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The Kähler-Ricci Flow on Kähler Manifolds with 2-Non-negative Traceless Bisectional Curvature Operator***

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Abstract The authors show that the 2-non-negative traceless bisectional curvature is preserved along the Kähler-Ricci flow. The positivity of Ricci curvature is also preserved along the Kähler-Ricci flow with 2-non-negative traceless bisectional curvature. As a corollary, the Kähler-Ricci flow with 2-non-negative traceless bisectional curvature will converge to a Kähler-Ricci soliton in the sense of Cheeger-Gromov-Hausdorff topology if complex dimension $n \geq 3$.

Keywords Kähler-Ricci flow, 2-Non-negative bisectional curvature
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1 Introduction

In 1982, in a famous paper [10], R. Hamilton proved that in a 3-dimensional compact manifold, if the initial metric has positive Ricci curvature, then this positivity condition will be preserved under the Ricci flow. He showed that the underlying manifold must be diffeomorphic to the standard S^3 or its finite quotient. Following this paper, there are intensive active researches on Ricci flow, and many works are devoted to study when certain convex cones of curvature pinching conditions are preserved by the Ricci flow. In [11], R. Hamilton proved that the positive curvature operator is preserved under the Ricci flow in all dimensions. H. Chen [5] further showed that a weaker notion, that the sum of any two eigenvalues is positive, is preserved under the Ricci flow. In 2004, L. Ni [14] constructed an example in a complete Riemannian manifold where the positivity of sectional curvature is not preserved by the Ricci flow. On the other hand, in the Kähler setting, it is well-known that the positive bisectional curvature is preserved under the Kähler-Ricci flow through the work of S. Bando [1] for complex dimension $n = 3$, and later N. Mok [15] for general dimensions. Following the argument of N. Mok, in an unpublished work of Cao-Hamilton, they proved that the orthogonal bisectional curvature (cf. Definition 3.2) is preserved under the Kähler-Ricci flow. There are other convex cones of curvature pinching conditions which are preserved, for instance [3, 13]. A more complete reference on this topic can be found in [12] .

In analyzing the evolution equation (2.4) of the Ricci tensor, it is somewhat unfortunate that the parabolic Laplacian of the Ricci tensor involves the full sectional curvature. It is then

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no surprise that we only know the positivity of Ricci tensor is preserved in real dimensions 2 and 3 by the earlier work of R. Hamilton. A counter example to the possible extension of R. Hamilton's result on Ricci tensor in high dimensions seems to be difficult to construct. Recently, D. Knopf [9] constructed a counter example in the Kähler setting where the positivity of Ricci curvature is not preserved. Again, D. Knopf's example is in a complete Kähler manifold. Therefore, it is still an open question whether or not positive Ricci curvature is preserved under the Ricci flow in the case of compact manifolds. In particular, in the case of compact Kähler manifolds, there might be some hope that some form of lower bound of Ricci curvature will be preserved in [6] where the first named author showed, along with other results, that any metric with positive orthogonal bisectional curvature, even a negative lower bound of Ricci curvature, is preserved and improved under the Kähler-Ricci flow. (The main result proved in [6] is that any irreducible Kähler manifold with positive first Chern class, where the positive orthogonal bisectional curvature is preserved under the Kähler-Ricci flow, must be biholomorphic to $\mathbb{CP}^n$.)

In a compact Kähler manifold X, the bisectional curvature tensor acts as a symmetric bilinear form on the space of $(1,1)$ form (which we will denote as $\Lambda^{1,1}(X)$). Furthermore, this action preserves the traceless part of this space (which we will denote as $\Lambda_0^{1,1}(X)$). In a recent paper by Phong and Sturm [19], they observed that the condition that the sum of any two eigenvalues of the traceless bisectional curvature operator is positive, is preserved under the Kähler-Ricci flow in complex dimension 2. Note that this condition is different from the condition used by H. Chen, even though the main idea of proof is very similar. The main theorem they proved in [19] is that, if this curvature condition holds, then the positivity of Ricci curvature will be preserved under the Kähler-Ricci flow in complex surfaces. The proof there is difficult and intriguing.

The 2-positive traceless bisectional curvature is certainly different to the popular notion of positive bisectional curvature. For instance, when this curvature condition holds, the Ricci curvature might not be positive. In [7, 8], the first named author and G. Tian studied the convergence of Kähler-Ricci flow in Kähler Einstein manifolds where the initial metric has positive bisectional curvature and showed that the Kähler-Ricci flow must converge to the Fubini-Study metric exponentially over the flow. The present work can be viewed as a continuation of [7, 8] in the sense that the curvature condition is relaxed in some subtle way. However, one of more immediate motivations of the present work is [5] and more recently [19]. The interest of the first named author in this type of special curvature conditions was certainly re-invigorated by this elegantly written paper [19]. Together with the second named author, we start to investigate systematically geometrical properties of this 2-non-negative traceless bisectional curvature operator on any Kähler manifold. Our first result is

Theorem 1.1 *Let X be a compact Kähler manifold with $c_1(M) > 0$, of complex dimension $n \geq 2$. Along the Kähler-Ricci flow, we have*

(1) *If the initial metric has non-negative traceless bisectional curvature operator, then the evolved metrics also have non-negative traceless bisectional curvature operator. If it is positive at one point initially, then it is positive everywhere for all $t > 0$.*

(2) *If the initial metric has 2-non-negative traceless bisectional curvature operator, then the evolved metrics also have 2-non-negative traceless bisectional curvature. If it is positive at one point initially, then it is positive everywhere for all $t > 0$.*

Under either of these two conditions, the positivity of Ricci tensor is preserved under the

Kähler-Ricci flow.

The relation between 2-positive traceless bisectional curvature and the notion of positive orthogonal bisectional curvature is much more subtle. They are defined in a completely different manner and the action of bisectional curvature operator on the space of (1,1) forms is very complicated. It is hard to visualize what 2-positive traceless bisectional curvature really is. A somewhat surprising result we prove in this paper is that (cf. Theorem 1.2) any Kähler metric which has 2-positive traceless bisectional curvature must also have positive orthogonal bisectional curvature. The last part of the preceding theorem follows directly from the application of Hamilton's maximal principle for tensors to the evolution equation of the Ricci tensor. Compared with the main theorem in [19], our theorem is for all dimensions and our proof is simpler and more straightforward, even in complex surfaces.

Theorem 1.2 *In a Kähler manifold with* 2*-non-negative traceless bisectional curvature operator, the orthogonal bisectional curvature must be non-negative. If, in addition, the scalar curvature is uniformly bounded from above and the dimension* $n \geq 3$*, then the bisectional curvature is uniformly bounded. Moreover, if we assume that the traceless bisectional curvature operator is non-negative, then the sum of any two eigenvalues of the Ricci tensor is non-negative.*

Remark 1.1 We point out that the condition $n \geq 3$ can not be removed. In Section 4, we construct a Kähler surface which has non-negative traceless bisectional curvature while the scalar curvature cannot bound the bisectional curvature.

Remark 1.2 In the special case of complex surfaces, similar estimate was derived in [20]. However, they need to assume also the non-negativity of Ricci curvature. In an unpublished work of G. Perelman, the scalar curvature is uniformly bounded along the Kähler-Ricci flow. Combining this with Theorem 1.2, we conclude that the full bisectional curvature is uniformly bounded over the Kähler-Ricci flow when the initial metric has 2-non-negative traceless bisectional curvature.

Following Remark 1.2 and a general theorem on the Kähler-Ricci flow (cf. [21, 22]), we get the following

Corollary 1.1 *Let* X *be a compact Kähler manifold with* $c_1(M) > 0$*, of complex dimension* $n \geq 3$*. Along the Kähler-Ricci flow, if the initial metric has* 2*-non-negative traceless bisectional curvature operator, then the flow converges by sequences to some Kähler-Ricci soliton in the limit in the sense of Cheeger-Gromov-Hausdorff topology.*

Similar results was also proved by Phong-Sturm [20] in complex surfaces with additional assumption that the initial metric has non-negative Ricci curvature.

2 Basic Kähler Geometry

2.1 Setup of notations

Let X be an n-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form ω on X. In local coordinates $z_1, \cdots, z_n$, this ω is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\overline{j}} \,\mathrm{d}\, z^i \wedge \mathrm{d}\, z^{\overline{j}} > 0,$$

where $\{g_{i\overline{j}}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that ω be a closed positive (1,1)-form. In other words, the following holds

$$\frac{\partial g_{i\overline{k}}}{\partial z^j} = \frac{\partial g_{j\overline{k}}}{\partial z^i} \quad \text{and} \quad \frac{\partial g_{k\overline{i}}}{\partial z^{\overline{j}}} = \frac{\partial g_{k\overline{j}}}{\partial z^{\overline{i}}}, \quad \forall\, i,j,k = 1,2,\cdots,n.$$

The Kähler metric corresponding to ω is given by

$$\sqrt{-1}\sum_{1}^{n} g_{\alpha\overline{\beta}}\, \mathrm{d}\, z^{\alpha} \otimes \mathrm{d}\, z^{\overline{\beta}}.$$

For simplicity, in the following, we will often denote by ω the corresponding Kähler metric. The Kähler class of ω is its cohomology class $[\omega]$ in $H^2(X,\mathbb{R})$. By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$\omega_{\varphi} = \omega + \sqrt{-1}\sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z^i \partial z^{\overline{j}}} > 0$$

for some real valued function φ on X. The functional space in which we are interested (often referred to as the space of Kähler potentials) is

$$\mathcal{P}(X,\omega) = \{\varphi \in C^{\infty}(X,\mathbb{R}) \mid \omega_{\varphi} = \omega + \sqrt{-1}\,\partial\overline{\partial}\varphi > 0 \text{ on } X\}.$$

Given a Kähler metric ω, its volume form is

$$\omega^n = n!(\sqrt{-1}\,)^n \det(g_{i\overline{j}})\mathrm{d}\, z^1 \wedge \mathrm{d}\, z^{\overline{1}} \wedge \cdots \wedge \mathrm{d}\, z^n \wedge \mathrm{d}\, z^{\overline{n}}.$$

Its Christoffel symbols are given by

$$\Gamma^k_{i\,j} = \sum_{l=1}^{n} g^{k\overline{l}} \frac{\partial g_{i\overline{l}}}{\partial z^j} \quad \text{and} \quad \Gamma^{\overline{k}}_{\overline{i}\,\overline{j}} = \sum_{l=1}^{n} g^{\overline{k}l} \frac{\partial g_{l\overline{i}}}{\partial z^{\overline{j}}}, \quad \forall\, i,j,k = 1,2,\cdots n.$$

The curvature tensor is

$$R_{i\overline{j}k\overline{l}} = -\frac{\partial^2 g_{i\overline{j}}}{\partial z^k \partial z^{\overline{l}}} + \sum_{p,q=1}^{n} g^{p\overline{q}} \frac{\partial g_{i\overline{q}}}{\partial z^k} \frac{\partial g_{p\overline{j}}}{\partial z^{\overline{l}}}, \quad \forall\, i,j,k,l = 1,2,\cdots n.$$

We say that ω is of nonnegative bisectional curvature if

$$R_{i\overline{j}k\overline{l}} v^i v^{\overline{j}} w^k w^{\overline{l}} \geq 0$$

for all non-zero vectors v and w in the holomorphic tangent bundle of X. The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature of ω is locally given by

$$R_{i\overline{j}} = -\frac{\partial^2 \log\det(g_{k\overline{l}})}{\partial z_i \partial \overline{z}_j}.$$

So its Ricci curvature form is

$$\mathrm{Ric}(\omega) = \sqrt{-1}\sum_{i,j=1}^{n} R_{i\overline{j}}(\omega)\mathrm{d}\, z^i \wedge \mathrm{d}\, z^{\overline{j}} = -\sqrt{-1}\,\partial\overline{\partial}\log\det(g_{k\overline{l}}).$$

It is a real, closed (1,1)-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to $\lambda\,\omega$, for some constant λ. In our setting, we require $\lambda = 1$. The trace of Ricci curvature is the scalar curvature, which is given by

$$R = g^{i\overline{j}} R_{i\overline{j}}.$$

2.2 The Kähler-Ricci flow

Now we assume that the first Chern class $c_1(X)$ is positive. The normalized Ricci flow (cf. [10, 11]) on a Kähler manifold X is of the form

$$\frac{\partial g_{i\bar{j}}}{\partial t} = g_{i\bar{j}} - R_{i\bar{j}}, \quad \forall\, i, j = 1, 2, \cdots, n. \tag{2.1}$$

If we choose the initial Kähler metric ω with $c_1(X)$ as its Kähler class. The flow (2.1) preserves the Kähler class $[\omega]$. It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega, \tag{2.2}$$

where h_ω is defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\overline{\partial} h_\omega \quad \text{and} \quad \int_X (\mathrm{e}^{h_\omega} - 1)\omega^n = 0.$$

Then the evolution equation for bisectional curvature is

$$\begin{aligned}\frac{\partial}{\partial t} R_{i\bar{j}k\bar{l}} = {}& \triangle R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}} R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}} R_{q\bar{p}k\bar{j}} + R_{i\bar{j}k\bar{l}} \\ & - \frac{1}{2}(R_{i\bar{p}} R_{p\bar{j}k\bar{l}} + R_{p\bar{j}} R_{i\bar{p}k\bar{l}} + R_{k\bar{p}} R_{i\bar{j}p\bar{l}} + R_{p\bar{l}} R_{i\bar{j}k\bar{p}}).\end{aligned} \tag{2.3}$$

Here Δ is the complex Laplacian with respect to the metric $g(t)$, and it acts on the bisectional curvature by

$$\Delta R_{i\bar{j}k\bar{l}} = \frac{1}{2} g^{\alpha\overline{\beta}}(R_{i\bar{j}k\bar{l},\alpha\overline{\beta}} + R_{i\bar{j}k\bar{l},\overline{\beta}\alpha}).$$

The evolution equation for Ricci curvature and scalar curvature are

$$\frac{\partial R_{i\bar{j}}}{\partial t} = \triangle R_{i\bar{j}} + R_{i\bar{j}p\bar{q}} R_{q\bar{p}} - R_{i\bar{p}} R_{p\bar{j}}, \tag{2.4}$$

$$\frac{\partial R}{\partial t} = \triangle R + R_{i\bar{j}} R_{j\bar{i}} - R. \tag{2.5}$$

By direct computations and using the evolved frames, we obtain the following evolution equation for the bisectional curvature:

$$\frac{\partial R_{i\bar{j}k\bar{l}}}{\partial t} = \Delta R_{i\bar{j}k\bar{l}} - R_{i\bar{j}k\bar{l}} + R_{i\bar{j}m\bar{n}} R_{n\overline{m}k\bar{l}} - R_{i\overline{m}k\overline{n}} R_{m\bar{j}n\bar{l}} + R_{i\bar{l}m\overline{n}} R_{n\overline{m}k\bar{l}}. \tag{2.6}$$

As usual, the flow equation (2.1) or (2.2) is referred to as the Kähler-Ricci flow on X. It is proved by Cao [2], who followed Yau's celebrated work [23], that the Kähler-Ricci flow exists globally for any smooth initial Kähler metric.

3 The Traceless Bisectional Curvature Operator

3.1 Definitions and the evolution equations

In Riemannian geometry, the curvature tensor for Riemannian metric can always be decomposed orthogonally into three parts: $Rm = W + V + U$, where W is the Weyl tensor and V, U are the traceless Ricci part and the scalar curvature part respectively. The decomposition for

Kähler case is slightly different. The bisectional curvature tensor can also be decomposed into orthogonal parts as well.

Set

$$S_{i\bar{j}} = R_{i\bar{j}} - \frac{1}{n} R g_{i\bar{j}} = R^0_{i\bar{j}}, \tag{3.1}$$

$$S_{a\bar{b}c\bar{d}} = R_{a\bar{b}c\bar{d}} - \frac{1}{n}(S_{a\bar{b}} g_{c\bar{d}} + S_{c\bar{d}} g_{a\bar{b}}) - \frac{1}{n^2} R g_{a\bar{b}} g_{c\bar{d}}. \tag{3.2}$$

As in the Riemannian case, the "Weyl" part $S_{a\bar{b}c\bar{d}}$ is also trace free:

$$S_{a\bar{b}c\bar{d}} = S_{c\bar{d}a\bar{b}}, \quad g^{a\bar{b}} S_{a\bar{b}c\bar{d}} = 0.$$

As in the previous subsection, under some evolved moving frame, we can rewrite the evolution equation for curvature as follows (cf. [19])

Proposition 3.1 *Along the Kähler-Ricci flow the evolution equation relating the traceless bisectional curvature operator are as follows:*

$$\frac{\partial R}{\partial t} = \Delta R - R + \frac{1}{n} R^2 + S_{\alpha\bar{\beta}} S_{\beta\bar{\alpha}}, \tag{3.3}$$

$$\frac{\partial S_{a\bar{b}}}{\partial t} = \Delta S_{a\bar{b}} + \frac{1}{n}(R-n) S_{a\bar{b}} + S_{a\bar{b}i\bar{j}} S_{j\bar{i}}, \tag{3.4}$$

$$\frac{\partial S_{a\bar{b}c\bar{d}}}{\partial t} = \Delta S_{a\bar{b}c\bar{d}} - S_{a\bar{b}c\bar{d}} + S_{a\bar{b}i\bar{j}} S_{j\bar{i}c\bar{d}} + S_{a\bar{i}j\bar{d}} S_{i\bar{b}c\bar{j}} - S_{a\bar{i}c\bar{j}} S_{i\bar{b}j\bar{d}} + \frac{1}{n} S_{a\bar{b}} S_{c\bar{d}}. \tag{3.5}$$

The bisectional curvature operator can be viewed as a symmetric operator on the space of real $(1,1)$ forms $\Lambda^{1,1}(X)$. For any pair of $(1,1)$ forms η, τ, the action of the bisectional curvature is

$$\mathcal{R}(\eta, \tau) = R_{i\bar{j}k\bar{l}}\, \eta_{a\bar{b}}\, \tau_{c\bar{d}}\, g^{i\bar{b}} g^{a\bar{j}} g^{k\bar{d}} g^{c\bar{l}}.$$

If we decompose the space $\Lambda^{1,1}(X)$ into the line which consists of the multiple of the Kähler form and its orthogonal complementary subspace $\Lambda^{1,1}_0(X)$, then the action of $S_{i\bar{j}k\bar{l}}$ preserves $\Lambda^{1,1}_0(X)$. Denote the action of $S_{i\bar{j}k\bar{l}}$ by $\mathcal{S}$. In some special basis, we will use M to denote the matrix of the operator $\mathcal{S}$. We often refer $\mathcal{S}$ as the traceless bisectional curvature operator. Moreover, there is a nice decomposition formula for the bisectional curvature operator in $\Lambda^{1,1}(X)$:

$$\begin{pmatrix} R & \mathrm{Ric}^0 \\ \mathrm{Ric}^{0t} & \mathcal{S} \end{pmatrix}. \tag{3.6}$$

If the action of $\mathcal{S}$ in $\Lambda^{1,1}_0(X)$ is non-negative, then we call the underlying Kähler metric has a non-negative traceless bisectional curvature operator. If the action of $\mathcal{S}$ in $\Lambda^{1,1}_0(X)$ has a property that the sum of any two eigenvalues is non-negative, then we say that the underlying Kähler metric has a 2-non-negative traceless bisectional curvature operator.

3.2 Geometric properties of the traceless bisectional curvature operator

In this subsection, we derive some geometric properties of the traceless bisectional curvature operator. First, in any local coordinate, after fixing a frame such that the metric tensor at the origin is an identity matrix, there is a natural orthonormal basis for $\Lambda^{1,1}(X)$ at the origin point (here $i, j = 1, 2, \cdots n$):

$$\{\sqrt{-1}\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^i, \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j - \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^i, \sqrt{-1}\,(\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j + \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^i)\}.$$

For convenience, we use the following notations.

Definition 3.1 *The space $\Lambda_0^{1,1}(X)$ is locally spanned by the following elements:*

$$\begin{aligned} A^{ij} &= \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^i - \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^j, \\ B^{ij} &= \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j + \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^i, \\ C^{ij} &= -\sqrt{-1}\,(\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j - \mathrm{d}z^j \wedge \mathrm{d}\overline{z}^i), \end{aligned}$$

where $1 \leq i \neq j \leq n$.

One remarks that this is not an orthonomal basis since $\{A^{1i},\ 2 \leq i \leq n\}$ are not orthogonal to each other. However, A is orthonomal to both B and C while B, C are an orthonomal basis for some subspace.

In this paper, we often use the following definition.

Definition 3.2 *An orthogonal bisectional curvature is a holomorphic bisectional curvature which acts on two orthogonal holomorphic planes.*

Proposition 3.2 *If the traceless bisectional curvature operator is 2-non-negative, then the orthogonal bisectional curvature is nonnegative. If the traceless bisectional curvature operator is nonnegative, then we have the following inequalities:*

$$R_{i\bar{i}i\bar{i}} + R_{j\bar{j}j\bar{j}} \geq 2R_{i\bar{i}j\bar{j}} \geq 0, \quad R_{i\bar{i}} + R_{j\bar{j}} \geq 0$$

for any $i \neq j$.

Proof (1) If A is a symmetric matrix and the sum of two lowest eigenvalues of A is nonnegative, then $A_{ii} + A_{jj} \geq 0$ if $i \neq j$. To see this, assuming that $m_1 \leq m_2 \leq \cdots \leq m_n$ are the eigenvalues of A, we have

$$m_1 + m_2 = \inf\{A(x, x) + A(y, y) \mid |x| = |y| = 1,\ x \perp y\} \geq 0.$$

Then for any $i \neq j$, we have

$$A_{ii} + A_{jj} = A(e_i, e_i) + A(e_j, e_j) \geq 0,$$

where $\{e_i\}$ are the standard basis of $\mathbb{R}^n$.

(2) Assume that $\mathcal{S}$ is 2-non-negative. Since the matrix of $\mathcal{S}$ is the same as the matrix of curvature operator Rm when acting on the space $\Lambda_0^{1,1}(X)$, we have

$$R(B^{ij}, B^{ij}) + R(C^{ij}, C^{ij}) \geq 0,$$

which implies $R_{i\bar{i}j\bar{j}} \geq 0,\ \forall\, i \neq j$.

(3) Assume that the traceless bisectional curvature operator is nonnegative,

$$R(A^{ij}, A^{ij}) = R_{i\bar{i}i\bar{i}} + R_{j\bar{j}j\bar{j}} - 2R_{i\bar{i}j\bar{j}} \geq 0.$$

Thus, we have

$$\begin{aligned}R_{i\bar{i}}+R_{j\bar{j}} &= R_{i\bar{i}i\bar{i}}+\sum_{\alpha\neq i}R_{\alpha\overline{\alpha}i\bar{i}}+R_{j\bar{j}j\bar{j}}+\sum_{\beta\neq j}R_{\beta\overline{\beta}j\bar{j}}\\ &\geq 2R_{i\bar{i}j\bar{j}}+\sum_{\alpha\neq i}R_{\alpha\overline{\alpha}i\bar{i}}+\sum_{\beta\neq j}R_{\beta\overline{\beta}j\bar{j}}\\ &\geq 0,\end{aligned}$$

where $i\neq j$. The proposition is proved.

4 Proof of Theorem 1.2

We follow notations in the previous section. Note that Proposition 3.2 already implies the first and last parts of Theorem 1.2, so it suffices to prove the following

Theorem 4.1 *Let X be a compact Kähler manifold of dimension $n\geq 3$. If the traceless bisectional curvature is 2-non-negative and the scalar curvature is bounded from above, then the bisectional curvature is uniformly bounded.*

Proof Choose a local coordinate at any point $x\in X$ as in Subsection 3.2, by the definition of scalar curvature, we have

$$R=\sum_{k=1}^{n}R_{k\overline{k}k\overline{k}}+2\sum_{i<j}R_{i\bar{i}j\bar{j}}. \tag{4.1}$$

Since $\mathcal{S}$ is 2-non-negative, by Proposition 3.2 the orthogonal bisectional curvature is nonnegative. Thus we have

$$R\geq\sum_{k=1}^{n}R_{k\overline{k}k\overline{k}}. \tag{4.2}$$

For fixed i and j , one notes that A^{ij} is orthonomal to both B^{ij} and C^{ij}. Following part (1) of the proof of Proposition 3.2, we have (since $\mathcal{S}$ is 2-non-negative)

$$\begin{aligned}&R(A^{ij},A^{ij})+R(B^{ij},B^{ij})\geq 0,\\ &R(A^{ij},A^{ij})+R(C^{ij},C^{ij})\geq 0\end{aligned}$$

for all $i\neq j$. Thus

$$(R(A^{ij},A^{ij})+R(B^{ij},B^{ij}))+(R(A^{ij},A^{ij})+R(C^{ij},C^{ij}))\geq 0.$$

This implies

$$R_{i\bar{i}i\bar{i}}+R_{j\bar{j}j\bar{j}}\geq 0,\quad \forall\, i\neq j. \tag{4.3}$$

Thus, we have

$$\sum_{k=1}^{n}R_{k\overline{k}k\overline{k}}\geq 0. \tag{4.4}$$

Combining this with (4.1), for all $i \neq j$, we have

$$0 \leq R_{i\bar{i}j\bar{j}} \leq \frac{R}{2}. \tag{4.5}$$

Note that $n \geq 3$, (4.2) and (4.3) imply that for all k,

$$R_{k\bar{k}k\bar{k}} \leq R. \tag{4.6}$$

Claim 4.1 *For any $1 \leq k \leq n$, we have $|R_{k\bar{k}k\bar{k}}| \leq 2R$.*

Proof Assume that the holomorphic bisectional curvatures satisfy the following inequalities

$$R_{1\bar{1}1\bar{1}} \leq R_{2\bar{2}2\bar{2}} \leq R_{3\bar{3}3\bar{3}} \leq \cdots \leq R_{n\bar{n}n\bar{n}}.$$

By (4.3), we have $R_{2\bar{2}2\bar{2}} \geq 0$. By (4.6), it suffices to show $R_{1\bar{1}1\bar{1}} \geq -2R$. In fact, since $\mathcal{S}$ is 2-non-negative, we have

$$R(A^{12}, A^{12}) + R(B^{12}, B^{12}) + R(C^{12}, C^{12}) \geq 0.$$

This implies

$$R_{1\bar{1}1\bar{1}} + R_{2\bar{2}2\bar{2}} + 2R_{1\bar{1}2\bar{2}} \geq 0.$$

By (4.5) and (4.6), we have

$$R_{1\bar{1}1\bar{1}} \geq -(R_{2\bar{2}2\bar{2}} + 2R_{1\bar{1}2\bar{2}}) \geq -2R.$$

The claim is proved.

Since all the curvature like $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ are bounded by the scalar curvature by Claim 4.1, other curvature tensors are also bounded. This can be seen from the following claim.

Claim 4.2 *For all i, j, k, l, we have $|R_{i\bar{j}k\bar{l}}| \leq cR$, where c is a universal constant.*

Proof The idea of the proof is to write the curvature tensors as some linear combinations of curvatures like $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ and then use Claim 4.1. Here we assume that i, j, k, l are different from each other. Setting

$$\mathrm{e}^{\alpha} = \mathrm{d}\, z^i + \mathrm{d}\, z^j, \quad \mathrm{e}^{\beta} = \mathrm{d}\, z^i + \sqrt{-1}\, \mathrm{d}\, z^j, \quad \epsilon^{\gamma} = \mathrm{d}\, z^i - \mathrm{d}\, z^j,$$

we can check

$$R_{i\bar{j}k\bar{k}} = \frac{1}{2}(R_{\alpha\overline{\alpha}k\bar{k}} - R_{i\bar{i}k\bar{k}} - R_{j\bar{j}k\bar{k}} + \sqrt{-1}\,(R_{\beta\overline{\beta}k\bar{k}} - R_{i\bar{i}k\bar{k}} - R_{j\bar{j}k\bar{k}})).$$

Thus, $|R_{i\bar{j}k\bar{k}}| \leq cR$. For $R_{i\bar{j}i\bar{j}}$, we can use the following identity:

$$\mathrm{Re}(R_{i\bar{j}i\bar{j}}) = \frac{1}{4}(R_{\alpha\overline{\alpha}\alpha\overline{\alpha}} + R_{\gamma\overline{\gamma}\gamma\overline{\gamma}} - 2R_{i\bar{i}i\bar{i}} - 8R_{i\bar{i}j\bar{j}} - 2R_{j\bar{j}j\bar{j}}).$$

Thus, $|\mathrm{Re}(R_{i\bar{j}i\bar{j}})| \leq cR$. Similarly, we can prove $|\mathrm{Im}(R_{i\bar{j}i\bar{j}})| \leq cR$, and so $|R_{i\bar{j}i\bar{j}}| \leq cR$. Using the same method, we see that other curvatures like $R_{i\bar{i}i\bar{j}}, R_{i\bar{j}i\bar{l}}, R_{i\bar{j}k\bar{l}}$ are also bounded by R. The claim is proved.

In summary, all the bisectional curvature tensors are bounded by the scalar curvature R if $\mathcal{S}$ is 2-non-negative. The theorem is proved.

Example 4.1 In complex surfaces, if $\mathcal{S}$ is 2-non-negative, the scalar curvature may not control the bisectional curvature. In fact, we consider the Kähler surface $S^2 \times S^2$, with a product Kähler metric

$$\omega = \sqrt{-1}\, F(z)\mathrm{d}z \wedge \mathrm{d}\overline{z} + \sqrt{-1}\, G(w)\mathrm{d}w \wedge \mathrm{d}\overline{w},$$

where z, w are a local coordinate of the first and second factor of $S^2 \times S^2$ respectively. Let $U \times U$ be an open neighborhood of $(0,0) \in S^2 \times S^2$. Set

$$F(z) = \frac{1}{(1+\lambda^2|z|^2)^2} \quad \text{and} \quad G(w) = \frac{1}{(1-\lambda^2|w|^2)^2}, \quad (z,w) \in U \times U,$$

where $\lambda \in \mathbb{R} \setminus \{0\}$. Now we calculate the curvature. Choose an orthonormal basis $\{e^1, e^2\}$ at any $(z,w) \in U \times U$, where

$$e^1 = \frac{1}{1+\lambda^2|z|^2}\mathrm{d}z, \quad e^2 = \frac{1}{1-\lambda^2|w|^2}\mathrm{d}w,$$

and set $R_{i\overline{j}k\overline{l}} = R(e^i, \overline{e^j}, e^k, \overline{e^l})$. Then for all points in $U \times U$ we have

$$R_{1\overline{1}1\overline{1}} = 2\lambda^2, \quad R_{2\overline{2}2\overline{2}} = -2\lambda^2$$

and other curvatures are zero. Thus, $\mathcal{S} = 0$ is non-negative and the scalar curvature $R = 0$, but the bisectional curvature is obviously unbounded.

5 Proof of Theorem 1.1

In this section, we are ready to prove Theorem 1.1. Note that in [11], the positivity of curvature operator is preserved and in [5] the 2-positivity of curvature operator is preserved along the Ricci flow. One can also see both from [12]. Our proof here is similar to theirs.

Now we begin to prove Theorem 1.1.

Proof (1) Define

$$[\phi^\lambda, \phi^\mu]_{a\overline{b}} = \phi^\lambda_{a\overline{m}}\phi^\mu_{m\overline{b}} - \phi^\mu_{a\overline{m}}\phi^\lambda_{m\overline{b}} = C^{\lambda\mu}_\rho \phi^\rho_{a\overline{b}},$$

where $\{\phi^\lambda\}$ is a basis of $\Lambda^{1,1}_0(X)$, which is some linear combinations of $\{A^{ij}, B^{ij}, C^{ij},\ i \neq j\}$ (cf. Definition 3.1). The coefficients $C^{\alpha\beta}_\gamma$ here are pure imaginary or zero in our notation, since one can check that $[A^{ij}, B^{ij}] = 2\sqrt{-1}\, C^{ij}$, etc. Now observe

$$\begin{aligned}
S_{a\overline{m}n\overline{d}}S_{m\overline{b}c\overline{n}} - S_{a\overline{m}c\overline{n}}S_{m\overline{b}n\overline{d}} &= M_{\alpha\beta}\phi^\alpha_{a\overline{m}}\phi^\beta_{n\overline{d}}M_{\gamma\delta}\phi^\gamma_{m\overline{b}}\phi^\delta_{c\overline{n}} - M_{\alpha\beta}\phi^\alpha_{a\overline{m}}\phi^\beta_{c\overline{n}}M_{\gamma\delta}\phi^\gamma_{m\overline{b}}\phi^\delta_{n\overline{d}} \\
&= M_{\alpha\beta}M_{\gamma\delta}\phi^\alpha_{a\overline{m}}\phi^\gamma_{m\overline{b}}(\phi^\beta_{n\overline{d}}\phi^\delta_{c\overline{n}} - \phi^\beta_{c\overline{n}}\phi^\delta_{n\overline{d}}) \\
&= M_{\alpha\beta}M_{\gamma\delta}\phi^\alpha_{a\overline{m}}\phi^\gamma_{m\overline{b}}C^{\delta\beta}_\rho\phi^\rho_{c\overline{d}} \\
&= -\frac{1}{2}C^{\alpha\gamma}_q C^{\beta\delta}_p M_{\alpha\beta}M_{\gamma\delta}\phi^q_{a\overline{b}}\phi^p_{c\overline{d}},
\end{aligned}$$

where M is the matrix of $\mathcal{S}$ under the basis $\{\phi^\lambda\}$. Define

$$M^\#_{qp} = C^{\alpha\gamma}_q C^{\beta\delta}_p M_{\alpha\beta}M_{\gamma\delta}. \tag{5.1}$$

Then we have

$$S_{a\overline{m}n\overline{d}}S_{m\bar{b}c\overline{n}} - S_{a\overline{m}c\overline{n}}S_{m\bar{b}n\overline{d}} = -\frac{1}{2}M^{\#}_{qp}\phi^{q}_{a\bar{b}}\phi^{p}_{c\bar{d}}$$

and

$$\frac{\partial M}{\partial t} = -M + M^2 - \frac{1}{2}M^{\#} + \frac{1}{n}T. \tag{5.2}$$

Now we have the following lemma.

Lemma 5.1 *If all $C_q^{\alpha\gamma}$ are real and $M \geq 0$, then $M^{\#} \geq 0$.*

Proof Without loss of generality, we may choose a basis $\{\phi^\alpha\}$ which diagonalizes M, so that $M_{\alpha\beta} = \delta_{\alpha\beta}M_{\alpha\alpha}$. For any $v = v^\alpha\phi^\alpha$, we have

$$M^{\#}(v,v) = (v^\alpha C_\alpha^{ai})(v^\beta C_\beta^{bj})M_{ab}M_{ij} = (v^\alpha C_\alpha^{ai})^2 M_{aa}M_{ii} \geq 0.$$

The lemma is then proved.

Now we return to the proof of Theorem 1.1 again. Since in our case all $C_\rho^{\lambda\mu}$ are zero or pure imaginary numbers, we have $M^{\#} \leq 0$ if $M \geq 0$. Since T is always non-negative, we have

$$\frac{\partial M}{\partial t} = -M + M^2 - \frac{1}{2}M^{\#} + \frac{1}{n}T \geq 0,$$

when $M = 0$. Note that $M \geq 0$ is convex and $M(0) \geq 0$, we have $M(t) \geq 0$ for all $t > 0$. In other words, the nonnegative traceless bisectional curvature operator is preserved. By the strong maximum principle, if M is positive at one point at time $t = 0$, M is positive everywhere for all time $t > 0$.

(2) We want to prove that the 2-non-negative traceless bisectional curvature operator is preserved along the Kähler-Ricci flow. Let us assume that the eigenvalues of the traceless bisectional curvature operator on $\Lambda_0^{1,1}(X)$ are $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_m$, where $m = n^2 - 1$. From (5.1), (5.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_1 + \lambda_2) &\geq \frac{\mathrm{d}}{\mathrm{d}t}(M_{11} + M_{22}) \\ &\geq -(\lambda_1 + \lambda_2) + (\lambda_1^2 + \lambda_2^2) - \frac{1}{2}\sum_{p,q}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q. \end{aligned} \tag{5.3}$$

Note that the right-hand side

$$\begin{aligned}\frac{1}{2}\sum_{p,q}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q &= \sum_{p<q}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q \\ &= \sum_{q\geq 3}(C_1^{2q})^2(\lambda_1 + \lambda_2)\lambda_q + \sum_{p,q\geq 3}((C_1^{pq})^2 + (C_2^{pq})^2)\lambda_p\lambda_q.\end{aligned}$$

Note that $\lambda_m \geq \cdots \geq \lambda_2 \geq 0$. If $\lambda_1 + \lambda_2 = 0$, then the right-hand side of (5.3) is nonnegative. Since $\lambda_1 + \lambda_2$ is a concave function on X, $\lambda_1 + \lambda_2 \geq 0$ is preserved. By the strong maximum principle again, if $\lambda_1 + \lambda_2 > 0$ is positive at one point at time $t = 0$, then $\lambda_1 + \lambda_2 > 0$ is positive everywhere for all time $t > 0$.

(3) Now we prove the last part of Theorem 1.1. If the traceless bisectional curvature positive is non-negative or 2-non-negative, Theorem 1.2 implies that

$$R_{i\bar{i}j\bar{j}} \geq 0, \quad \forall\, i \neq j.$$

Let us assume that initially the Ricci curvature is positive and after finite time $t_0 > 0$, at some point $p \in X$, $R_{i\bar{j}}$ vanishes at least at one direction. For convenience, set this direction as $\frac{\partial}{\partial z_1}$ and diagonalize the Ricci curvature at this point. Then

$$\frac{\partial R_{1\bar{1}}}{\partial t}\Big|_{t_0} \geq R_{1\bar{1}j\bar{j}}R_{j\bar{j}} - R_{1\bar{1}}R_{1\bar{1}} = \sum_{j=2}^{n} R_{1\bar{1}j\bar{j}}R_{j\bar{j}} \geq 0.$$

By Hamilton's maximum principle for tensors, this is enough to show that the positivity of Ricci curvature is preserved under the condition.

6 Proof of Corollary 1.1

To prove Corollary 1.1, it suffices to prove that the Kähler-Ricci flow converges by sequences to some Kähler-Ricci soliton when the bisectional curvature is uniformly bounded by Theorem 1.2. In [21], N. Sesum proved that τ-flow converges by sequences to some Ricci soliton when the curvature operator and the diameter are uniformly bounded. In [22], Sesum also proved that the Kähler-Ricci flow converges by sequences to some Kähler-Ricci soliton outside some isolated points on any Kähler surface. The idea is more or less standard, and we include a proof here.

First let us recall Perelman's no local collapsing theorem.

Definition 6.1 (cf. [16]) *Let $g_{ij}(t)$ be a smooth solution to the Ricci flow $(g_{ij})_t$ on $[0, T)$ on a Riemannian manifold X of dimension n. We say that $g_{ij}(t)$ is locally collapsing at T, if there is a sequence of times $t_k \to T$ and a sequence of metric balls $B_k = B_k(p_k, r_k)$ at times t_k, such that $\frac{r_k^2}{t_k}$ is bounded, $|\mathrm{Rm}|(g_{ij})(t_k) \leq r_k^{-2}$ in B_k and $\frac{\mathrm{Vol}(B_k)}{r_k^n} \to 0$.*

Lemma 6.1 (cf. [16]) *If X is closed and $T < \infty$, then $g_{ij}(t)$ is not locally collapsing at T.*

Now we begin to prove Corollary 1.1.

Proof First we are ready to prove that the injectivity radii have a uniformly positive lower bound along the Kähler-Ricci flow. If the traceless bisectional curvature operator is 2-nonnegative and the scalar curvature is bounded along the flow, Theorem 1.2 implies that the curvature tensor is uniformly bounded.

Claim 6.1 *The injectivity radius has a uniformly positive lower bound along the flow.*

Proof Let $(X, g_{i\bar{j}})$ be the Kähler-Ricci flow. Fix $T > 0$. Now we re-scale the metric

$$\overline{g}_{i\bar{j}}(s) = (T-s)g_{i\bar{j}}\Big(-\log\Big(\frac{T-s}{T}\Big)\Big), \quad s \in [0, T). \tag{6.1}$$

Then, $\overline{g}_{i\bar{j}}(s)$ is a solution with finite maximal existence interval to the unnormalized Kähler-Ricci flow $\frac{\partial g_{i\bar{j}}}{\partial s} = -R_{i\bar{j}}$. Lemma 6.2 implies that $(X, \overline{g}_{i\bar{j}}(s))$ is not locally collapsing. In other words, for any sequence of times $s_k \to T$, any sequence of metric balls $B_k = B_k(x_k, r_k)$ at times

s_k, such that $\frac{r_k^2}{s_k}$ is bounded and $|\mathrm{Rm}|(\overline{g}_{ij})(s_k) \leq r_k^{-2}$ in B_k, there exists a constant $\delta > 0$ such that

$$\frac{\mathrm{Vol}(B_k)}{r_k^{2n}} \geq \delta. \tag{6.2}$$

Since $|\mathrm{Rm}|(g_{i\bar{j}}(t))$ is uniformly bounded along the Kähler-Ricci flow, for the un-normalized flow, we have

$$|\mathrm{Rm}|(\overline{g}_{i\bar{j}}(s)) \leq \frac{C}{T-s}.$$

We claim that there exists a constant $\epsilon > 0$ such that $\mathrm{inj}(\overline{g}(s)) \geq \sqrt{T-s}\,\epsilon$. We prove this by contradiction. Assume that there exists a sequence of times $s_k \to T$, such that

$$\frac{\mathrm{inj}(\overline{g}(s_k))}{\sqrt{T-s_k}} \to 0.$$

We re-scale the metric

$$h(s_k) = \frac{1}{T-s_k}\overline{g}(s_k).$$

Let $r_k^2 = T - s_k$. Then

$$|\mathrm{Rm}|(h(s_k)) \leq C, \quad \mathrm{inj}(h(s_k)) \to 0. \tag{6.3}$$

From (6.2), we have

$$\mathrm{Vol}(B_{h(s_k)}(x_k, 1)) \geq \delta. \tag{6.4}$$

Then, (6.3), (6.4) contradict J. Cheeger's injectivity radius estimate (cf. [18]). Thus, we have $\mathrm{inj}(\overline{g}(s)) \geq \sqrt{T-s}\,\epsilon$. Together with (6.1), we have

$$\mathrm{inj}(g(t)) \geq \epsilon > 0.$$

Claim 6.2 *The diameter has a uniformly upper bound along the flow.*

Proof To see this, we assume that there are N points $p_1, p_2, \cdots, p_N$ such that

$$\mathrm{dist}_{g(t)}(p_i, p_j) \geq 2\epsilon, \quad \forall\, 1 \leq i \neq j \leq N,$$

where $\epsilon > 0$ is the uniformly lower bound on the injectivity radius from Claim 6.1. Hence, the balls $B_{g(t)}(p_i, \epsilon)$ are embedded and pairwisely disjoint. Since the curvature operator is uniformly bounded, from the volume comparison theorem,

$$V \geq \sum_{i=1}^{N} \mathrm{Vol}(B_{g(t)}(p_i, \epsilon)) \geq NC\epsilon^{2n}.$$

Since the volume V is fixed along the flow, N is bounded from above. Consequently the diameter has a uniformly upper bound along the flow.

Now we return to the proof of Corollary 1.1. Since we have uniformly bounds on curvature tensor and uniformly lower bound on the injectivity radius, by Hamilton's compactness theorem, for every $t_k \to \infty$ as $k \to \infty$, there exists a subsequence such that $(X, g(t_k + t))$ converges to $(X, h(t))$, in the sense that there exist diffeomorphisms $\phi_i : X \to X$, such that $\phi_i^* g(t_k + t)$ converge uniformly together with their covariant derivatives to metrics $h(t)$ on any compact subsets. For every sequence of times $t_k \to \infty$, there exists a subsequence, such that the $(X, g(t_k + t))$ converges to a Kähler Ricci soliton as $k \to \infty$. The readers are referred to [21, 22] for details. The corollary is proved.

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