\cdot\Delta y-y\cdot\Delta x\quad \textit{for } x,y\in L\ (\textit{compatibility condition}). \tag{1.7}$$

Denote by $\mathcal{U}$ the universal enveloping algebra of L and by 1 the identity element of $\mathcal{U}$. For an element $r=\sum\limits_i a_i\otimes b_i\in L\otimes L$, we define r^{ij}, $c(r)$, $i,j=1,2,3$ to be elements of $\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}$ by (where the bracket in (1.8) is the commutator):

$$\begin{aligned}&c(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}],\\ &r^{12}=\sum_i a_i\otimes b_i\otimes 1,\quad r^{13}=\sum_i a_i\otimes 1\otimes b_i,\quad r^{23}=\sum_i 1\otimes a_i\otimes b_i.\end{aligned} \tag{1.8}$$

Definition 1.2 (1) *A coboundary Lie bialgebra is a 4-tuple (L,δ,Δ,r), where (L,δ,Δ) is a Lie bialgebra and $r\in\mathrm{Im}(1-\tau)\subset L\otimes L$ such that $\Delta=\Delta_r$ is a coboundary of r, where Δ_r is defined by*

$$\Delta_r(x)=x\cdot r\quad \textit{for } x\in L. \tag{1.9}$$

(2) *A coboundary Lie bialgebra (L,δ,Δ,r) is called triangular if it satisfies the following classical Yang-Baxter Equation (CYBE):*

$$c(r)=0. \tag{1.10}$$

Now let us formulate the main result below. Let G be any nonzero additive subgroup of $\mathbb{F}$ with $\mathbb{Z}\subset G$. The Lie algebras considered in this paper are the Block Lie algebras $B=B(G)$ with basis $\{\partial,\, x^{a,i}\,|\,a\in G,\, i\in\mathbb{Z}\}$ and brackets

$$[\partial,x^{b,j}]=bx^{b,j}, \tag{1.11}$$

$$[x^{a,i},x^{b,j}]=((a-1)j-(b-1)i)x^{a+b,i+j-1}. \tag{1.12}$$

The main result of this paper is the following

Theorem 1.1 *Every Lie bialgebra structure on B is a triangular coboundary Lie bialgebra.*

2 Proof of the Main Result

The following result can be found in [1, 2, 6].

Lemma 2.1 *Let L be a Lie algebra and $r \in \mathrm{Im}(1-\tau) \subset L \otimes L$.*

(1) *The tripple $(L, [\cdot\,,\cdot], \Delta_r)$ is a Lie bialgebra if and only if r satisfies CYBE* (1.10).

(2) *We have*

$$(1+\xi+\xi^2)\cdot(1\otimes\Delta)\cdot\Delta(x) = x\cdot c(r) \quad \textit{for all } x \in L. \tag{2.1}$$

Now consider the Lie algebra B. We denote

$$B' = [B, B] = \mathrm{span}\{x^{a,i} \mid (a,i) \in G \times \mathbb{Z}\} \qquad \text{(the derived subalgebra of } B\text{)}. \tag{2.2}$$

Note that $C = x^{1,0}$ is a central element of B', and $B'/\mathbb{F}C$ is a simple Lie algebra (in this case B is called a central simple Lie algebra). For convenience, we use the following convention.

Convention 2.1 *If an undefined symbol appears in an expression, we always regard it as zero.*

Lemma 2.2 *Let $B[n] = B \otimes \cdots \otimes B$ be the tensor product of n copies of B, and regard $B[n]$ as a B-module under the adjoint diagonal action of B.*

(1) *Suppose that $c \in B[n]$ satisfies $a \cdot c = 0$ for all $a \in B$. Then $c = 0$.*

(2) *Suppose that $c \in B[n]$ satisfies $a \cdot c = 0$ for all $a \in B'$. Then $c \in \mathbb{F}(C \otimes \cdots \otimes C)$.*

Proof It can be proved by using the similar arguments as in the proof of [10, Lemma 2.2].

An element $r \in \mathrm{Im}(1-\tau) \subset B \otimes B$ is said to satisfy the modified Yang-Baxter Equation (MYBE) if

$$x \cdot c(r) = 0 \quad \text{for all } x \in B. \tag{2.3}$$

As a conclusion of Lemma 2.2, one immediately obtains

Corollary 2.1 *An element $r \in \mathrm{Im}(1-\tau) \subset B \otimes B$ satisfies CYBE* (1.10) *if and only if it satisfies MYBE* (2.3).

Regard $V = B \otimes B$ as a B-module under the adjoint diagonal action. Denote by $\mathrm{Der}(B, V)$ the set of derivations $D : B \to V$, namely, D is a linear map satisfying

$$D([x,y]) = x \cdot D(y) - y \cdot D(x) \quad \text{for } x, y \in B, \tag{2.4}$$

and by $\mathrm{Inn}(B, V)$ the set consisting of the derivations a_{inn}, $a \in V$, where a_{inn} is the inner derivation defined by

$$a_{\mathrm{inn}} : x \mapsto x \cdot a \quad \text{for } x \in B. \tag{2.5}$$

Then it is well-known that

$$H^1(B,V) \cong \mathrm{Der}(B,V)/\mathrm{Inn}(B,V), \tag{2.6}$$

where $H^1(B,V)$ is the first cohomology group of a Lie algebra B with coefficients in the B-module V.

Proposition 2.1 $\mathrm{Der}(B,V)=\mathrm{Inn}(B,V)$, *equivalently,* $H^1(B,V)=0$.

Proof Note that $B=\bigoplus_{a\in G} B_a$ and $V=B\otimes B=\bigoplus_{a\in G} V_a$ are G-graded (but not finitely graded), with

$$B_a=\mathrm{Span}\{x^{a,i}\,|\,i\in\mathbb{Z}\}\oplus\delta_{a,0}\mathbb{F}\partial \quad \text{and} \quad V_a=\sum_{\substack{b,c\in G\\ b+c=a}} B_b\otimes B_c \quad \text{for } a\in G. \tag{2.7}$$

A derivation $D\in\mathrm{Der}(B,V)$ is homogeneous of degree $a\in G$ if $D(B_b)\subset V_{a+b}$ for all $b\in G$. Denote

$$\mathrm{Der}(B,V)_a=\{D\in\mathrm{Der}(B,V)\,|\,\deg D=a\} \quad \text{for } a\in G.$$

Let $D\in\mathrm{Der}(B,V)$. For $a\in G$, we define the linear map $D_a: B\to V$ as follows: For any $\mu\in B_b$ with $b\in G$, write $D(\mu)=\sum_{c\in G}\mu_c$ with $\mu_c\in V_c$, then we set

$$D_a(\mu)=\mu_{a+b}.$$

Obviously, $D_a\in\mathrm{Der}(B,V)_a$ and we have

$$D=\sum_{a\in G} D_a, \tag{2.8}$$

which holds in the sense that for every $u\in B$, only finitely many $D_a(u)\neq 0$, and $D(u)=\sum_{a\in G} D_a(u)$ (we call such a sum in (2.8) summable).

We shall prove this proposition by several claims.

Claim 2.1 *If* $0\neq a\in G$, *then* $D_a\in\mathrm{Inn}(B,V)$.

For $a\neq 0$, denote $\gamma=a^{-1}D_a(\partial)\in V_a$. Then for any $x^{b,j}\in B_b$ with $b\in G$, applying D_a to $[\partial,x^{b,j}]=bx^{b,j}$ and using $D_a(x^{b,j})\in V_{a+b}$, we have

$$(a+b)D_a(x^{b,j})-x^{b,j}\cdot D_a(\partial)=\partial\cdot D_a(x^{b,j})-x^{b,j}\cdot D_a(\partial)=bD_a(x^{b,j}), \tag{2.9}$$

i.e., $D_a(x^{b,j})=\gamma_{\rm inn}(x^{b,j})$. Thus $D_a=\gamma_{\rm inn}$ is inner.

Claim 2.2 $D_0(\partial)=D_0(x^{1,0})=0$.

Applying D_0 to $[\partial,x]=bx$ for $x\in B_b$ with $b\in G$, as in (2.9) we obtain $x\cdot D_0(\partial)=0$. Thus by Lemma 2.2(1), $D_0(\partial)=0$. Next, applying D_0 to $[x^{b,j},x^{1,0}]=0$ for any $x^{b,j}\in B'$, we obtain $x^{b,j}\cdot D_0(x^{1,0})=0$. Thus by Lemma 2.2(2), $D_0(x^{1,0})\in\mathbb{F}(C\otimes C)$. But $C\otimes C\in V_2$, while $D_0(x^{1,0})\in V_1$, we have $D_0(x^{1,0})=0$ (recall Convention 2.1).

Claim 2.3 *Replacing D_0 by $D_0 - u_{\rm inn}$ for some $u \in V_0$, we can suppose $D_0(x^{a,i}) = 0$ for $(a, i) \in G \times \mathbb{Z}$.*

We can write $D_0(x^{a,j})$ as

$$D_0(x^{a,j}) = \sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{a,j}_{p,q,r} x^{p,q} \otimes x^{-p+a,r} + \sum_{s\in\mathbb{Z}} d^{a,j}_s \partial \otimes x^{a,s} + \sum_{t\in\mathbb{Z}} e^{a,j}_t x^{a,t} \otimes \partial \tag{2.10}$$

for all $a \in G$ and some $d^{a,j}_{p,q,r}, d^{a,j}_s, d^{a,j}_t \in \mathbb{F}$, where $\{(p, q, r) \in G \times \mathbb{Z} \times \mathbb{Z} \,|\, d^{a,j}_{p,q,r} \neq 0\}$, $\{s \in \mathbb{Z} \,|\, d^{a,j}_s \neq 0\}$ and $\{t \in \mathbb{Z} \,|\, e^{a,j}_t \neq 0\}$ are finite sets.

Applying D_0 to $[x^{1,0}, x^{a,j}] = 0$, we obtain

$$\sum_{s\in\mathbb{Z}} d^{a,j}_s x^{1,0} \otimes x^{a,s} + \sum_{t\in\mathbb{Z}} e^{a,j}_t x^{a,t} \otimes x^{1,0} = 0. \tag{2.11}$$

Comparing the coefficients of $x^{1,0} \otimes x^{a,s}$ and $x^{a,t} \otimes x^{1,0}$, we obtain

$$d^{a,j}_s = e^{a,j}_t = 0, \quad (a, s), (a, t) \neq (1, 0) \quad \text{and} \quad d^{1,j}_0 = -e^{1,j}_0. \tag{2.12}$$

Hence we can rewrite (2.10) as

$$D_0(x^{a,j}) = \sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{a,j}_{p,q,r} x^{p,q} \otimes x^{-p+a,r}, \qquad a \neq 1, \tag{2.13}$$

$$D_0(x^{1,j}) = \sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{1,j}_{p,q,r} x^{p,q} \otimes x^{-p+1,r} + d^{1,j}_0 (\partial \otimes x^{1,0} - x^{1,0} \otimes \partial), \quad j \neq 0. \tag{2.14}$$

That is,

$$D_0(x^{a,j}) = \sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{a,j}_{p,q,r} x^{p,q} \otimes x^{-p+a,r} + \delta_{a,1} d^{1,j}_0 (\partial \otimes x^{1,0} - x^{1,0} \otimes \partial). \tag{2.15}$$

Subclaim *Replacing D_0 by $D_0 - u_{\rm inn}$ for some $u \in V_0$, we can suppose $D_0(x^{a,j}) = 0$ for all $a \in \mathbb{Z}$, $j \in \mathbb{Z}$.*

We can write

$$D_0(x^{0,1}) = \sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{0,1}_{p,q,r}\, x^{p,q} \otimes x^{-p,r} \tag{2.16}$$

for some $d^{0,1}_{p,q,r} \in \mathbb{F}$, where $\{(p, q, r) \in G \times \mathbb{Z} \times \mathbb{Z} \,|\, d^{0,1}_{p,q,r} \neq 0\}$ is a finite set. Note that

$$x^{0,1} \cdot x^{p,q} \otimes x^{-p,r} = (2 - q - r) x^{p,q} \otimes x^{-p,r}.$$

Using this, by replacing D_0 by $D_0 - u_{\rm inn}$, where u is a combination of some $x^{p,q} \otimes x^{-p,r}$ for $q + r \neq 2$, we can rewrite (2.16) as

$$D_0(x^{0,1}) = \sum_{\substack{q+r=2\\ p\in G\\ q,r\in\mathbb{Z}}} d^{0,1}_{p,q,r}\, x^{p,q} \otimes x^{-p,r}. \tag{2.17}$$

Furthermore, from the following facts

$$x^{0,1}\cdot(x^{p,0}\otimes x^{-p,2})=0=x^{0,1}\cdot(x^{p,2}\otimes x^{-p,0}),$$
$$x^{0,0}\cdot(x^{p,0}\otimes x^{-p,2})=-2x^{p,0}\otimes x^{-p,1},$$
$$x^{0,0}\cdot(x^{p,2}\otimes x^{-p,0})=-2x^{p,1}\otimes x^{-p,0},$$

by replacing D_0 by D_0-u_{inn}, where u is a combination of $x^{p,0}\otimes x^{-p,2}$ and $x^{p,2}\otimes x^{-p,0}$ (this replacement does not affect (2.17)), we can suppose

$$d^{0,0}_{p,0,1}=d^{0,0}_{p,1,0}=0. \tag{2.18}$$

Applying D_0 to $[x^{0,0},x^{0,1}]=-x^{0,0}$, we obtain

$$\sum_{\substack{q+r=2\\ p\in G\\ q,r\in\mathbb{Z}}} d^{0,1}_{p,q,r}(-qx^{p,q-1}\otimes x^{-p,r}-rx^{p,q}\otimes x^{-p,r-1})=\sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}}(1-q-r)d^{0,0}_{p,q,r}x^{p,q}\otimes x^{-p,r}.$$

That is,

$$\begin{aligned}&\sum_{\substack{q+r=1\\ p\in G\\ q,r\in\mathbb{Z}}}(-(q+1)d^{0,1}_{p,q+1,r}x^{p,q}\otimes x^{-p,r}-(r+1)d^{0,1}_{p,q,r+1}x^{p,q}\otimes x^{-p,r})\\ &=\sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}}(1-q-r)d^{0,0}_{p,q,r}x^{p,q}\otimes x^{-p,r}.\end{aligned}$$

Comparing the coefficients of $x^{p,q}\otimes x^{-p,r}$, we obtain

$$2d^{0,1}_{p,0,2}=-d^{0,1}_{p,1,1}=2d^{0,1}_{p,2,0}, \tag{2.19}$$
$$d^{0,0}_{p,q,r}=0,\quad q+r\neq 1. \tag{2.20}$$

By (2.15) and (2.17)–(2.20), $D_0(x^{0,0})$ and $D_0(x^{0,1})$ can be respectively rewritten as

$$D_0(x^{0,0})=0\,, \tag{2.21}$$
$$D_0(x^{0,1})=\sum_{p\in G}d^{0,1}_{p,0,2}(x^{p,0}\otimes x^{-p,2}-2x^{p,1}\otimes x^{-p,1}+x^{p,2}\otimes x^{-p,0}). \tag{2.22}$$

Applying D_0 to $[x^{0,1},x^{-1,0}]=2x^{-1,0}$, we obtain

$$\begin{aligned}&\sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{-1,0}_{p,q,r}((1-q-r)x^{p,q}\otimes x^{-p-1,r})\\ &=\sum_{p\in G}4(d^{0,1}_{p,0,2}-d^{0,1}_{p+1,0,2})(x^{p,1}\otimes x^{-p-1,0}-x^{p,0}\otimes x^{-p-1,1}).\end{aligned}$$

Comparing the coefficients of $x^{p,q}\otimes x^{-p-1,r}$, we obtain

$$d^{-1,0}_{p,q,r}=0,\quad q+r\neq 1, \tag{2.23}$$
$$\sum_{p\in G}(d^{0,1}_{p+1,0,2}-d^{0,1}_{p,0,2})x^{p,0}\otimes x^{-p-1,1}=0, \tag{2.24}$$
$$\sum_{p\in G}(d^{0,1}_{p,0,2}-d^{0,1}_{p+1,0,2})x^{p,1}\otimes x^{-p-1,0}=0. \tag{2.25}$$

From the equation (2.24) or (2.25), we have

$$d^{0,1}_{p+1,0,2} = d^{0,1}_{p,0,2} \quad \text{for any } p \in G. \tag{2.26}$$

According to the fact that the set $\{p \in G \mid d^{0,1}_{p,0,2} \neq 0\}$ is of finite order, we obtain

$$d^{0,1}_{p,0,2} = 0 \quad \text{for any } p \in G. \tag{2.27}$$

Combining (2.22) and (2.27), we can safely deduce that

$$D_0(x^{0,1}) = 0. \tag{2.28}$$

Applying D_0 to $[x^{0,1}, x^{a,j}] = (1-a-j)x^{a,j}$, we obtain

$$\begin{aligned} &\sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} (2-a-q-r)d^{a,j}_{p,q,r}x^{p,q}\otimes x^{-p+a,r} \\ &= (1-a-j)\Big(\sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} d^{a,j}_{p,q,r}x^{p,q}\otimes x^{-p+a,r} + \delta_{a,1}d^{1,j}_0(\partial\otimes x^{1,0} - x^{1,0}\otimes\partial)\Big). \end{aligned} \tag{2.29}$$

That is,

$$\sum_{\substack{p\in G\\ q,r\in\mathbb{Z}}} (1-q-r+j)d^{a,j}_{p,q,r}x^{p,q}\otimes x^{-p+a,r} - (1-a-j)\delta_{a,1}d^{1,j}_0(\partial\otimes x^{1,0} - x^{1,0}\otimes\partial) = 0.$$

Thus we can deduce $d^{a,j}_{p,q,r} = 0$ for any $a \in G$, $j \in \mathbb{Z}$ unless $q + r = j + 1$ and $d^{1,j}_0 = 0$ for $0 \neq j \in \mathbb{Z}$. But we have proved $D_0(x^{1,0}) = 0$ in Claim 2.2. Hence

$$d^{1,j}_0 = 0 \quad \text{for all } j \in \mathbb{Z}. \tag{2.30}$$

Then (2.10) can be rewritten as

$$D_0(x^{a,j}) = \sum_{\substack{p\in G\\ j+1\geqslant q\in\mathbb{Z}}} d^{a,j}_{p,q}x^{p,q}\otimes x^{-p+a,j+1-q} \quad \text{for all } a \in G \tag{2.31}$$

for some $d^{a,j}_{p,q} \in \mathbb{F}$, where $\{(p,q) \in G\times\mathbb{Z},\ q \leqslant j+1 \,|\, d^{a,j}_{p,q} \neq 0\}$ is a finite set for any $a \in G$.

According to (2.31), for any $a \in G$, we can write $D_0(x^{a,0})$ as

$$D_0(x^{a,0}) = \sum_{p\in G}(d^{a,0}_{p,0}x^{p,0}\otimes x^{-p+a,1} + d^{a,0}_{p,1}x^{p,1}\otimes x^{-p+a,0}). \tag{2.32}$$

Applying D_0 to $[x^{a,0}, x^{0,0}] = 0$, we obtain

$$\sum_{p\in G}(d^{a,0}_{p,0}\ x^{p,0}\otimes x^{-p+a,0} + d^{a,0}_{p,1}\ x^{p,0}\otimes x^{-p+a,0}) = 0.$$

Comparing the coefficients of $x^{p,0}\otimes x^{-p+a,0}$, we obtain

$$d^{a,0}_{p,0} + d^{a,0}_{p,1} = 0. \tag{2.33}$$

According to (2.32) and (2.33), we can rewrite $D_0(x^{a,0})$ as

$$D_0(x^{a,0}) = \sum_{p\in G} d^{a,0}_{p,0}(x^{p,0} \otimes x^{-p+a,1} - x^{p,1} \otimes x^{-p+a,0}). \tag{2.34}$$

In particular, for $a = -1$ and $a = 2$, one has

$$D_0(x^{-1,0}) = \sum_{p\in G} d^{-1,0}_{p,0}(x^{p,0} \otimes x^{-p-1,1} - x^{p,1} \otimes x^{-p-1,0}), \tag{2.35}$$

$$D_0(x^{2,0}) = \sum_{p\in G} d^{2,0}_{p,0}(x^{p,0} \otimes x^{-p+2,1} - x^{p,1} \otimes x^{-p+2,0}). \tag{2.36}$$

Applying D_0 to $[x^{-1,0}, x^{2,0}] = 0$, we obtain

$$\begin{aligned}&\sum_{p\in G}(-2d^{2,0}_{p,0}x^{p,0} \otimes x^{-p+1,0} + 2d^{2,0}_{p+1,0}x^{p,0} \otimes x^{-p+1,0})\\ &= \sum_{p\in G}(d^{-1,0}_{p,0}x^{p,0} \otimes x^{-p+1,0} - d^{-1,0}_{p-2,0}x^{p,0} \otimes x^{-p+1,0}).\end{aligned}$$

Comparing the coefficients of $x^{p,0} \otimes x^{-p+1,0}$, we have

$$2d^{2,0}_{p+1,0} - 2d^{2,0}_{p,0} + d^{-1,0}_{p-2,0} - d^{-1,0}_{p,0} = 0. \tag{2.37}$$

According to (2.31), we can write $D_0(x^{0,2})$ as

$$D_0(x^{0,2}) = \sum_{p\in G}(d^{0,2}_{p,0}x^{p,0} \otimes x^{-p,3} + d^{0,2}_{p,1}x^{p,1} \otimes x^{-p,2} + d^{0,2}_{p,2}x^{p,2} \otimes x^{-p,1} + d^{0,2}_{p,3}x^{p,3} \otimes x^{-p,0}). \tag{2.38}$$

Applying D_0 to $[x^{0,0}, x^{0,2}] = -2x^{0,1}$, we obtain

$$\begin{aligned}&\sum_{p\in G}(3d^{0,2}_{p,0}x^{p,0} \otimes x^{-p,2} + d^{0,2}_{p,1}x^{p,0} \otimes x^{-p,2} + 2d^{0,2}_{p,1}x^{p,1} \otimes x^{-p,1}\\ &+ 2d^{0,2}_{p,2}x^{p,1} \otimes x^{-p,1} + d^{0,2}_{p,2}x^{p,2} \otimes x^{-p,0} + 3d^{0,2}_{p,3}x^{p,2} \otimes x^{-p,0}) = 0.\end{aligned}$$

Comparing the coefficients of $x^{p,0} \otimes x^{-p,2}$, $x^{p,2} \otimes x^{-p,0}$ and $x^{p,1} \otimes x^{-p,1}$, we obtain

$$3d^{0,2}_{p,0} + d^{0,2}_{p,1} = 0, \quad d^{0,2}_{p,2} + 3d^{0,2}_{p,3} = 0, \quad d^{0,2}_{p,1} + d^{0,2}_{p,2} = 0. \tag{2.39}$$

According to equations (2.38) and (2.39), we can rewrite $D_0(x^{0,2})$ as

$$D_0(x^{0,2}) = \sum_{p\in G} d^{0,2}_{p,0}(x^{p,0} \otimes x^{-p,3} - 3x^{p,1} \otimes x^{-p,2} + 3x^{p,2} \otimes x^{-p,1} - x^{p,3} \otimes x^{-p,0}). \tag{2.40}$$

Using the following facts

$$x^{0,1} \cdot (x^{p,0} \otimes x^{-p,2} + x^{p,2} \otimes x^{-p,0} - 2x^{p,1} \otimes x^{-p,1}) = 0, \tag{2.41}$$

$$x^{0,0} \cdot (x^{p,0} \otimes x^{-p,2} + x^{p,2} \otimes x^{-p,0} - 2x^{p,1} \otimes x^{-p,1}) = 0, \tag{2.42}$$

and

$$\begin{aligned}&x^{0,2} \cdot (x^{p,0} \otimes x^{-p,2} + x^{p,2} \otimes x^{-p,0} - 2x^{p,1} \otimes x^{-p,1})\\ &= 2p(x^{p,0} \otimes x^{-p,3} - 3x^{p,1} \otimes x^{-p,2} + 3x^{p,2} \otimes x^{-p,1} - x^{p,3} \otimes x^{-p,0}),\end{aligned} \tag{2.43}$$

and replacing D_0 by $D_0 - u_{\text{inn}}$, where u is a combination of $x^{p,0}\otimes x^{-p,2}+x^{p,2}\otimes x^{-p,0}-2x^{p,1}\otimes x^{-p,1}$ for $p \neq 0$ (this replacement does not affect (2.17) and (2.18)), we can rewrite (2.38) as

$$D_0(x^{0,2}) = d_{0,0}^{0,2}(x^{0,0} \otimes x^{0,3} - 3x^{0,1} \otimes x^{0,2} + 3x^{0,2} \otimes x^{0,1} - x^{0,3} \otimes x^{0,0}). \tag{2.44}$$

According to the following facts

$$x^{0,1} \cdot (x^{0,0} \otimes x^{0,2} + x^{0,2} \otimes x^{0,0} - 2x^{0,1} \otimes x^{0,1}) = 0,$$
$$x^{0,0} \cdot (x^{0,0} \otimes x^{0,2} + x^{0,2} \otimes x^{0,0} - 2x^{0,1} \otimes x^{0,1}) = 0,$$
$$x^{0,2} \cdot (x^{0,0} \otimes x^{0,2} + x^{0,2} \otimes x^{0,0} - 2x^{0,1} \otimes x^{0,1}) = 0,$$

and

$$\begin{aligned} &x^{-1,0} \cdot (x^{0,0} \otimes x^{0,2} + x^{0,2} \otimes x^{0,0} - 2x^{0,1} \otimes x^{0,1}) \\ &= -4(x^{0,0} \otimes x^{-1,1} - x^{0,1} \otimes x^{-1,0}) + 4(x^{-1,0} \otimes x^{0,1} - x^{-1,1} \otimes x^{0,0}), \end{aligned}$$

and replacing D_0 by $D_0 - u_{\text{inn}}$, where u is a combination of $x^{0,0}\otimes x^{0,2}+x^{0,2}\otimes x^{0,0}-2x^{0,1}\otimes x^{0,1}$ (this replacement does not affect (2.17), (2.18) and (2.44)), we can suppose

$$d_{0,0}^{-1,0} = 0. \tag{2.45}$$

Hence we can rewrite (2.35) as

$$D_0(x^{-1,0}) = \sum_{0\neq p\in G} d_{p,0}^{-1,0}(x^{p,0} \otimes x^{-p-1,1} - x^{p,1} \otimes x^{-p-1,0}). \tag{2.46}$$

For any $a, b \in \mathbb{Z}, a, b, a+b \neq 0, 1$, applying D_0 to

$$(a+b-1)[x^{a,0},[x^{b,0},x^{0,2}]] = 2(a-1)(b-1)[x^{a+b,0},x^{0,1}],$$

we obtain

$$\begin{aligned} &\sum_{p\in G}((a-1)(b+2a-2p)d_{p-a,0}^{b,0} + (a-1)(1+2p-2b)d_{p,0}^{b,0} + (b-1)(1-p+b)d_{p-b,0}^{a,0} \\ &+ (b-1)(p-a+b)d_{p,0}^{a,0} + (a-1)(b-1)d_{p,0}^{a+b,0})x^{p,0} \otimes x^{-p+a+b,1} \\ &+ \sum_{p\in G}((a-1)(b-2p)d_{p,0}^{b,0} - (a-1)(1-2p+2a)d_{p-a,0}^{b,0} - (b-1)(2b-p)d_{p-b,0}^{a,0} \\ &- (b-1)(p+1-a)d_{p,0}^{a,0} - (a-1)(b-1)d_{p,0}^{a+b,0})x^{p,1} \otimes x^{-p+a+b,0} \\ &+ 3(a-1)(b-1)d_{0,0}^{0,2}(-x^{0,0} \otimes x^{a+b,1} + x^{b,0} \otimes x^{a,1} + x^{a,0} \otimes x^{b,1} - x^{a+b,0} \otimes x^{0,1} \\ &- x^{b,1} \otimes x^{a,0} - x^{a,1} \otimes x^{b,0} + x^{0,1} \otimes x^{a+b,0} + x^{a+b,1} \otimes x^{0,0}) = 0. \end{aligned} \tag{2.47}$$

Comparing the coefficients of $x^{p,0} \otimes x^{-p+a+b,1}$ and $x^{p,1} \otimes x^{-p+a+b,0}$ where $p \neq 0, a, b, a+b$ in (2.47), we obtain

$$\begin{aligned} 0 &= (a-1)(b+2a-2p)d_{p-a,0}^{b,0} + (a-1)(1+2p-2b)d_{p,0}^{b,0} + (b-1)(1-p+b)d_{p-b,0}^{a,0} \\ &\quad + (b-1)(p-a+b)d_{p,0}^{a,0} + (a-1)(b-1)d_{p,0}^{a+b,0}, \end{aligned} \tag{2.48}$$

$$\begin{aligned} 0 &= (a-1)(b-2p)d_{p,0}^{b,0} - (a-1)(1-2p+2a)d_{p-a,0}^{b,0} - (b-1)(2b-p)d_{p-b,0}^{a,0} \\ &\quad - (b-1)(p+1-a)d_{p,0}^{a,0} - (a-1)(b-1)d_{p,0}^{a+b,0}. \end{aligned} \tag{2.49}$$

Replacing a, b with b, a in both equations (2.48) and (2.49), we obtain

$$\begin{aligned}0 &= (b-1)(a+2b-2p)d^{b,0}_{p-b,0} + (b-1)(1+2p-2a)d^{a,0}_{p,0} + (a-1)(1-p+a)d^{b,0}_{p-a,0} \\ &\quad + (a-1)(p-b+a)d^{b,0}_{p,0} + (b-1)(a-1)d^{a+b,0}_{p,0}, \qquad (2.50)\\ 0 &= (b-1)(a-2p)d^{a,0}_{p,0} - (b-1)(1-2p+2b)d^{a,0}_{p-b,0} - (a-1)(2a-p)d^{b,0}_{p-a,0} \\ &\quad - (a-1)(p+1-b)d^{b,0}_{p,0} - (b-1)(a-1)d^{a+b,0}_{p,0}. \qquad (2.51)\end{aligned}$$

Adding (2.49) to (2.48), we obtain

$$2(b-1)((a-1)d^{b,0}_{p-a,0} + (1-a)d^{b,0}_{p,0} + (1-3b)d^{a,0}_{p-b,0} + (1-b)d^{a,0}_{p,0}) = 0. \qquad (2.52)$$

Adding (2.51) to (2.48), we obtain

$$\begin{aligned}0 &= 2((ab+p-b-ap)d^{b,0}_{p-a,0} + (b+ap-ab-p)d^{b,0}_{p,0} \\ &\quad + (3b+bp-3b^2-p)d^{a,0}_{p-b,0} + (b^2+p-b-bp)d^{a,0}_{p,0}). \qquad (2.53)\end{aligned}$$

Multiplying (2.53) by $(a-1)$, (2.52) by $-2(ab+p-b-ap)$, and then adding both results together, one has

$$-8(a-1)(b-1)b(p+b-1)d^{a,0}_{p,0} = 0. \qquad (2.54)$$

According to (2.54), for $a \neq 0, 1$, we have

$$d^{a,0}_{p,0} = 0, \quad \text{unless } p = 0, a. \qquad (2.55)$$

For $a, b, a+b \neq 0, 1$, $a \neq b$, comparing the coefficients of $x^{0,0} \otimes x^{a+b,1}$, $x^{b,0} \otimes x^{a+b,1}$ and $x^{a+b,0} \otimes x^{a+b,1}$ in (2.47), we respectively obtain

$$\begin{aligned}&(a-1)(2a+b)d^{b,0}_{-a,0} + (a-1)(1-2b)d^{b,0}_{0,0} + (b-1)(1+b)d^{a,0}_{-b,0} \\ &\quad + (b-1)(-a+b)d^{a,0}_{0,0} + (a-1)(b-1)d^{a+b,0}_{0,0} - 3(a-1)(b-1)d^{0,2}_{0,0} = 0, \qquad (2.56)\\ &(a-1)(2a-b)d^{b,0}_{b-a,0} + (a-1)d^{b,0}_{b,0} + (b-1)d^{a,0}_{0,0} \\ &\quad - a(b-1)d^{a,0}_{b,0} + (a-1)(b-1)d^{a+b,0}_{b,0} + 3(a-1)(b-1)d^{0,2}_{0,0} = 0. \qquad (2.57)\end{aligned}$$

Combining equations (2.55) and (2.56), we get

$$d^{0,2}_{0,0} = 0. \qquad (2.58)$$

Then according to (2.44), one has

$$D_0(x^{0,2}) = 0. \qquad (2.59)$$

Hence, combining equations (2.55) and (2.57)–(2.58), one has

$$(a-1)d^{b,0}_{b,0} + (b-1)d^{a,0}_{0,0} = 0. \qquad (2.60)$$

According to equation (2.45), and taking $a = -1, b = 3$ in (2.60), we have

$$d^{3,0}_{3,0} = 0. \qquad (2.61)$$

For $a=b\neq 0,\pm 1$, comparing the coefficients of $x^{a,1}\otimes x^{a,0}$ in (2.47), one has

$$(a+1)d_{a,0}^{a,0}+(a+1)d_{0,0}^{a,0}+(a-1)d_{a,0}^{2a,0}+6(a-1)d_{0,0}^{0,2}=0. \tag{2.62}$$

Taking $a=3$ in (2.62), by (2.55), (2.58) and (2.61) we can deduce

$$d_{0,0}^{3,0}=0. \tag{2.63}$$

According to equation (2.63), and taking $a=3, b=-1$ in (2.60), we have

$$d_{-1,0}^{-1,0}=0. \tag{2.64}$$

Finally, by equations (2.55), (2.45), (2.61), (2.63) and (2.64), we deduce

$$D_0(x^{-1,0})=D_0(x^{3,0})=0. \tag{2.65}$$

Note that $\{x^{a,j}\,|\,(a,j)\in\mathbb{Z}\times\mathbb{Z}\}$ can be generated by the set $\{x^{-1,0},x^{0,2},x^{3,0}\}$. According to the facts that we have proved in (2.59) and (2.65), we can easily deduce that $D_0(x^{a,j})=0$ for $(a,j)\in\mathbb{Z}\times\mathbb{Z}$.

Now we can finish the proof of Claim 2.3 as follows.

Applying D_0 to $[x^{0,0},x^{a,0}]=0$ and $[x^{-1,0},x^{a,0}]=0$ respectively, using (2.32) we can deduce that

$$d_{p,0}^{a,0}=-d_{p,1}^{a,0}\quad\text{and}\quad d_{p,0}^{a,0}=-d_{p+1,1}^{a,0}. \tag{2.66}$$

That is, $d_{p,1}^{a,0}=d_{p+1,1}^{a,0}$. According to the fact that the set $\{p\in G\,|\,d_{p,1}^{a,0}\neq 0\}$ is of finite order, we obtain

$$d_{p,1}^{a,0}=0\quad\text{for any } p\in G. \tag{2.67}$$

Then by (2.66), we also have

$$d_{p,0}^{a,0}=0\quad\text{for any } p\in G. \tag{2.68}$$

Thus $D_0(x^{a,0})=0$ for any $a\in G$. Since, for any element $a\in G$ and $i\in\mathbb{Z}$, we always have

$$[x^{a,0},x^{0,i+1}]=(a-1)(i+1)x^{a,i},$$

it follows that, for any element $a\in G$ and $i\in\mathbb{Z}$,

$$D_0(x^{a,i})=0.$$

This proves Claim 2.3.

Claim 2.4 $D_0=0$.

By Claims 2.1–2.3, we have $D_0(B)\subseteq\mathbb{F}(C\otimes C)$. Since $[B,B]=B$, we obtain

$$D_0(B)\subseteq B\cdot(D_0(B))=0.$$

We can obtain $D_0(x^{a,j}) = 0$ for any $a \in G,\ j \in \mathbb{Z}$. Then, Claim 2.4 follows.

Claim 2.5 *For every $D \in \mathrm{Der}(B, V)$, (2.8) is a finite sum.*

By the above claims, we can suppose $D_a = (v_a)_{\mathrm{inn}}$ for some $v_a \in V_a$ and $a \in G$. If $G' = \{a \in G \backslash \{0\} \mid v_a \neq 0\}$ is an infinite set, we see that

$$D(\partial) = \sum_{a \in G' \cup \{0\}} \partial \cdot v_a = \sum_{a \in G'} a v_a$$

is an infinite sum, which is not an element in V, contradicting the fact that D is a derivation from B to V. This proves Claim 2.5 and the proposition.

Lemma 2.3 *Suppose $v \in V$ such that $b \cdot v \in \mathrm{Im}(1-\tau)$ for all $b \in B$. Then $v \in \mathrm{Im}(1-\tau)$.*

Proof (cf. [10]) First note that $B \cdot \mathrm{Im}(1-\tau) \subset \mathrm{Im}(1-\tau)$. We shall prove that after a number of steps in each of which v is replaced by $v - u$ for some $u \in \mathrm{Im}(1-\tau)$, the zero element is obtained and thus $v \in \mathrm{Im}(1-\tau)$ is proved. Write

$$v = \sum_{x \in G} v_x.$$

Obviously,

$$v \in \mathrm{Im}(1-\tau) \Leftrightarrow v_x \in \mathrm{Im}(1-\tau) \quad \text{for all } x \in G. \tag{2.69}$$

Then

$$\sum_{x \in G} x v_x = \partial \cdot v \in \mathrm{Im}(1-\tau).$$

By (2.69), $x v_x \in \mathrm{Im}(1-\tau)$, in particular,

$$v_x \in \mathrm{Im}(1-\tau), \qquad \text{if } x \neq 0.$$

Thus by replacing v by $v - \sum_{0 \neq x \in G} v_x$, we can suppose

$$v = v_0 \in V_0.$$

Now we can write

$$v = \sum_{p,q,r} w_{p,q,r} x^{p,q} \otimes x^{-p,r} \tag{2.70}$$

for some $w_{p,q,r} \in \mathbb{F}$. Choose any total order on G compatible with its additive group structure. Since

$$u_{p,q,r} := x^{p,q} \otimes x^{-p,r} - x^{-p,r} \otimes x^{p,q} \in \mathrm{Im}(1-\tau),$$

replacing v by $v - u$, where u is a combination of some $u_{p,q,r}$, we can suppose

$$w_{p,q,r} \neq 0 \Rightarrow p > 0 \text{ or } p = 0. \tag{2.71}$$

First assume that $w_{p,q,r} \neq 0$ for some $p > 0, q, r$ when $(p,q) \neq (1,0)$. Choose $s, t > 0$ such that

$$(s-1)q - t(p-1) \neq 0.$$

Then we see that the term $x^{p+s,q+t-1} \otimes x^{-p,r}$ appears in $x^{s,t} \cdot v$, but (2.71) implies that the term $x^{-p,r} \otimes x^{p+s,q+t-1}$ does not appear in $x^{s,t} \cdot v$, which is in contradiction with the fact that $x^{s,t} \cdot v \in \mathrm{Im}(1-\tau)$. Then assume that $w_{0,q,r} \neq 0$ for some q, r. Choose $s < 0, t > 0$ such that

$$(s-1)r + t \neq 0.$$

Then we see that the term $x^{0,q} \otimes x^{s,t+r-1}$ appears in $x^{s,t} \cdot v$, but (2.71) implies that the term $x^{s,t+r-1} \otimes x^{0,q}$ does not appear in $x^{s,t} \cdot v$, which is again in contradiction with the fact that $x^{s,t} \cdot v \in \mathrm{Im}(1-\tau)$. By now, we can write

$$v = \sum_r w_r x^{1,0} \otimes x^{-1,r}. \tag{2.72}$$

We have to prove $w_r = 0$ for all $r \in \mathbb{Z}$. If there is some $r_0 \in \mathbb{Z}$ such that $w_{r_0} \neq 0$, then there is some $s, t > 0, (s,t) \neq (2, 1-r_0)$ satisfying

$$(s-1)r_0 + 2t \neq 0.$$

That is, there is some $x^{s,t} \in B$ such that

$$(1+\tau)(x^{s,t} \cdot (x^{1,0} \otimes x^{-1,r_0})) \neq 0.$$

This contradicts the facts that $\mathrm{Im}(1-\tau) \subset \mathrm{Ker}(1+\tau)$ and $b \cdot v \in \mathrm{Im}(1-\tau)$ for all $b \in B$. Thus

$$v \in \mathrm{Im}(1-\tau).$$

This proves the lemma.

Proof of Theorem 1.1 Let $(B, [\cdot,\cdot], \Delta)$ be a Lie bialgebra structure on B. By (1.7), (2.4) and Proposition 2.1, $\Delta = \Delta_r$ is defined by (1.9) for some $r \in B \otimes B$. By (1.3), $\mathrm{Im}\,\Delta \subset \mathrm{Im}(1-\tau)$. Thus by Lemma 2.3, $r \in \mathrm{Im}(1-\tau)$. Then (1.4), (2.1) and Corollary 2.1 show that $c(r) = 0$. Thus Definition 1.2 says that $(B, [\cdot,\cdot], \Delta)$ is a triangular coboundary Lie bialgebra.

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